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Feasible and Continuous Double Implementation of Constrained Walrasian Allocations*

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This paper provides a simple, feasible, and continuous mechanism whose Nash allocations and strong Nash allocations coincide with constrained Walrasian allocations for pure exchange economies. This mechanism allows not only preferences and initial endowments but also coalition patterns to be privately observed, a feature missing from much recent work in implementation theory. Unlike most mechanisms proposed in the literature, our mechanism works not only for three or more agents, but also for two-agent economies, and thus it is a unified mechanism which is irrespective of the number of agents. In addition, our implementation result is obtained without modifying individuals' preferences and thus it improves the mechanism proposed in Peleg (1996) without having to define artificial preference relations on price messages announced by individuals. *Journal of Economic Literature* Classification Numbers: C72, D61, D78.

Key Words: Incentive mechanism design; Walrasian allocations; Implementation.

1. INTRODUCTION

This paper considers the problem of double implementation of constrained Walrasian allocations in Nash and strong Nash equilibria using a "market type" feasible and continuous mechanism for pure exchange economies when coalition patterns, preferences, and endowments are unknown to the designer. The important reasons for preferring double implementation over Nash implementation and strong Nash implementation are two-fold: (1) The double implementation covers the case where in some coalitions agents

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may cooperate and in others they may not, and such information is unknown to the designer. (2) This combining solution concept, which characterizes agents' strategic behavior, may yield a state that is easy to reach and hard to leave.

There are numerous papers on implementation of (constrained) Walrasian allocations in various solution concepts in implementation literature, including Hong (1995), Hurwicz (1972, 1979), Hurwicz, Maskin, and Postlewaite (1995), Nakamura (1990), Peleg (1996), Postlewaite and Wettstein (1989), Schmeidler (1980), Suh (1994), Tian (1989, 1990, 1992), and Tian and Li (1995). Most of these papers, however, only considered implementation of Walrasian allocations by using a Nash equilibrium as a solution concept to describe individuals' self-interested behavior. Nash equilibrium is a strictly noncooperative notion and is only concerned with single individual deviations at which no one can be improved by a unilateral deviation from a prescribed strategy profile. No cooperation among agents is allowed. Although a Nash equilibrium may be easy to reach, it may not be stable in the sense that there may exist a group of agents who can be improved by forming a coalition. Thus it is natural to adopt strong Nash equilibrium which allows possible cooperation (coalitions) among agents. However, while a strong Nash equilibrium may result in a more stable equilibrium outcome, it requires more information about communication network and other agents' characteristics in order to eliminate those outcome that can upset by coalitionary action. Thus, in order to have a solution concept combining the properties of Nash and strong Nash equilibria, it is desirable to construct a mechanism which doubly implements a social choice rule by Nash and strong Nash equilibria so that its equilibrium outcomes are not only easy to reach, but also hard to leave. By double implementation it also covers the situation where agents in some coalitions will cooperate and in some other coalitions will not. Thus the designer does not need to know which coalitions are permissible and consequently it allows the possibility for agents to manipulate coalition patterns.

Schmeidler (1980) was the first to consider double implementation of (constrained) Walrasian allocations in Nash and strong Nash equilibria. However, Schmeidler's mechanism has two properties one may consider undesirable. One is that his outcome function is not individually feasible: out of equilibria, some outcome allocations may not be in the consumption set; although at equilibrium, they are necessarily in the consumption set. The other is that the outcome function is not continuous: small changes in an agent's strategy choice may lead to large jumps in the resulting allocations (so it is not robust with respect to some misspecifications). In fact, there does not exist any feasible mechanism which Nash-implements the Walrasian correspondence because the correspondence violates Maskin's monotonicity condition when a boundary equilibrium allocations occur.

However, it is possible to design a mechanism which doubly implements a slightly larger set than Walrasian allocations, namely, constrained Walrasian allocations which are Pareto-efficient and individually rational (cf. Tian (1988)). Maskin (1979) has proved that the social choice correspondence which selects all Pareto-efficient and individual allocations are doubly implementable in Nash and strong Nash equilibria.

Suh (1994) provided a necessary and sufficient condition for a social choice correspondence to be doubly implementable in Nash and strong Nash equilibrium on a restricted domain of economic environments. By applying this characterization result, Suh (1994) investigated the double implementation of (constrained) Walrasian allocations in Nash and strong Nash equilibria. However, due to the general nature of the social choice rules under consideration, the implementing mechanisms turn out to be quite complex. Characterization results show what is possible for the implementation of a social choice rule (correspondence), but not what is realistic. Thus, like most characterization results in the literature, Suh's mechanism is not continuous and has a large message space of infinite dimension.

Recently, Peleg (1996) gave a feasible and continuous mechanism which doubly implements constrained Walrasian allocations. But a main drawback of his mechanism is that it is not a pure mechanism in the sense that a preference relation is artificially introduced for individuals to rank announced prices. In other words, his implementation result is obtained not based on the original preferences defined on the allocation space but based on the re-defined preferences on the outcome/message space which consists of allocations and price determinations (see equation (3.8) in Peleg (1996)). In Peleg's approach, an individual is not only a player, but also an *inside* auctioneer. Unlike the traditional auctioneer defined in the literature, the auctioneers defined in Peleg (1996) not only announce and adjust prices, but also are assigned a preference relation on the level of prices announced. In an incentive mechanism design, the preferences of agents should be given, not be assigned, otherwise any social choice rule may be implementable in any equilibrium solution concept by signing a preference for some components of messages announced by agents. In our opinion, the designer cannot vary an agent's preference behavior since it is in fact determined by the agent himself. Thus, it still leaves a question of whether there is a pure mechanism which is feasible and continuous, and further it doubly implements the constrained Walrasian correspondence in Nash and strong Nash equilibria.

A similar situation prevailed with regard to double implementation of the Lindahl correspondence in Nash and strong Nash equilibria until Tian (2000) presented a pure continuous and feasible mechanism which doubly implements the Lindahl correspondence in Nash and strong Nash equilibria.

This paper will answer this question affirmatively by giving a pure feasible and continuous mechanism which doubly implements the constrained Walrasian correspondence. Our implementation result is obtained without changing individuals' preferences and thus improves on the mechanism proposed in Peleg (1996) without having to introduce artificial preference relations for individuals on the outcome space. In addition, our mechanism works not only for three or more agents, but also for a two-agent world. While most mechanisms need to distinguish the case of two agents from that of three or more agents, this paper gives a unified mechanism which is irrespective of the number of agents. Further, our implementation result holds on a very large domain of economic environments, including some non-neoclassical economic environments. Only the strict monotonicity condition is assumed in our feasible and continuous mechanism and thus no continuity and convexity assumptions on preferences are needed, and further, preferences may be nontotal or nontransitive. Finally, our mechanism is simple and natural. It is a type of "market game" and thus it is similar to the Walras rule: the strategies of the mechanism are "prices" and "quantities", and agents' consumption is chosen from their budget sets. The "natural" mechanism design provides at least a partial response to a common concern about much of the implementation literature, namely that the implementing mechanisms are highly unrealistic and impossible for a real player to use. Besides, we allow preferences of agents to be discontinuous and nontotal-nontransitive.

The remainder of the paper is organized as follows. Section 2 presents notions, definitions, and solution concepts which will be used in the paper. Section 3 presents a feasible and continuous mechanism which doubly implements the constrained Walrasian correspondence. Section 4 proves the equivalence among Nash allocations, strong Nash allocations, and constrained Walrasian allocations. Finally, some concluding remarks are given in Section 5.

2. NOTATION AND DEFINITIONS

2.1. Economic Environments

We consider a class of pure exchange economies where there are n $(n \geq 2)$ agents who consume L private goods. Denote by $N = \{1, 2, \dots, n\}$ the set of agents. Each agent's characteristic is denoted by $e_i = (\mathring{w}_i,\ P_i)$, where $\mathring{w}_i \in \mathbb{R}^L_+$ is agent i's initial endowment and P_i is the strict (irreflexive) preference defined on $\mathbb{R}^L_+$ which is strictly monotonically increasing may be nontotal or nontransitive.

An economy is the full vector $e = (e_1, \dots, e_n)$ and the set of all such economies is denoted by E.

2.2. The Constrained Walrasian Allocations

An allocation $x^* = (x_1^*, x_2^*, \ldots, x_n^*) \in \mathbb{R}_+^{nL}$ is a *constrained Walrasian allocation* for an economy e if there is a price vector $p^* \in \mathbb{R}_+^L$ such that

(1) $p^* \cdot x_i^* \leq p^* \cdot \mathring{w}_i$ for all $i \in N$,
(2) for all $i \in N$, there does not exist $x_i \in \mathbb{R}_+^L$ such that

(2.a) $x_i \; P_i \; x_i^*$;
(2.b) $p^* \cdot x_i \leq p^* \cdot \mathring{w}_i$;
(2.c) $x_i \leq \sum_{j=1}^n \mathring{w}_j$,

(3) $\sum_{j=1}^n x_j \leq \sum_{j=1}^n \mathring{w}_j$.

Denote by $W_c(e)$ the set of all such allocations.

Note that a constrained Walrasian allocation differs from a Walrasian allocation only in a way that each agent maximizes his preferences not only subject to his budget constraint, but also subject to total endowments available to the economy. It can be easily shown that every ordinary Walrasian allocation (competitive equilibrium allocation) is a constrained Walrasian allocation and every interior constrained Walrasian allocation is a Walrasian allocation (cf. Tian (1988)).

An allocation x is *Pareto-efficient* with respect to strict preference profile $P = (P_1, \ldots, P_n)$ if it is feasible (i.e., $x \in \mathbb{R}_+^{nL}$ and $\sum_{j=1}^n x_j \leq \sum_{j=1}^n \mathring{w}_j$) and there does not exist another feasible allocation x' such that $x_i' \; P_i \; x_i$ for all $i \in N$.

An allocation x is *individually rational* with respect to P if it is not true that $\mathring{w}_i \; P_i \; x_i$ for all $i \in N$.

It can be easily shown that every constrained Walrasian allocation is Pareto-efficient and individually rational.

An *coalition* C is a non-empty subset of N.

A group of agents (a coalition) $C \subset N$ is said to block an allocation x if there exists some allocation (x', y') such that

(i) $\sum_{i \in C} x_i' \leq \sum_{i \in C} \mathring{w}_i$,
(ii) $x_i' P_i x_i$ for all $i \in C$.

A feasible allocation x is said to be in the *core* of e if there does not exist any coalition C that can improve upon x.

Note that an allocation cannot be improved upon by N if and only if it is Pareto efficient, and an allocation cannot be improved upon by any single person if and only if it is individually rational. Also every constrained Walrasian allocation is in the core of e.

2.3. Mechanism

Let M_i denote the i-th message (strategy) domain. Its elements are written as m_i and called messages. Let $M = \prod_{i=1}^n M_i$ denote the message (strategy) space. Let $X : M \to Z$ denote the outcome function, or more explicitly, $X_i(m)$ is the i-th agent's outcome at m. A mechanism consists of $\langle M, X\rangle$, which is defined on E. A message $m^* = (m_1^*, \dots, m_n^*) \in M$ is a *Nash equilibrium* (NE) of the mechanism $\langle M, X\rangle$ for an economy $e \in E$ if for all $i \in N$ and $m_i \in M_i$, it is not true that

$$X_i(m_i, m_{-i}^*) \; P_i \; X_i(m^*), \tag{1}$$

where $(m_i, m_{-i}^*) = (m_1^*, \dots, m_{i-1}^*, m_i, m_{i+1}^*, \dots, m_n^*)$. The outcome $X(m^*)$ is then called a *Nash (equilibrium) allocation*. Denote by $V_{M,X}(e)$ the set of all such Nash equilibria and by $N_{M,X}(e)$ the set of all such Nash (equilibrium) allocations.

A mechanism $\langle M, X\rangle$ *Nash-implements* the constrained Walrasian correspondence W_c on E if for all $e \in E$, $N_{M,X}(e) = W_c(e)$.

A message $m^* = (m_1^*, \dots, m_n^*) \in M$ is said to be a *strong Nash equilibrium* of the mechanism $\langle M, X\rangle$ for an economy $e \in E$ if there does not exist any coalition C and $m_C \in \prod_{i \in C} M_i$ such that for all $i \in C$,

$$X_i(m_C, m_{-C}^*) \; P_i \; X_i(m^*). \tag{2}$$

$X(m^*)$ is then called a *strong Nash (equilibrium) allocation* of the mechanism for the economy e. Denote by $SV_{M,X}(e)$ the set of all such strong Nash equilibria and by $SN_{M,X}(e)$ the set of all such strong Nash (equilibrium) allocations.

The mechanism $\langle M, h\rangle$ is said to doubly *implement* the constrained Walrasian correspondence W_c on E, if, for all $e \in E$, $SN_{M,N}(e) = N_{M,X}(e) = W_c(e)$.

A mechanism $\langle M, X\rangle$ is *feasible* if for all $m \in M$, $X(m) \in \mathbb{R}_+^{nL}$ and

$$\sum_{j=1}^n X_j(m) \leq \sum_{j=1}^n \mathring{w}_j. \tag{3}$$

3. A FEASIBLE AND CONTINUOUS MECHANISM

In this section, we present a simple feasible and continuous mechanism which doubly implements the constrained Walrasian correspondence on E.

For each $i \in N$, let the message domain of agent i be of the form

$$M_i = (0, \mathring{w}_i] \times \Delta_{++}^L \times \mathbb{R}^{nL}, \tag{4}$$

where $\Delta_{++}^L = \{(p \in \mathbb{R}_{++}^L : \sum_{l=1}^L p^l = 1\}$.

A generic element of M_i is $m_i = (w_i, p_i, x_{i1}, ..., x_{in})$ whose components have the following interpretations. The component w_i denotes a profession of agent i's endowment, the inequality $0 < w_i \leq \mathring{w}_i$ means that the agent cannot overstate his own endowment; on the other hand, the endowment can be understated, but the claimed endowment w_i must be positive. Note that, although the true endowment is the upper bound of the reported endowment, the designer does not need to know this upper bound. This is because whenever an agent claims an endowment of a certain amount, the designer can ask him to *exhibit* it (one may, for instance, imagine that the rules of the game require that the agent 'put on the table' the reported amount w_i). The component p_i is the price vector proposed by agent i and is used as a price vector of agent $i-1$, where $i-1$ is read to be n when $i=1$. The component x_{ij} is interpreted as the contribution that agent i is willing to make to agent j (a negative x_{ij} means agent i wants to get $-x_{ij}$ amount of goods from agent j).

Define agent i's price vector $p_i : M \to \mathbb{R}^L_{++}$ by

$$p_i(m) = p_{i+1}, \tag{5}$$

where $n+1$ is to be read as 1. Note that although $p_i(\cdot)$ is a function of proposed price vector announced by agent $i+1$, for simplicity, we can write $p(\cdot)$ as a function of m without loss of generality. Also it may be remarked that the construction of $p_i(m)$ is much simpler than the one used in Postlewaite and Wettstein (1989) and Tian (1992), in which it is determined by proposed price vector of all individuals, while ours is only involved one person's proposed price.

Define a feasible correspondence $B : M \to\to \mathbb{R}^{nL}_+$ by

$$\begin{aligned} B(m) \;=\; \{x \in \mathbb{R}^{nL}_+ : & \sum_{i=1}^{n} x_i \leq \sum_{i=1}^{n} w_i \;\& \\ & p_i(m) \cdot x_i \leq \frac{1}{1 + \|p_i - p_i(m)\|} p_i(m) \cdot w_i \;\forall\; i \in N\}, \end{aligned} \tag{6}$$

which is clearly nonempty compact convex (by the total resource constraints) for all $m \in M$. We will show the following lemma in the Appendix.

LEMMA 1. *$B(\cdot)$ is continuous on M.*

Let $\tilde{x}_j = \sum_{i=1}^{n} x_{ij}$ which is the sum of contributions that agents are willing to make to agent j and $\tilde{x} = (\tilde{x}_1, \tilde{x}_2, ..., \tilde{x}_n)$.

The outcome function $X\colon M \to \mathbb{R}^{nL}_+$ is given by

$$X(m) = \{y \in \mathbb{R}^{nL}_+ : \min_{y \in B(m)} \|y - \tilde{x}\|\}, \tag{7}$$

which is the closest to $\tilde{x}$. Then X is single-valued and continuous on M. [1] Also, since $X(m) \in \mathbb{R}^{nL}_+$ and

$$\sum_{i=1}^{n} X_i(m) \leq \sum_{i=1}^{n} \mathring{w}_i \tag{8}$$

for all $m \in M$, the mechanism is feasible and continuous.

Remark 3.1. Note that the above mechanism does not depends on the number of agents. Thus it is a unified mechanism which works for two-agent economies as well as for economies with three or more agents. For two-agents economies, only feasible and continuous mechanism which Nash implements the constrained Walrasian correspondence was given by Nakamura (1990). Here we give an even simpler feasible and continuous mechanism which implements the constrained Walrasian correspondence not only in Nash equilibrium, but also in strong Nash equilibrium.

4. RESULTS

The remainder of this paper is devoted to the proof of equivalence among Nash allocations, strong Nash allocations, and constrained Walrasian allocations. Proposition 4.1 below proves that every Nash allocation is a constrained Walrasian allocation. Proposition 4.2 below proves that every constrained Walrasian allocation is a Nash allocation. Proposition 4.3 below proves that every Nash equilibrium is a strong Nash equilibrium. To show these results, we first prove the following lemmas.

LEMMA 2. *If $m^* \in V_{M,X}(e)$, then $p_1^* = p_2^* = \ldots = p_n^*$, and thus $p_1(m^*) = p_2(m^*) = \ldots = p_n(m^*) = p^*$ for some $p^* \in \Delta^L_{++}$.*

Proof. Suppose, by way of contradiction, that $p_i^* \neq p_{i+1}^*$ (i.e., $p_i^* \neq p_i(m^*)$) for some $i \in N$. Then $p_i(m^*) \cdot X_i(m^*) \leq \frac{1}{1+\|p_i^* - p_i(m^*)\|} p_i(m^*) \cdot w_i^* < p_i(m^*) \cdot w_i^*$, and thus there is $x_i \in \mathbb{R}^L_+$ such that $p_i(m^*) \cdot x_i \leq p_i(m^*) \cdot w_i^*$ and $x_i\ P_i\ X_i(m^*)$ by strict monotonicity of preferences. Now

[1] This is because X is an upper semi-continuous correspondence by Berge's Maximum Theorem (see Debreu (1959), p.19]) and single-valued (see Mas-Colell (1985), p.28]).

if agent i chooses $p_i = p_i(m^*)$, $x_{ii} = x_i - \sum_{t \neq i} x^*_{ti}$, $x_{ij} = -\sum_{t \neq i} x^*_{tj}$ for $j \neq i$, and keeps w^*_i unchanged, then $(0, \ldots, 0, x_i, 0, \ldots, 0) \in B(m_i, m^*_{-i})$, and thus $X_i(m_i, m^*_{-i}) = x_i$. Therefore, $X_i(m_i, m^*_{-i})\ P_i\ X_i(m^*)$. This contradicts $X(m^*) \in N_{M,X}(e)$. Thus we must have $p^*_1 = p^*_2 = \ldots = p^*_n$, and therefore $p_1(m^*) = p_2(m^*) = \ldots = p_n(m^*) = p^*$ for some $p^* \in \Delta^L_+$. ▮

LEMMA 3. *If $m^* \in N_{M,X}(e)$, then $w^*_i = \mathring{w}_i$ for all $i \in N$.*

Proof. Suppose, by way of contradiction, that $w^*_i \neq \mathring{w}_i$ for some $i \in N$. Then $p_i(m^*) \cdot X_i(m^*) \leq p_i(m^*) \cdot w^*_i < p_i(m^*) \cdot \mathring{w}_i$, and thus there is $x_i \in \mathbb{R}^L_+$ such that $p_i(m^*) \cdot x_i \leq p_i(m^*) \cdot \mathring{w}_i$ and $x_i\ P_i\ X_i(m^*)$ by strict monotonicity of preferences. Now if agent i chooses $w_i = \mathring{w}_i$, $x_{ii} = x_i - \sum_{t \neq i} x^*_{ti}$, $x_{ij} = -\sum_{t \neq i} x^*_{tj}$ for $j \neq i$, and keeps p^*_i unchanged, then $(0, \ldots, 0, x_i, 0, \ldots, 0) \in B(m_i, m^*_{-i})$, and thus $X_i(m_i, m^*_{-i}) = x_i$. Hence, $X_i(m_i, m^*_{-i})\ P_i\ X_i(m^*)$. This contradicts $X(m^*) \in N_{M,X}(e)$ and thus $w^*_i = \mathring{w}_i$ for all $i \in N$. ▮

LEMMA 4. *If $X(m^*) \in N_{M,X}(e)$, then $p_i(m^*) \cdot X_i(m^*) = p_i(m^*) \cdot \mathring{w}_i$.*

Proof. Suppose, by way of contradiction, that $p_i(m^*) \cdot X_i(m^*) < p_i(m^*) \cdot \mathring{w}_i$ for some $i \in N$. Then there is $x_i \in \mathbb{R}^L_+$ such that $p_i(m^*) \cdot x_i \leq p_i(m^*) \cdot \mathring{w}_i$ and $x_i\ P_i\ X_i(m^*)$ by strict monotonicity of preferences. Now if agent i chooses $x_{ii} = x_i - \sum_{t \neq i} x^*_{ti}$, $x_{ij} = -\sum_{t \neq i} x^*_{tj}$ for $j \neq i$, and keeps p^*_i and w^*_i unchanged, then $(0, \ldots, 0, x_i, 0, \ldots, 0) \in B(m_i, m^*_{-i})$, and thus $X_i(m_i, m^*_{-i}) = x_i$. Hence, $X_i(m_i, m^*_{-i})\ P_i\ X_i(m^*)$. This contradicts $X(m^*) \in N_{M,X}(e)$. ▮

PROPOSITION 1. *If the mechanism $\langle M, X \rangle$ defined above has a Nash equilibrium m^* for $e \in E$, then $X(m^*)$ is a constrained Walrasian allocation with p^* as a competitive equilibrium price vector, i.e., $N_{M,X}(e) \subset W_c(e)$ for all $e \in E$.*

Proof. Let m^* be a Nash equilibrium. Then $X(m^*)$ is a Nash equilibrium allocation. We wish to show that $X(m^*)$ is a constrained Walrasian allocation. By Lemmas 4.1-4.3, $p_1(m^*) = \ldots = p_n(m^*) = p^*$ for some $p^* \in \Delta^L_+$, $w^*_i = \mathring{w}_i$, and $p(m^*) \cdot X_i(m^*) = p(m^*) \cdot \mathring{w}_i$ for all $i \in N$. Also, by the construction of the mechanism, we know that $X(m^*) \in \mathbb{R}^{nL}_+$ and $\sum_{j=1}^n X_j(m^*) \leq \sum_{j=1}^n \mathring{w}_j$. So we only need to show that each individual is maximizing his/her preferences. Suppose, by way of contradiction, that for some agent i, there exists some $\tilde{x}_i \in \mathbb{R}^L_+$ such that $\tilde{x}_i \leq \sum_{j=1}^n \mathring{w}_j$, $p(m^*) \cdot \tilde{x}_i \leq p(m^*) \cdot \mathring{w}_i$, and $\tilde{x}_i\ P_i\ X_i(m^*)$. Let $x_{ii} = \tilde{x}_i - \sum_{t \neq i} x^*_{ti}$, $x_{ij} = -\sum_{t \neq i} x^*_{tj}$ for $j \neq i$, and keep p^*_i and w^*_i unchanged, then $(0, \ldots, 0, \tilde{x}_i, 0, \ldots, 0) \in B(m_i, m^*_{-i})$, and thus $X_i(m_i, m^*_{-i}) = \tilde{x}_i$. Therefore, we have $X_i(m_i, m^*_{-i})\ P_i\ X_i(m^*)$. This contradicts $X(m^*) \in N_{M,X}(e)$. So $X(m^*)$ is a constrained Walrasian allocation. ▮

PROPOSITION 2. *If $x^* = (x^*_1, x^*_2, ..., x^*_n)$ is a constrained Walrasian allocation with a competitive equilibrium price vector $p* \in \Delta^L_+$ for $e \in E$, then there exists a Nash equilibrium m^* of the mechanism $\langle M, X\rangle$ defined above such that $X_i(m^*) = x^*_i$, $p_i(m^*) = p*$, for all $i \in N$, i.e., $W_c(e) \subset N_{M,X}(e)$ for all $e \in E$.*

Proof. Since preferences satisfy the strict monotonicity condition and x^* is a constrained Walrasian allocation, we must have $p^* \in \Delta^L_{++}$, $\sum_{j=1}^n x^*_j \leq \sum_{j=1}^n \mathring{w}_j$ and $p^* \cdot x^*_i = p^* \cdot \mathring{w}_i$ for $i \in N$. Now for each $i \in N$, let $m^*_i = (p*, x^*_{i1}, ..., x^*_{in})$, where $x^*_{ii} = x^*_i$ and $x^*_{ij} = 0$ for $j \neq i$.

Then x^* is an outcome with p^* as a price vector, i.e., $X_i(m^*) = x^*_i$ for all $i \in N$, and $p_i(m^*) = p^*$. We show that m^* yields this allocation as a Nash allocation. In fact, agent i cannot change $p_i(m^*)$ by changing his proposed price (i.e., $p_i(m_i, m^*_{-i}) = p_i(m^*)$ for all $m_i \in M_i$). Announcing a different message m_i by agent i may yield an allocation $X(m_i, m^*_{-i})$ such that $X_i(m_i, m^*_{-i}) \in \mathbb{R}^L_+$ and

$$p(m^*) \cdot X_i(m_i, m^*_{-i}) \leq p(m^*) \cdot \mathring{w}_i. \tag{9}$$

Now suppose, by way of contradiction, that m^* is not a Nash equilibrium. Then there are $i \in N$ and m_i such that $X_i(m_i, m^*_{-i})\ P_i\ X_i(m^*)$. Since $X_i(m_i, m^*_{-i}) \leq \sum_{i=1}^n \mathring{w}_i$, we must have, by the definition of the constrained Walrasian allocation, $p(m^*) \cdot X_i(m_i, m^*_{-i}) > p(m^*) \cdot \mathring{w}_i$. But this contradicts the budget constraint (9). Thus we have shown that agent i cannot improve his/her utility by changing his/her own message while the others' messages remain fixed for all $i \in N$. Hence x^* is a Nash allocation. ▮

PROPOSITION 3. *Every Nash equilibrium m^* of the mechanism defined above is a strong Nash equilibrium, that is, $N_{M,X}(e) \subseteq SN_{M,X}(e)$.*

Proof. Let m^* be a Nash equilibrium. By Proposition 4.1, we know that $X(m^*)$ is a constrained Walrasian allocation with $p(m^*)$ as a price vector. Then $X(m^*)$ is Pareto optimal and thus the coalition N cannot be improved upon by any $m \in M$. Now for any coalition C with $\emptyset \neq C \neq N$, choose $i \in C$ such that $i + 1 \notin C$. Then no strategy played by C can change the budget set of i since $p_i(m)$ is determined by p_{i+1}. Furthermore, because $X(m^*) \in W_c(e)$, it is the preference maximizing consumption with respect to the budget set of i, and thus C cannot improve upon $X(m^*)$. ▮

Since every strong Nash equilibrium is clearly a Nash equilibrium, then by combining Propositions 4.1-4.3, we have the following theorem.

THEOREM 1. *For the class of exchange economies E, there exists a feasible and continuous mechanism which doubly implements the constrained Walrasian correspondence. That is,* $N_{M,X}(e) = SN_{M,X}(e) = W_c(e)$ *for all* $e \in E$.

5. CONCLUDING REMARKS

This paper gives a simple, feasible, and continuous mechanism which doubly implements the constrained Walrasian correspondence in Nash and strong Nash equilibrium for economies without total, transitive, continuous, and convex preferences when coalition patterns, preferences and endowments are unknown to the designer. The implementation result is obtained without changing individuals' preferences and thus it improves the mechanism proposed in Peleg (1996) without assigning artificial preference relations for individuals on outcome space. In addition, unlike most mechanisms proposed in the literature, it gives a unified mechanism which is irrespective of the number of agents.

It may be remarked that the implementation of the constrained Walrasian allocations does not lose much generality, compared to implementing any other social choice correspondence which guarantees Pareto-efficiency and individual rationality. A slightly modified version of Theorems 1 and 2 of Hurwicz (1979a) states that for any mechanism, if all of its Nash allocations for a given environment are Pareto-efficient and individually rational, then every constrained Walrasian allocation is a Nash allocation, and every interior Nash allocation is a constrained Walrasian allocation. Thus, the implementation of the constrained Walrasian correspondence at most loses some possible boundary Pareto-efficient and individually rational allocations.

Also note that, by the construction of the mechanism, one can see that the mechanism constructed in the paper only yields weakly balanced allocations in the sense that equation (8) may not hold with equality for all messages; although at equilibrium, it necessarily holds in equality. However, since the above mechanism is a unified mechanism which deals with both cases of two-agent economies and economies with three or more agents, a result given by Kwan and Nakamura (1987) showed that it is impossible to find another mechanism which Nash implements constrained Walrasian allocations by a balanced feasible and continuous mechanism that works for two-agent economic environments.[2] Consequently, it is impossible to have a mechanism which doubly implements the constrained Walrasian correspondence in Nash and strong Nash equilibria by a balanced feasible and continuous mechanism that also works for two-agent economic environments. An open question is whether or not there exists a balanced feasible and continuous mechanism which doubly implements the constrained Walrasian correspondence in Nash and strong Nash equilibrium for economies with three or more agents.

Though this paper only considers double implementation of the constrained Walrasian correspondence in Nash and strong Nash equilibria, we think some of the techniques developed in the paper can be applied to doubly implement other social choice rules which results in Pareto efficient and individually rational allocations in Nash and strong Nash equilibrium.

APPENDIX

Proof of Theorem 1: It is clear that $B(\cdot)$ has closed graph by the continuity of $p_i(\cdot)$. Since the range space of the correspondence $B(\cdot)$ is bounded by the total endowments $\sum_{i=1}^n w_i$, it is compact. Thus, $B(\cdot)$ is upper hemicontinuous on M. So we only need to show that $B(m)$ is also lower hemicontinuous at every $m \in M$. Let $m \in M$, $x = (x_1, \ldots, x_n) \in B(m)$, and let $\{m_k\}$ be a sequence such that $m_k \to m$, where $m_k = (m_1^k, \ldots, m_n^k)$ and $m_i^k = (w_i^k, p_i^k, z_{i1}^k, \ldots, z_{in}^k)$. We want to prove that there is a sequence $\{x_k\}$ such that $x_k \to x$, and, for all k, $x_k \in B(m_k)$, i.e., $x_k = (x_{1k}, \ldots, x_{nk}) \in \mathbb{R}_+^{nL}$, $p_i(m_k) \cdot x_{ik} \leq \frac{1}{1+\|p_i^k - p_i(m_k)\|} p_i(m_k) \cdot w_i^k$ for all $i \in N$, and $\sum_{i \in N} x_{ik} \leqq \sum_{i \in N} w_i^k$. We first prove that there is a sequence $\{\hat{x}_k\}$ such that $\hat{x}_k \to x$, and, for all k, $\hat{x}_k \in \mathbb{R}_+^{nL}$ and $p_i(m_k) \cdot \hat{x}_{ik} \leq \frac{p_i(m_k) \cdot w_i^k}{1+\|p_i^k - p_i(m_k)\|}$ for all $i \in N$. For each $i \in N$, two cases will be considered.

Case 1. $p_i(m) \cdot x_i < \frac{p_i(m) \cdot w_i}{1+\|p_i - p_i(m)\|}$. Hence, for all k larger than a certain integer k', we have $p_i(m_k) \cdot x_i < \frac{p_i(m_k) \cdot w_i^k}{1+\|p_i^k - p_i(m_k)\|}$ by noting that $p_i(\cdot)$ is continuous. Let $\hat{x}_{ik} = x_i$ for all $k > k'$ and $\hat{x}_{ik} = 0$ for $k \leq k'$. Then, we have $p_i(m_k) \cdot \hat{x}_{ik} < \frac{p_i(m_k) \cdot w_i^k}{1+\|p_i^k - p_i(m_k)\|}$.

Case 2. $p_i(m) \cdot x_i = \frac{p_i(m) \cdot w_i}{1+\|p_i - p_i(m)\|}$. Note that, since $p_i(m) > 0$ and $w_i > 0$ for all i, we must have $x_i > 0$. Let $\omega_i = \frac{p_i(m) \cdot w_i}{1+\|p_i - p_i(m)\|}$ and $\omega_{ik} =$

[2] For economies with three or more agents, however, it is possible to find a mechanism which Nash implements the constrained Walrasian allocation by a balanced feasible and continuous mechanism. Tian (1992) gave such a mechanism.

$\frac{p_i(m_k)\cdot w_i^k}{1+\|p_i^k-p_i(m_k)\|}$. Define $\hat{x}_{ik}$ as follows:

$$\hat{x}_{ik} = \begin{cases} \frac{\omega_{ik}}{p_i(m_k)\cdot x_i} x_i, & \text{if } \frac{\omega_{ik}}{p_i(m_k)\cdot x_i} \leq 1; \\ x_i, & \text{otherwise.} \end{cases}$$

Then $\hat{x}_{ik} \leq x_i$, and $p_i(m_k) \cdot \hat{x}_{ik} \leq \frac{p_i(m_k)\cdot w_i^k}{1+\|p_i^k-p_i(m_k)\|}$. Also, since $\frac{\omega_{ik}}{p_i(m_k)\cdot x_i} \to \frac{\omega_i}{p_i(m)\cdot x_i} = 1$, we have $\hat{x}_{ik} \to x_i$. Thus, in both cases, there is a sequence $\{\hat{x}_k\}$ such that $\hat{x}_k \to x$, and, for all k, $\hat{x}_k \in \mathbb{R}_+^{nL}$ and $p_i(m_k) \cdot \hat{x}_{ik} \leq \frac{p_i(m_k)\cdot w_i^k}{1+\|p_i^k-p_i(m_k)\|}$ for all $i \in N$.

We now show that there is a sequence $\{\bar{x}_k\}$ such that $\bar{x}_k \to x$, and, for all k, $\bar{x}_k \in \mathbb{R}_+^{nL}$ and $\sum_{i\in N} \bar{x}_{ik} \leq \sum_{i\in N} w_i^k$. For each $l = 1, \ldots, L$, two cases will be considered.

Case 1. $\sum_{i\in N} x_i^l < \sum_{i\in N} w_i^l$. Hence, for all k larger than a certain integer k', we have $\sum_{i\in N} x_i^l < \sum_{i\in N} w_i^{lk}$. For each $i \in N$, let $\bar{x}_{ik}^l = x_i^l$ for all $k > k'$ and $\hat{x}_{ik}^l = 0$ for $k \leq k'$. Then, we have $\sum_{i\in N} x_{ik}^l < \sum_{i\in N} w_i^{lk}$.

Case 2. $\sum_{i\in N} x_i^l = \sum_{i\in N} w_i^l$. Note that, since $w_i > 0$ for all i, we must have $\sum_{i\in N} x_i^l > 0$. For each $i \in N$, define $\bar{x}_{ik}^l$ as follows:

$$\bar{x}_{ik}^l = \begin{cases} \frac{\sum_{i\in N} w_i^{lk}}{\sum_{i\in N} x_i^l} x_i, & \text{if } \frac{\sum_{i\in N} w_i^{lk}}{\sum_{i\in N} x_i^l} \leq 1; \\ x_i^l, & \text{otherwise.} \end{cases}$$

Then $\hat{x}_{ik}^l \leq x_i^l$, and $\sum_{i\in N} \bar{x}_{ik}^l \leq \sum_{i\in N} w_i^{lk}$. Also, since $\frac{\sum_{i\in N} w_i^{lk}}{\sum_{i\in N} x_i^l} \to \frac{\sum_{i\in N} w_i^{lk}}{\sum_{i\in N} x_i^l} = 1$, we have $\hat{x}_{ik}^l \to x_i^l$. Thus, in both cases, there is a sequence $\{\hat{x}_k\}$ such that $\hat{x}_k \to x$, and, for all k, $\hat{x}_k \in \mathbb{R}_+^{nL}$ and $\sum_{i\in N} x_i^k \leq \sum_{i\in N} w_i^k$. Here $\hat{x}_k = (\hat{x}_k^1, \ldots, \hat{x}_k^L)$.

Finally, let $x_k' = \min(\bar{x}_k, \hat{x}_k)$ with $x_{ik}' = \min(\bar{x}_{ik}, \hat{x}_{ik})$ for $i = 1, \ldots, n$. Then $x_k' \to x$ since $\bar{x}_k \to x$ and $\hat{x}_k \to x$. Also, for every k larger than a certain integer $\bar{k}$, we have $x_k' \geq 0$, $\sum_{i\in N} x_{ik}' \leq \sum_{i\in N} w_i^k$ because $x_k' \leq \bar{x}_k$ and $\sum_{i\in N} \bar{x}_{ik} \leq \sum_{i\in N} w_i^k$, and $p_i(m_k) \cdot x_{ik}' \leq \frac{p_i(m_k)\cdot w_i^k}{1+\|p_i^k-p_i(m_k)\|}$ for all $i \in N$ by noting that $x_{ik}' \leq \hat{x}_{ik}$. Let $x_k = x_k'$ for all $k > \bar{k}$ and $x_k = 0$ for $k \leqq \bar{k}$. Then, $x_k \to x$, and $x_k \in B(m_k)$ for all k. Therefore, the sequence $\{x_k\}$ has all the desired properties. So $B_x(m)$ is lower hemi-continuous at every $m \in M$. ▌

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