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Growth Optimal Portfolio in a Market Driven by a Jump-Diffusion-Like Process or a Lévy Process *

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It is shown that in a market modeled by a vector-valued semimartingale, when we choose the wealth process of an admissible self-financing strategy as a numeraire such that the historical probability measure becomes a martingale measure, then this numeraire must be the wealth process of a growth optimal portfolio. As applications of this result, the growth optimal portfolio in a market driven by a jump-diffusion-like process or a Lévy process is worked out.
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1. INTRODUCTION

The arbitrage pricing for contingent claims is one of major issues in mathematical finance. Martingale method provides a convenient and powerful tool for arbitrage pricing. When the market is incomplete, the martingale measure is not unique. In this case, the arbitrage pricing cannot provide a unique valuation rule for non-replicatable contingent claim. To get round this problem several pricing rules have been proposed. One approach, called the numeraire portfolio approach, was initiated by J. Long (1990) and developed by Bajeux and Portait (1995a, 1995b). The starting point of this approach is to search for a suitable derivative asset as the numeraire such that the denominated price processes of primitive assets are martingales under the historical probability measure. It turns out that this numeraire must be the wealth process of the growth optimal portfolio (see Bajeux and Portait (1995a, 1995b)). For a market with asset returns being diffusion processes the growth optimal portfolio is well known (see Karatzas and Shreve, 1998).

Option pricing when underlying stock returns are discontinuous was first studied by Merton (1976). In recent years, many researchers have used more general discontinuous process (e.g., jump-diffusion process or Lévy process) to model the asset returns (see references). This model has the feature that it allows random jumps in asset values. The purpose of this paper is to work out the growth optimal portfolio in a market with asset returns being a jump-diffusion-like process or a Lévy process.

2. THE GROWTH OPTIMAL WEALTH PROCESS

In this section we show that in a market modeled by a vector-valued semimartingale, when we choose the wealth process of an admissible self-financing strategy as a numeraire such that the historical probability measure becomes a martingale measure, then this numeraire must be the wealth process of a growth optimal portfolio. This result is stated in Bajeux and Portait (1995a, 1995b) without giving a proof. For reader's convenience we give below a complete proof for this result.

We fix a finite time-horizon $[0,T]$ and consider a security market which consists of $m+1$ assets whose price processes $(S_t^i), i=0,\cdots,m$, are assumed to be strictly positive semimartingales, defined on a filtered probability space $(\Omega,\mathcal{F},(\mathcal{F}_t),\mathbb{P})$ satisfying the usual conditions. Moreover, we assume that $\mathcal{F}_0$ is the trivial σ-algebra. We take arbitrarily an asset, say asset 0, as the numeraire asset. We set $\gamma_t\hat{=}(S_t^0)^{-1}$ and call γ_t the *deflater* at time t. We set $S_t=(S_t^1,\cdots,S_t^m)$ and $\widetilde{S}_t=(\widetilde{S}_t^1,\cdots,\widetilde{S}_t^m)$, where $\widetilde{S}_t^i=\gamma_t S_t^i, 1\le i\le m$. We call $(\widetilde{S}_t)$ the *deflated* price process of the assets. Note that the deflated price process of asset 0 is the constant 1.

The continuous trading is modeled by a stochastic integral. In order to be able to define a trading strategy we need the notion of integration w.r.t. a vector-valued semimartingale (see Jacod, 1980). If a vector valued $\mathcal{F}_t$-predictable process $(H^0, \cdots, H^m)$ is integrable w.r.t. a semimartingale $(X^0, \cdots, X^m)$, and if H^0 is integrable w.r.t. X^0, then we have

$$(H^0, H).(X^0, X) = H^0.X^0 + H.X, \tag{1}$$

where $H = (H^1, \cdots, H^m), X = (X^1, \cdots, X^m)$, and $H.X$ stands for the integral of H w.r.t. X.

A *trading strategy* is a $\mathbb{R}^{m+1}$-valued $\mathcal{F}_t$-predictable process $\phi = \{\theta^0, \theta\}$ such that ϕ is integrable w.r.t semimartingale (S_t^0, S_t), where

$$\theta(t) = (\theta^1(t), \cdots, \theta^m(t)), \quad S_t = (S_t^1, \cdots, S_t^m),$$

and $\theta^i(t)$ represents the numbers of units of asset i held at time t. The wealth $V_t(\phi)$ at time t of a trading strategy $\phi = \{\theta^0, \theta\}$ is

$$V_t(\phi) = \theta^0(t) S_t^0 + \theta(t) \cdot S_t, \tag{2}$$

where $\theta(t) \cdot S_t = \sum_{i=1}^m \theta^i(t) S_t^i$. The deflated wealth at time t is $\widetilde{V}_t(\phi) = V_t(\phi)\gamma_t$. A trading strategy $\{\theta^0, \theta\}$ is said to be *self-financing*, if

$$V_t(\phi) = V_0(\phi) + \int_0^t \phi(u) d(S_u^0, S_u). \tag{3}$$

Here and henceforth we use the notation $\int_0^t H_u dX_u$ or $(H.X)_t$ to denote the integral of H w.r.t. X over the interval $(0, t]$. In particular, we have $(H.X)_0 = 0$.

It is easy to see that for any given $\mathbb{R}^m$-valued predictable process θ which is integrable w.r.t (S_t) and a real number x, there exists a real-valued predictable process (θ_t^0) such that $\{\theta^0, \theta\}$ is a self-financing strategy with the initial wealth x.

A security market is said to be *fair* if there exists a probability measure $\mathbb{Q}$ equivalent to the historical probability measure $\mathbb{P}$ such that the deflated price processes $(\widetilde{S}_t)$ is a (vector-valued) $\mathbb{Q}$-martingale. We call such a $\mathbb{Q}$ an *equivalent martingale measure* for the market.

We denote by $\mathcal{M}^j$ the set of all equivalent martingale measures for the market, if asset j is taken as the numeraire asset. It is shown in Yan (1998) that the fairness of a market is invariant under the change of numeraire.

A strategy is said to be *admissible*, if its wealth process is non-negative. A strategy is said to be *allowable*, if there exists a positive constant c such that its wealth process V_t is bounded from below by $-c \sum_{i=0}^m S_t^i$. An admissible strategy is allowable.

The following lemma can be found in Yan (1998).

LEMMA 2.1. *A strategy $\phi = \{\theta^0, \theta\}$ is self-financing if and only if its wealth process (V_t) satisfies*

$$d\widetilde{V}_t = \theta(t) d\widetilde{S}_t,$$

where $\widetilde{V}_t = V_t \gamma_t$. In particular, the deflated wealth process of an allowable self-financing strategy is a local $\mathbb{Q}$-martingale and a $\mathbb{Q}$-supermartingale for any $\mathbb{Q} \in \mathcal{M}^0$.

DEFINITION 2.1. An admissible self-financing strategy $\{\theta^0, \theta\}$ is called a growth optimal strategy, if its wealth process V_t satisfies the following condition: for the wealth process (X_t) of any other admissible self-financing strategy with the same initial wealth V_0, we have

$$\mathbb{E}[\log X_t] \leq \mathbb{E}[\log V_t], \quad t \geq 0. \tag{4}$$

We call (V_t) a growth optimal wealth process.

According to Bajeux and Portait (1995a), the following well-known theorem can be traced to Samuelson (1963). For reader's convenience we include its proof.

THEOREM 2.1. *Let (V_t) be the wealth process of an admissible self-financing strategy with $V_0 = 1$. Assume that for every $0 \leq j \leq m$, $\left(V_t^{-1} S_t^j\right)$ is a $\mathbb{P}$-martingale. Then (V_t) is the unique growth optimal wealth process with initial value 1.*

Proof. We fix arbitrarily a j. By assumption we can define a probability measure $\mathbb{P}^j$ such that

$$M_t^j \widehat{=} \left.\frac{d\mathbb{P}^j}{d\mathbb{P}}\right|_{\mathcal{F}_t} = V_t^{-1} S_t^j (S_0^j)^{-1}. \tag{5}$$

Since $\frac{d\mathbb{P}}{d\mathbb{P}^j}|_{\mathcal{F}_t} = (M_t^j)^{-1}$, we know that for each $0 \leq i \leq m$, $S_t^i (S_t^j)^{-1} S_0^j = (M_t^j)^{-1} S_t^i V_t^{-1}$ is a $\mathbb{P}^j$-martingale, thus $\mathbb{P}^j \in \mathcal{M}^j$.

Now we assume that (X_t) is the wealth process of an admissible self-financing strategy with $X_0 = 1$. By Lemma 2.1, $\left(X_t (S_t^j)^{-1}\right)$ is a non-negative $\mathbb{P}^j$-local martingale, hence a $\mathbb{P}^j$-supermartingale. Thus we have

$$\mathbb{E}\left[\frac{X_t}{V_t}\right] = \mathbb{E}^{(j)}\left[\frac{X_t}{V_t} \left.\frac{d\mathbb{P}}{d\mathbb{P}^j}\right|_{\mathcal{F}_t}\right] = \mathbb{E}^{(j)}\left[X_t (S_t^j)^{-1}\right] S_0^j \leq 1. \tag{6}$$

By Jensen's inequality, this implies

$$\mathbb{E}\left[\log \frac{X_t}{V_t}\right] \leq \log \mathbb{E}\left[\frac{X_t}{V_t}\right] \leq 0. \tag{7}$$

Consequently, we have $\mathbb{E}[\log X_t] \leq \mathbb{E}[\log V_t]$. This means that V_t is a growth optimal wealth process. Its uniqueness can be proved as follows. Assume that (X_t) is a growth optimal wealth process with initial value 1. Then we have $\mathbb{E}\log X_t \geq \mathbb{E}\log V_t$, i.e., $\log \mathbb{E}\frac{X_t}{V_t} \geq 0$, which together with (7) implies $\mathbb{E}[\log \frac{X_t}{V_t}] = \log \mathbb{E}[\frac{X_t}{V_t}] = 0$. Consequently, both X_t/V_t and $\log(X_t/V_t)$ are $\mathbb{P}$-martingales, because they are already known to be $\mathbb{P}$-supermartingale. Therefore, $X_t = V_t$ *a.s.*. ▮

Recall that the relative entropy $I_{\mathbb{Q}}(\mathbb{P})$ of a probability measure $\mathbb{P}$ with respect to $\mathbb{Q}$ is defined by

$$I_{\mathbb{Q}}(\mathbb{P}) = \mathbb{E}_{\mathbb{Q}}\left[\frac{d\mathbb{P}}{d\,\mathbb{Q}} \log \frac{d\mathbb{P}}{d\,\mathbb{Q}}\right] = -\mathbb{E}_{\mathbb{P}}\left[\log \frac{d\,\mathbb{Q}}{d\,\mathbb{P}}\right].$$

$I_{\mathbb{Q}}(\mathbb{P})$ is a quantitative measure of the difference between $\mathbb{Q}$ and $\mathbb{P}$. The following theorem shows that the objective measure $\mathbb{P}$ has minimum relative entropy with respect to the measure $\mathbb{P}^j$ within $\mathcal{M}^j$. We refer the reader to Chan (1999) for a similar result (in a "dual form") about the Esscher transform.

THEOREM 2.2. *We have*

$$\mathbb{E}\left[\log\left(\frac{d\mathbb{P}^j}{d\mathbb{P}}\bigg|_{\mathcal{F}_t}\right)\right] \geq \mathbb{E}\left[\log\left(\frac{d\,\mathbb{Q}}{d\mathbb{P}}\bigg|_{\mathcal{F}_t}\right)\right], \quad \forall\, \mathbb{Q} \in \mathcal{M}^j. \tag{8}$$

Proof. Let $\mathbb{Q} \in \mathcal{M}^j$. We put $N_t = \frac{d\,\mathbb{Q}}{d\mathbb{P}}\big|_{\mathcal{F}_t}$ and $L_t = \frac{d\,\mathbb{Q}}{d\mathbb{P}^j}\big|_{\mathcal{F}_t}$. Since V_t is the wealth process of an admissible self-financing strategy, $V_t(S_t^j)^{-1}$ must be a non-negative $\mathbb{Q}$-local martingale, hence a $\mathbb{Q}$-supermartingale. Thus $L_t V_t^{-1} S_t^j$ must be a $\mathbb{P}^j$-supermartingale. Consequently, L_t is a $\mathbb{P}$-supermartingale, and we have $\mathbb{E}[L_t] \leq \mathbb{E}[L_0] = 1$. Thus, By Jensen's inequality, we have $\mathbb{E}[\log L_t] \leq 0$. But we have

$$\mathbb{E}[\log N_t] = \mathbb{E}[\log L_t] + \mathbb{E}[\log(V_t^{-1} S_t^j)].$$

So (8) is proved. ▮

3. GROWTH OPTIMAL PORTFOLIO IN A MARKET DRIVEN BY A JUMP-DIFFUSIN-LIKE PROCESS

In this section we will introduce a financial market with a jump-diffusion-like process as asset returns and will work out a portfolio such that, if we take its wealth process as a numeraire, then the historical probability measure becomes a martingale measure. By Theorem 2.1 this portfolio must be the growth optimal portfolio.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ be a filtered probability space satisfying the usual conditions. Consider a financial market which consists of a risk-free asset (savings account) and a risky asset (stock) whose prices S_t^0 and S_t satisfy

$$\begin{aligned} d\,S_t^0 &= S_t^0 r(t)\,dt, \quad S_0(0)=1, \\ d\,S_t &= S_{t-}\left\{b(t)dt + \sigma(t)d\,B_t + \phi(t)\,d\,N_t\right\}, \end{aligned} \tag{9}$$

where (B_t) is an $(\mathcal{F}_t, \mathbb{P})$-standard Brownian motion, $N(t)$ is $\mathcal{F}_t$-adapted counting process with intensity $\lambda(t) > 0$, and $r(t)$, $b(t)$, $\sigma(t)$, $\phi(t)$, and $\lambda(t)$ are assumed to be bounded $\mathcal{F}_t$-predictable processes. Furthermore, we assume that $\phi(t) > -1$ and $|\sigma(t)|$, $|\phi(t)|$, and $\lambda(t)$ are uniformly bounded from below by a positive constant. If $b(t), \sigma(t)$, and $\phi(t)$ are of the forms $b(t, S_t)$, $\sigma(t, S_t)$, and $\phi(t, S_t)$ with b, σ, and ϕ being deterministic functions and $\lambda(t)$ being a constant, S_t is a jump-diffusion process.

Let $\{\theta^0(t), \theta(t)\}$ be an admissible self-financing strategy. Its wealth process V_t satisfies

$$d\,V_t = \theta(t)dS_t + \theta^0(t)S_t^0 r(t)dt. \tag{10}$$

Set $\widetilde{V}_t = e^{-\int_0^t r(s)ds} V_t$ and $\widetilde{S}_t = e^{-\int_0^t r(s)ds} S_t$. By (9) we have

$$\begin{aligned} d\,\widetilde{V}_t &= \theta(t)d\,\widetilde{S}_t \\ &= \theta(t)\widetilde{S}_{t-}\left[\bar{b}(t)dt + \sigma(t)dB_t + \phi(t)dN_t\right] \\ &= \widetilde{V}_{t-}\pi(t)\left[\bar{b}(t)dt + \sigma(t)dB_t + \phi(t)dN_t\right], \end{aligned} \tag{11}$$

where $\pi(t) = \theta(t)S_{t-}/V_{t-}$ and $\bar{b}(t) = b(t) - r(t)$. $(\widetilde{V}_t)$ is strictly positive if $\pi(t)\phi(t) > -1$ for each t. In the following we assume that (V_t) is strictly positive.

LEMMA 3.1. *The following equality holds:*

$$\widetilde{V}_t^{-1} - \widetilde{V}_{t-}^{-1} + \widetilde{V}_{t-}^{-2}\Delta\widetilde{V}_t = \widetilde{V}_{t-}^{-1}\frac{\pi(t)^2\phi(t)^2}{1+\pi(t)\phi(t)}\Delta N_t, \tag{12}$$

here $\Delta X = X - X_-$.

Proof. By (11), $\Delta\widetilde{V}_t = \widetilde{V}_{t-}\pi(t)\phi(t)\Delta N_t$. Thus we have

$$\begin{aligned}
\widetilde{V}_t^{-1} - \widetilde{V}_{t-}^{-1} + \widetilde{V}_{t-}^{-2}\Delta\widetilde{V}_t &= -\frac{\Delta\widetilde{V}_t}{\widetilde{V}_t\widetilde{V}_{t-}} + \widetilde{V}_{t-}^{-1}\pi(t)\phi(t)\Delta N_t \\
&= -\frac{\pi(t)\phi(t)\Delta N_t}{\widetilde{V}_{t-}(1+\pi(t)\phi(t)\Delta N_t)} + \widetilde{V}_{t-}^{-1}\pi(t)\phi(t)\Delta N_t \\
&= \widetilde{V}_{t-}^{-1}\frac{\pi(t)^2\phi(t)^2}{1+\pi(t)\phi(t)}\Delta N_t.
\end{aligned}$$

∎

THEOREM 3.1. *If*

$$\pi = \frac{-(\sigma^2 - \phi\bar{b}) + \sqrt{(\sigma^2+\phi\bar{b})^2 + 4\sigma^2\phi^2\lambda}}{2\sigma^2\phi}, \tag{13}$$

then $(V_t^{-1}S_t^0)$ *(i.e.,* $(\widetilde{V}_t^{-1})$*) and* $(V_t^{-1}S_t)$ *are* $\mathbb{P}$*-martingales. In particular,* (π_t) *is the growth optimal portfolio. Moreover, we have*

$$d\widetilde{V}_t^{-1} = \widetilde{V}_{t-}^{-1}\left[-\pi(t)\sigma(t)dB_t + \left((1+\pi(t)\phi(t))^{-1} - 1\right)dM_t\right], \tag{14}$$

where (M_t) *is the following* $(\mathcal{F}_t,\mathbb{P})$*-martingale:*

$$M_t = N_t - \int_0^t \lambda(s)ds. \tag{15}$$

Proof. In the following we write $dX_t \sim dY_t$ to stand for the fact that $(X_t - Y_t)$ is a $\mathbb{P}$-local martingale. By Itô's formula and (12),

$$\begin{aligned}
d\widetilde{V}_t^{-1} &= -\widetilde{V}_{t-}^{-2}d\widetilde{V}_t + \widetilde{V}_{t-}^{-3}d\left\langle\widetilde{V}^c,\widetilde{V}^c\right\rangle_t + d\sum_{0<s\le t}\left(\widetilde{V}_s^{-1} - \widetilde{V}_{s-}^{-1} + \widetilde{V}_{s-}^{-2}\Delta\widetilde{V}_s\right) \\
&\sim -\widetilde{V}_{t-}^{-1}[\pi(t)\bar{b}(t)dt + \pi(t)\phi(t)\lambda(t)dt] + \widetilde{V}_{t-}^{-1}\pi(t)^2\sigma(t)^2 dt \\
&\quad +\widetilde{V}_{t-}^{-1}\frac{\pi(t)^2\phi(t)^2\lambda(t)}{1+\pi(t)\phi(t)}dt \\
&\sim \widetilde{V}_{t-}^{-1}\pi(t)\left(\bar{b}(t) + \phi(t)\lambda(t) - \pi(t)\sigma(t)^2 - \frac{\pi(t)\phi(t)^2\lambda(t)}{1+\pi(t)\phi(t)}\right)dt.
\end{aligned}$$

$$d\left(V_t^{-1}S_t\right) = d\left(\widetilde{V}_t^{-1}\widetilde{S}_t\right)$$

$$\begin{aligned}
&= \widetilde{S}_{t-}d\widetilde{V}^{-1} + \widetilde{V}_{t-}^{-1}d\widetilde{S}_t + d[\widetilde{V}^{-1}, \widetilde{S}]_t \\
&\sim \widetilde{S}_{t-}d\widetilde{V}^{-1} + \widetilde{V}_{t-}^{-1}\widetilde{S}_{t-}[\bar{b}(t)dt + \phi(t)\lambda(t)dt] - \widetilde{V}_{t-}^{-2}d[\widetilde{V}^{-1}, \widetilde{S}]_t \\
&\quad +d\sum_{0<s\le t}\left(\widetilde{V}_s^{-1} - \widetilde{V}_{s-}^{-1} + \widetilde{V}_{s-}^{-2}\Delta\widetilde{V}_s\right)\Delta\widetilde{S}_s \\
&= \widetilde{S}_{t-}d\widetilde{V}^{-1} + \widetilde{V}_{t-}^{-1}\widetilde{S}_{t-}[\bar{b}(t)dt + \phi(t)\lambda(t)dt] \\
&\quad -\widetilde{V}_{t-}^{-1}\widetilde{S}_{t-}[\pi(t)\sigma(t)^2dt + \pi(t)\phi(t)^2dN_t] \\
&\quad +d\sum_{0<s\le t}\left(\widetilde{V}_s^{-1} - \widetilde{V}_{s-}^{-1}\frac{\pi(s)^2\phi(s)^2}{1+\pi(s)\phi(s)}\phi(t)\Delta N_s\right) \\
&\sim \widetilde{S}_{t-}d\widetilde{V}^{-1} \\
&\quad +\widetilde{V}_{t-}^{-1}\widetilde{S}_{t-}\Bigg[\bar{b}(t) + \phi(t)\lambda(t) - \pi(t)\sigma(t)^2 - \pi(t)\phi(t)^2\lambda(t) \\
&\quad +\frac{\pi(t)^2\phi(t)^3\lambda(t)}{1+\pi(t)\phi(t)}\Bigg]dt \\
&= \widetilde{S}_{t-}d\widetilde{V}^{-1} + \widetilde{V}_{t-}^{-1}\widetilde{S}_{t-}\Bigg[\bar{b}(t) + \phi(t)\lambda(t) \\
&\quad -\pi(t)\sigma(t)^2 - \frac{\pi(t)\phi(t)^2\lambda(t)}{1+\pi(t)\phi(t)}\Bigg]dt.
\end{aligned}$$

Thus, in order that $(\widetilde{V}_t^{-1})$ and $(V_t^{-1}S_t)$ are $\mathbb{P}$-local martingales, the above "dt" terms must vanish, i.e., π must satisfy the following equation

$$\bar{b} - \pi\sigma^2 + \frac{\phi\lambda}{1+\pi\phi} = 0. \tag{16}$$

Equation (16) has two solutions. Only the solution given by (13) satisfies the condition $\pi\phi > -1$. From the above proof, it is easy to see that for such choice of π we have

$$d\widetilde{V}_t^{-1} = -\widetilde{V}_{t-}^{-1}[\pi(t)\sigma(t)dB_t + \pi(t)\phi(t)dM_t] + \widetilde{V}_{t-}^{-1}\frac{\pi(t)^2\phi(t)^2}{1+\pi(t)\phi(t)}dM_t,$$

from which (14) follows.

Now we are going to prove that $(\widetilde{V}_t^{-1})$ and $(V_t^{-1}S_t)$ are actually $\mathbb{P}$-martingales. In fact, it is easy to see that

$$\widetilde{V}_t^{-1} = V_0^{-1}\exp\left\{-\int_0^t \pi(s)\bar{b}(s)ds - \int_0^t \pi(s)\sigma(s)dB_s\right.$$

$$-\int_0^t \log(1+\pi(s)\phi(s))dN_s - \frac{1}{2}\int_0^t \pi(s)^2\sigma(s)^2 ds\Big\}$$

and

$$\begin{aligned} V_t^{-1}S_t &= \widetilde{V}_t^{-1}\widetilde{S}_t \\ &= V_0^{-1}S_0 \exp\Big\{\int_0^t (1-\pi(s))\bar{b}(s)ds + \int_0^t (1-\pi(s))\sigma(s)dB_s \\ &\quad + \int_0^t \log\frac{1+\phi(s)}{1+\pi(s)\phi(s)}dN_s - \frac{1}{2}\int_0^t (1-\pi(s)^2)\sigma(s)^2 ds\Big\}. \end{aligned}$$

Since the functions π, $\bar{b}$, σ, ϕ are all bounded, $(\widetilde{V}_t^{-1})$ and $(V_t^{-1}S_t)$ are uniformly integrable on $[0,\, T]$, $(\widetilde{V}_t^{-1})$ and $(V_t^{-1}S_t)$ are $\mathbb{P}$-martingales. ∎

Remark 3.1. From the above proof it is easy to see that $(\widetilde{V}_t^{-1})$ and $(V_t^{-1}S_t)$ are $\mathbb{P}$-martingales iff π is given by (13). For those $t \in [0,T]$ with $\bar{b}(t) + \phi(t)\lambda(t) = 0$ we have $\pi(t) = 0$.

THEOREM 3.2. *Let $\pi(t)$ be given by (13) and V_t be its corresponding wealth process with $V_0 = 1$. Put*

$$\frac{d\widehat{\mathbb{P}}}{d\mathbb{P}}\Big|_{\mathcal{F}_t} = \widetilde{V}_t^{-1}, \quad 0 \le t \le T.$$

Then under $\widehat{\mathbb{P}}$, $\widehat{B}_t = B_t + \int_0^t \pi(s)\sigma(s)ds$ is an $(\mathcal{F}_t)$-standard Brownian motion and N_t is an $(\mathcal{F}_t)$-adapted counting process with intensity $(1+\pi(t)\phi(t))^{-1}\lambda(t)$. Moreover,

$$d\widetilde{S}_t = \widetilde{S}_{t-}\left[\sigma(t)d\widehat{B}_t + \phi(t)[dN_t - ((1+\pi(t)\phi(t))^{-1}\lambda(t))dt]\right].$$

Proof. We have

$$\begin{aligned} d(\widehat{B}_t\widetilde{V}_t^{-1}) &\sim \widetilde{V}_t^{-1}d\widehat{B}_t + d[\widehat{B}, \widetilde{V}^{-1}]_t \\ &= \widetilde{V}_{t-}^{-1}(dB_t + \pi(t)\sigma(t)dt) - \widetilde{V}_{t-}^{-1}\pi(t)\sigma(t)dt \\ &= \widetilde{V}_{t-}^{-1}dB_t. \end{aligned}$$

This means that $(\widehat{B}_t\widetilde{V}_t^{-1})$ is a $\mathbb{P}$-local martingale, i.e., $(\widehat{B}_t)$ is a $\widehat{\mathbb{P}}$-local martingale. On the other hand, under $\mathbb{P}$, $[\widehat{B},\widehat{B}]_t = [B,B]_t = t$, and $[\widehat{B},\widehat{B}]$ is invariant under the equivalent change of probability. Therefore, by the well-known Lévy's theorem, $(\widehat{B}_t)$ is an $(\mathcal{F}_t, \widehat{\mathbb{P}})$-standard Brownian motion. Similarly, we can prove that (N_t) is an $(\mathcal{F}_t, \widehat{\mathbb{P}})$-counting process with intensity $(1+\pi(t)\phi(t))^{-1}\lambda(t)$. ∎

4. GROWTH OPTIMAL PORTFOLIO IN A MARKET DRIVEN BY A LÉVY PROCESS

Recently, Chan (1999) introduced a market model in which the stock price (S_t) is driven by a Lévy process:

$$dS_t = \sigma_t S_{t-} dX_t + b_t S_{t-} dt,$$

where σ_t and b_t are deterministic functions of t and X is a Lévy process of type $X_t = cB_t + N_t + \alpha t$ with (B_t) being a Brownian motion and N_t being a purely discontinuous martingale.

Our model is little more general. Let (X_t) be a càdlàg version of a Lévy process, i.e. a process with stationary and independent increments. Put

$$\mu(\omega, dt, dx) = \sum_{s>0} I_{[\Delta X_s(\omega)\neq 0]}(s)\delta_{(s,\Delta X_s(\omega))}.$$

We call μ the jump measure of X. For a Borel set Λ in $\mathbb{R}\backslash\{0\}$, we put

$$N_t(\omega,\Lambda) = \mu(\omega,[0,t]\times\Lambda) = \sum_{0<s\leq t} I_\Lambda(\Delta X_s(\omega)), \quad \nu(\Lambda) = E[N_1(\cdot,\Lambda)]. \tag{17}$$

Then for each $t \in \mathbb{R}_+$ and $\omega \in \Omega$, $N_t(\omega,\cdot)$ and ν are σ-finite measures on $\mathbb{R}\setminus\{0\}$. Moreover, for every $n \geq 2$, $N_t(\cdot,[\frac{1}{n},1))$ is an integrable increasing process, $M_t^{(n)} = \int_{[\frac{1}{n}\leq|x|<1]} xN_t(\cdot,dx) - t\int_{[\frac{1}{n}\leq|x|<1]} x\nu(dx)$ is a square-integrable martingale and the sequence $(M_t^{(n)})$ tends to a square-integrable martingale (M_t). We denote it by

$$M_t = \int_{[|x|<1]} x(N_t(\cdot,dx) - t\nu(dx)). \tag{18}$$

One should beware of that $\int_{[|x|<1]} xN_t(\cdot,dx)$ and $\int_{[|x|<1]} x\nu(dx)$ individually may make no sense or equal infinity. What we only know is the fact that $\int_{\mathbb{R}\backslash\{0\}}(x^2\wedge 1)\nu(dx) < \infty$. It is well-known that (X_t) has the following Lévy decomposition (see Protter, 1989)

$$X_t = \alpha t + cB_t + M_t + A_t, \tag{19}$$

where α and c are constants, (B_t) is a standard Brownian motion, and

$$A_t = \int_{[x|\geq 1]} xN_t(\cdot, dx) = \sum_{0<s\leq t} \Delta X_s I_{[|\Delta X_s|\geq 1]}. \tag{20}$$

In particular, (X_t) is a semimartingale.

Now we consider a security market in which there are two assets: a risky asset and a savings account. We assume that the price process S_t of the risky asset satisfies the following equation:

$$dS_t = S_{t-}[\sigma_t dX_t + b_t dt], \tag{21}$$

where σ_t and b_t are deterministic functions of t. We assume that σ_t is strictly positive. The value of the savings account at time t is $\beta_t = \int_0^t e^{r_s} ds$, and r_t is a deterministic function of t. We assume that there exist $c_1 \in [0, 1)$ and $0 < c_2 \leq \infty$ such that $-c_1 \leq \Delta X \leq c_2$. Consequently, the Lévy measure ν is supported by $[-c_1, c_2]$. We assume that $c_1\sigma_t < 1$ to ensure the strict positivity of S_t.

In the following we always assume $\alpha = 0$, otherwise we can replace b_t by $b_t + \alpha\sigma_t$.

Let (V_t) be the wealth process of an admissible self-financing strategy (θ_t^0, θ_t), i.e., $V_t = \theta_t S_t + \theta_t^0 \beta_t$. Then,

$$dV_t = \theta_t dS_t + \theta_t^0 r_t \beta_t dt. \tag{22}$$

Let $\widetilde{V}_t = \beta_t^{-1} V_t$, $\widetilde{S}_t = \beta_t^{-1} S_t$, and $\pi_t = \theta_t S_{t-}/V_{t-}$, then

$$d\widetilde{S}_t = \widetilde{S}_{t-}[(b_t - r_t)dt + \sigma_t(cdB_t + dM_t + dA_t)], \tag{23}$$

$$d\widetilde{V}_t = \theta_t d\widetilde{S}_t = \widetilde{V}_{t-}\pi_t[(b_t - r_t)dt + \sigma_t(cdB_t + dM_t + dA_t)]. \tag{24}$$

If $\pi_t\sigma_t$ takes values in $(-c_2^{-1}, c_1^{-1})$, then (V_t) is strictly positive.

LEMMA 4.1. *We have*

$$\int_{\mathbb{R}\setminus\{0\}} \frac{x^2}{(1+ax)^2}\nu(dx) < \infty, \quad a \in (-c_2^{-1}, c_1^{-1}). \tag{25}$$

For given $b, r,$ *and* $\sigma > 0$, *we put*

$$f(a) = b - r - a\sigma c^2 + \sigma \int_{\mathbb{R}\setminus\{0\}} \left(\frac{x}{1+ax} - xI_{[|x|<1]}\right)\nu(dx), \quad a \in (-c_2^{-1}, c_1^{-1}). \tag{26}$$

Then $f(a)=0$ has a unique solution a^* in $(-c_2^{-1},c_1^{-1})$ if and only if

$$\lim_{a\to -c_2^{-1}} f(a)>0,\quad \lim_{a\to c_1^{-1}} f(a)<0. \tag{27}$$

Proof. For $a\in(-c_2^{-1},c_1^{-1})$, $x\in[-c_1,c_2]$, we have

$$\frac{x}{1+ax}-xI_{[|x|<1]}=\frac{x}{1+ax}I_{[|x|\geq 1]}-\frac{ax^2}{1+ax}I_{[|x|<1]},$$

$$\frac{|ax^2|}{1+ax}I_{[|x|<1]}\leq \frac{|a|x^2}{1-ac_1}\wedge\frac{|a|}{1-ac_1},$$

$$\frac{|x|}{1+ax}\leq\frac{|x|}{1-ac_1}\wedge\frac{c_2}{1+ac_2},\quad \frac{|x|}{1+ax}I_{[|x|\geq 1]}\leq\frac{x^2}{1-ac_1}\wedge\frac{c_2}{1+ac_2}.$$

Since $\int_{\mathbb{R}\setminus\{0\}}(x^2\wedge 1)\nu(dx)<\infty$, we see that $f(a)$ is well-defined and (25) holds. Here $\frac{c_2}{1+ac_2}=\frac{1}{a}$ if $c_2=\infty$. Thus by the dominated convergence theorem it is easy to see that for each $a\in(-c_2^{-1},c_1^{-1})$, $f'(a)$ exists and

$$f'(a)=-\sigma c^2-\sigma\int_{\mathbb{R}\setminus\{0\}}\frac{x^2}{(1+ax)^2}\nu(dx)<0.$$

This implies that $f(a)=0$ has a unique solution $a^*\in(-c_2^{-1},c_1^{-1})$, if and only if (27) holds. ∎

In the sequel, for each $t\in[0,T]$ and $a\in(-c_2^{-1},c_1^{-1})$, we put

$$f_t(a)=b_t-r_t-a\sigma_t c^2+\sigma_t\int_{\mathbb{R}\setminus\{0\}}\Big(\frac{x}{1+ax}-xI_{[|x|<1]}\Big)\nu(dx). \tag{28}$$

We assume

$$\lim_{a\to -c_2^{-1}} f_t(a)>0,\quad \lim_{a\to c_1^{-1}} f_t(a)<0,\quad t\in[0,T]. \tag{29}$$

By Lemma 4.1 there exists a unique solution a_t^* of $f_t(a)=0$ in $(-c_2^{-1},c_1^{-1})$. We denote by (V_t) the wealth process corresponding to $\pi_t^*=\sigma_t^{-1}a_t^*$ with $V_0=1$.

THEOREM 4.1. *Assume (29) holds. If*

$$\int_0^T (a_s^*)^2ds<\infty,\quad \int_0^T(1-\pi_s^*)^2\sigma_s^2 ds<\infty, \tag{30}$$

$$\int_0^t \int_{\mathbb{R}\setminus\{0\}} \frac{(a_s^*)^2 x^2}{(1+a_s^* x)^2} \nu(dx)ds < \infty \tag{31}$$

and

$$\int_0^t \int_{\mathbb{R}\setminus\{0\}} \frac{(1-\pi_s^*)^2 \sigma_s^2 x^2}{(1+a_s^* x)^2} \nu(dx)ds < \infty, \tag{32}$$

then $(V_t^{-1}\beta_t$ *and* $(V_t^{-1}S_t)$ *are strictly positive square integrable martingales. In particular,* (π_t^*) *is the growth portfolio.*

Proof. Let (V_t) be the wealth process of an admissible self-financing strategy π such that for each $t \in [0,T]\, \sigma_t \pi_t$ takes values in $(-c_2^{-1}, c_1^{-1})$. We shall use the notation $dY_t \sim dZ_t$ to stand for the fact that $Y_t - Z_t$ is a local martingale. By Itô's formula, we have

$$\begin{aligned}
& d\widetilde{V}_t^{-1} \\
= \; & -\widetilde{V}_{t-}^{-2} d\widetilde{V}_t + \widetilde{V}_{t-}^{-3} d\left\langle V^c, V^c \right\rangle_t + d \sum_{0<s\leq t} \left(\widetilde{V}_s^{-1} - \widetilde{V}_{s-}^{-1} + \widetilde{V}_{s-}^{-2} \Delta \widetilde{V}_s\right) \\
\sim \; & -\widetilde{V}_{t-}^{-1} [\pi_t(b_t - r_t - \pi_t \sigma_t^2 c^2)dt + \pi_t \sigma_t dA_t] + d \sum_{0<s\leq t} \widetilde{V}_{s-}^{-1} \frac{\pi_s^2 \sigma_s^2 \Delta X_s^2}{1 + \pi_s \sigma_s \Delta X_s} \\
\sim \; & -\widetilde{V}_{t-}^{-1} \left[\pi_t(b_t - r_t - \pi_t \sigma_t^2 c^2) - \int_{\mathbb{R}\setminus\{0\}} \left(\frac{\pi_t^2 \sigma_t^2 x^2}{1+\pi_t \sigma_t x} - \pi_t \sigma_t x I_{[|x|\geq 1]}\right) \nu(dx)\right] dt \\
\sim \; & -\widetilde{V}_{t-}^{-1} \pi_t \left[b_t - r_t - \pi_t \sigma_t^2 c^2 + \sigma_t \int_{\mathbb{R}\setminus\{0\}} \left(\frac{x}{1+\pi_t \sigma_t x} - x I_{[|x|<1]}\right) \nu(dx)\right] dt.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
d\,(V_t^{-1} S_t) \; = \; & d(\widetilde{V}_t^{-1} \widetilde{S}_t) = \widetilde{V}_{t-}^{-1} d\widetilde{S}_t + d[\widetilde{V}^{-1}, \widetilde{S}]_t + \widetilde{S}_{t-} d\widetilde{V}_t^{-1} \\
\sim \; & \widetilde{V}_{t-}^{-1} \widetilde{S}_{t-} [(b_t - r_t)dt + \sigma_t dA_t] - \widetilde{V}_{t-}^{-2} d[\widetilde{V}, \widetilde{S}]_t \\
& + d \sum_{0<s\leq t} \left(\widetilde{V}_s^{-1} - \widetilde{V}_{s-}^{-1} + \widetilde{V}_{s-}^{-2} \Delta \widetilde{V}_s\right) \Delta \widetilde{S}_s + \widetilde{S}_{t-} d\widetilde{V}_t^{-1} \\
\sim \; & \widetilde{V}_{t-}^{-1} \widetilde{S}_{t-} (1 - \pi_t) \Bigg[b_t - r_t - \pi_t \sigma_t^2 c^2 \\
& + \sigma_t \int_{\mathbb{R}\setminus\{0\}} \left(\frac{x}{1+\pi_t \sigma_t x} - x I_{[|x|<1]}\right) \nu(dx)\Bigg] dt.
\end{aligned}$$

Thus, if (V_t) is the wealth process corresponding to $\pi_t^* = \sigma_t^{-1} a_t^*$, then $(V_t^{-1}\beta_t)$ and $(V_t^{-1}S_t)$ are local martingales.

Now we are going to prove that $(V_t^{-1}\beta_t)$ and $(V_t^{-1}S_t)$ are square integrable martingales. In the following for any semimartingale Y with $Y_0 = 0$ we denote by $\mathcal{E}(Y)$ the Doléans' exponential of Y, i.e.,

$$\mathcal{E}(Y)_t = \exp\{Y_t - \frac{1}{2}\langle Y^c, Y^c\rangle_t\} \prod_{0<s\le t}(1+\Delta Y_s)e^{-\Delta Y_s},$$

where Y^c is the continuous martingale part of Y. $\mathcal{E}(Y)$ is the unique solution of the equation $dW_t = W_{t-}dY_t$ with $W_0 = 1$.

Since $V_t^{-1}\beta_t = \widetilde{V}_t$, by (24) and Doléans exponential formula we have

$$\begin{aligned} V_t^{-1}\beta_t = & \exp\left\{-\int_0^t \pi_s^*[(b_s - r_s)ds + c\sigma_s dB_s - \tfrac{c^2\sigma_s^2\pi_s^*}{2}ds + \sigma_s(dM_s + dA_s)]\right\} \\ & \times \prod\nolimits_{0<s\le t}(1 + a_s^*\Delta X_s)^{-1}e^{a_s^*\Delta X_s} \\ = & \mathcal{E}(-c(a^*.B)_t)\exp\left\{-\int_0^t[\pi_s^*(b_s - r_s - \pi^*\sigma_s^2c^2)ds + a_s^*(dM_s + dA_s)]\right\} \\ & \times \prod\nolimits_{0<s\le t}(1 - \tfrac{a_s^*\Delta X_s}{1+a_s^*\Delta X_s})e^{\frac{a_s^*\Delta X_s}{1+a_s^*\Delta X_s} + \frac{a_s^{*2}(\Delta X_s)^2}{1+a_s^*\Delta X_s}}. \end{aligned} \tag{33}$$

Put

$$Z_t = \int_0^t \int_{\mathbb{R}\setminus\{0\}} \frac{\sigma_s x}{1 + a_s^* x}[\mu(\cdot, ds, dx) - \nu(dx)ds].$$

Then Z is well-defined local martingale. By (31), the stochastic integral $\pi^*.Z$ is well-defined and its oblique bracket process is

$$\langle \pi^*.Z, \pi^*.Z\rangle_t = \int_0^t \int_{\mathbb{R}\setminus\{0\}} \frac{(\sigma_s\pi_s^*)^2x^2}{(1+a_s^*x)^2}\nu(dx)ds < \infty.$$

Hence $\pi^*.Z$ is a square integrable martingale.

Since we have

$$\mathcal{E}(-\pi^*.Z)_t = e^{-(\pi^*.Z)_t}\prod_{0<s\le t}(1 - \frac{a_s^*\Delta X_s}{1+a_s^*\Delta X_s})e^{\frac{a_s^*\Delta X_s}{1+a_s^*\Delta X_s}},$$

from the fact that $f_t(a_t^*) = 0$ we obtain

$$V_t^{-1}\beta_t = \mathcal{E}(-\pi^*.(c\sigma.B + Z))_t.$$

The oblique bracket process at time T of the martingale $\pi^*.(c\sigma.B + Z)$

$$\langle \pi^*.(c\sigma.B + Z), \pi^*.(c\sigma.B + Z)\rangle_T = c^2\int_0^T (a_s^*)^2 ds + \langle \pi^*.Z, \pi^*.Z\rangle_T$$

is a finite constant. Thus, by a result of Lépingle-Mémin (1978), $\mathcal{E}(-\pi^*.(c\sigma.B+Z))$ is a square integrable martingale.

Similarly, we have

$$
\begin{aligned}
& V_t^{-1}S_t = \widetilde{V}_t^{-1}\widetilde{S}_t \\
= \; & \exp\left\{-\int_0^t [\pi_s^*(b_s - r_s) - \frac{(c\sigma_s\pi_s^*)^2}{2}]ds - (a^*.(cB+M+A))_t\right\} \\
& \times \prod_{0<s\le t} (1+a_s^*\Delta X_s)^{-1} e^{a_s^*\Delta X_s} \\
& \times \exp\left\{\int_0^t [b_s - r_s - \frac{(c\sigma_s)^2}{2}]ds + (\sigma.(cB+M+A))_t\right\} \\
& \times \prod_{0<s\le t} (1+\sigma_s\Delta X_s)e^{-\sigma_s\Delta X_s},
\end{aligned}
$$

from which and the fact that $f_t(a^*)=0$ it is easy to prove that

$$V_t^{-1}S_t = \mathcal{E}((1-\pi^*).(c\sigma.B+Z))_t.$$

Therefore, by (30), (32), and a result of Lépingle-Mémin (1978) we know that $(V_t^{-1}S_t)$ is also a square integrable martingale. ▮

Remark 4.1. The condition (27) is also necessary for (V_t) to be strictly positive, if $[-c_1, c_2]$ is exactly the support of the Lévy measure ν. This condition is easily checked for some concrete models (e.g., the compound Poisson process case).

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