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Viable Costs and Equilibrium Prices in Frictional Securities Markets

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This paper studies security markets with trading frictions, and offers a complete characterization of viable convex cost systems. For frictional markets that give rise to a convex-cone traded-payoff span and a sublinear payoff cost functional, the following three conditions are equivalent: viability, the extension property, and the absence of free lunches. Special cases in this class of markets include perfect-markets economies [Harrison and Kreps (1979)], economies with proportional transaction costs [Jouini and Kallal (1992, 1995)], economies with solvency constraints [Hindy (1995)], economies with no-short-selling, and economies with any combination of these frictions.

Key Words: Frictional markets; Viable price system; No arbitrage; Free lunches; Sublinear price functional; Market frictions.
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1. INTRODUCTION

In the theory of asset pricing, a securities market is said to be viable if some investor from certain preferences class can find an optimal trade. Viability is indeed a minimal condition that any well-functioning market should satisfy. In a perfect markets context, Harrison and Kreps (1979) elegantly argue that this condition may also be sufficient to yield an equilibrium for certain economies. They show that an asset price system is viable if and only if this price system can be extended via a positive continuous linear functional to the entire space of contingent claims. Moreover,

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they find that viability is equivalent to the absence of free lunches or general arbitrage opportunities.[1] The Harrison-Kreps characterization further formalizes the arbitrage arguments in Black and Scholes (1973), Cox and Ross (1976), and Ross (1978), and thus provides a formal foundation for the theory of asset pricing.

Recent attempts have been made to extend the results in Harrison and Kreps (1979), Kreps (1981), and Ross (1978) to economies with market frictions and/or transaction costs. For example, Jouini and Kallal (1992, 1995) examine the equivalent conditions of a viable "price system" when security transactions are costly and short sales are prohibited. Hindy (1995) studies viable prices under solvency constraints. Prisman (1986) and Ross (1987) extend the arbitrage argument to economies with income taxation. While these and other interesting attempts have produced significant insights into the structure of viable/arbitrage-free prices, it is not yet clear how far and under what other types of market friction the Harrison-Kreps results can be generalized.

In this paper, we do not assume specific types of market friction. Instead, we start with a sufficiently general setup so that the results we derive apply to a large class of frictional securities markets. Specifically, the discussion is cast on a single-period economy with uncertainty. Given an arbitrary number of traded securities, let M denote the set of all *feasible* payoffs, where a payoff is said to be feasible if it can be generated through a portfolio that satisfies whatever constraints the economy has. Examples of trading constraints include: no-short-selling, limits on short positions, lower bounds on asset holdings, upper bounds on holdings, solvency constraints, transaction costs, taxes, and so on. Defined on M is a *cost functional* ϕ such that, for any $x \in M$, the value $\phi(x)$ determines the minimum cost that it takes to acquire x, where $\phi(x)$ includes the portfolio purchase costs, transaction costs, and other applicable expenses. If $-x \in M$, then $-\phi(-x)$ determines how much an investor can receive from selling the payoff x (synthetically). A cost system, denoted by the pair (M, ϕ), is said to be *convex* if both M and ϕ are convex. Again, as the examples in Section 2 demonstrate, many types of frictional or frictionless securities markets give rise to such a cost system. It is also worth noting that, as will be shown later, a *cost system* and a *price system* are generally not the same in frictional economies. Here, a cost system determines how much an investor will have to pay in order to receive a payoff, whereas a price system is what security issuers can use to assign prices to securities, existing or new.

The purpose of this paper is to characterize a general convex cost system. In the spirit of Harrison and Kreps (1979), a cost system (M, ϕ) is said

[1]See, among others, Back and Pliska (1991), Chen and Knez (1995), Clark (1993), Duffie and Huang (1986), Hansen and Richard (1987), Hansen and Jagannathan (1991, 1997), Kreps (1981), and Ross (1978) for related developments and applications.

to be *viable* if some investor from the increasing, continuous and convex preferences class can find an optimal trade under (M, ϕ). It is said to satisfy the *extension property* if ϕ has a strictly positive extension to the entire consumption space such that this extension is linear if ϕ is linear, sublinear if ϕ is sublinear, and convex otherwise.[2] In this paper, the consumption space is taken to be any space L^p of random variables whose p-th absolute moment exists: $X = L^p$, for $1 \le p \le \infty$. A summary of our main findings follows:

1. A convex cost system (M, ϕ) is viable if and only if there is a strictly positive continuous linear functional ψ lying below ϕ: $\psi(x) \le \phi(x)$ for each $x \in M$. When $X = L^p$, for any p satisfying $1 \le p < \infty$, another equivalent condition is that there is some $d \in L^q$ such that (i) $d > 0$ almost surely and (ii) $E(x \cdot d) \le \phi(x)$ for each $x \in M$, where q is determined by $\frac{1}{q} + \frac{1}{p} = 1$ and $E(\cdot)$ is the expectation operator.

2. If a convex cost system (M, ϕ) is viable, then it satisfies the extension property and admits no free lunches.

3. Let $X = L^p$, for $1 \le p < \infty$. Suppose that M is convex and ϕ is sublinear. Then (M, ϕ) is viable if and only if it satisfies the extension property.

4. Let $X = L^p$, for $1 \le p < \infty$. Suppose that M is a convex cone and ϕ is sublinear. Then, (M, ϕ) is viable if and only if it admits no free lunches.

These results together offer a complete characterization of viable convex cost systems. When either M or ϕ is not convex, for instance, the proof of the first result will not follow through. For the class of frictional economies that gives rise to a convex cone M and a sublinear ϕ, the following three conditions are equivalent: viability, the extension property, and the absence of free lunches. Special cases in this class include: perfect-markets economies [Harrison and Kreps (1979)], economies with proportional transaction costs [Jouini and Kallal (1992, 1995)], economies with solvency constraints [Hindy (1995)], economies with no-short-selling, and economies with any combination of these frictions.

In addition to the above-mentioned characterizations, we offer an example economy in Section 6 to make two points. First, in frictional economies, the cost functional generally does not coincide with the equilibrium price functional. This means that in the presence of frictions one may not be able to use a hedging argument to even price a new security whose payoff is already marketed. This point shows the limitation of the arbitrage

[2]A functional ϕ defined on M is sublinear if (i) $\phi(x + y) \le \phi(x) + \phi(y)$ and (ii) $\phi(\lambda x) = \lambda\phi(x)$, for any $x, y \in M$ and $\lambda \ge 0$. Clearly, the class of linear functionals is contained in the class of sublinear functionals which is in turn contained in the class of convex functionals. This fact is useful for appreciating the results in this paper.

valuation approach. Second, when trading is not continuous [as assumed in Hindy (1995) and Jouini and Kallal (1992, 1995)], it is technically more demanding to characterize the minimum hedging cost problem or the cost functional ϕ via the Lagrange duality theorem. This point is useful when one tries to study solution properties of the hedging cost problem.

Besides the existing work cited above, some other papers also incorporate frictions to discuss asset pricing issues. For example, in a continuous-time framework, Cvitanic and Karatzas (1993) study the minimum hedging cost problem for contingent claims when asset holdings must lie in a convex set. Others include, for a partial list, Cocharane and Hansen (1992), Constantinides (1986), Dybvig and Huang (1988), Dybvig and Ross (1986), He and Modest (1995), He and Pearson (1991a,b), Heaton and Lucas (1996), Luttmer (1992), and Prisman (1986).

2. ECONOMIES WITH FRICTIONS

Consider a two-date economy in which decision-making takes place at time 0 and payoffs to securities are made at time 1. Uncertain time-1 events are described by a complete probability space $\{\Omega, \mathcal{F}, Pr\}$, where Ω has infinitely many states of nature. All agents share the same information as contained in the probability space. There is a single perishable consumption good used as the value numeraire.

Conceivable time-1 consumption bundles are taken to be in the space $X \equiv L^p(\Omega, \mathcal{F}, Pr)$, for any p satisfying $1 \leq p \leq \infty$, where $L^p(\Omega, \mathcal{F}, Pr)$, for $1 \leq p < \infty$, contains all random variables x such that $E(\mid x \mid^p) < \infty$ and $L^\infty(\Omega, \mathcal{F}, Pr)$ is the space of essentially bounded random variables. Denote the positive cone of X by $X_+ \equiv \{x \in X : Pr(x \geq 0) = 1\}$. It is useful to keep in mind that the norm dual of L^p is L^q, for $1 \leq p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$, and the norm dual of L^∞ is the space of bounded additive measures on $(\Omega, \mathcal{F})$ that are absolutely continuous with respect to Pr [Dunford and Schwartz (1966, p. 296)].[3]

Following convention, for any $x, y \in X$, we write "$x = y$" if $Pr(x = y) = 1$ (i.e. x and y are identified); "$x \geq y$" if $x - y \in X_+$; and "$x > y$" if $x \geq y$ and $Pr(x = y) < 1$. A real-valued linear functional ψ on X is *positive* if $\psi(x) \geq 0$ for each $x \in X_+$, and *strictly positive* if $\psi(x) > 0$ for every $x > 0$. Let Ψ denote the set of strictly positive and continuous linear functionals on X. A functional $g : X \rightarrow \Re$ is *sublinear* if, for every $x, y \in X$ and any scalar $\lambda \geq 0$, (i) $g(x + y) \leq g(x) + g(y)$ and (ii) $g(\lambda x) = \lambda g(x)$; and it is *convex* if, for every $x, y \in X$ and any scalar λ such that $0 \leq \lambda \leq 1$, $g(\lambda x + (1 - \lambda)y) \leq \lambda g(x) + (1 - \lambda)g(y)$.

[3]See Bewley (1972) for a brief discussion on L^∞ and its norm dual.

Investors are characterized by their preference orderings, $\succeq$, over the net trade space $\Re \times X$, where a pair $(c_0, x) \in \Re \times X$ denotes c_0 units of time-0 consumption and $x(\omega)$ units of time-1 consumption in state $\omega \in \Omega$. Let τ be the product topology of the Euclidean topology on $\Re$ and the norm topology on X. Each investor's $\succeq$ is assumed to satisfy the following conditions:

- $\succeq$ is convex in the sense that for any $(c_0', x') \in \Re \times X$, the set $\{(c_0, x) \in \Re \times X : (c_0, x) \succeq (c_0', x')\}$ is convex;
- $\succeq$ is τ-continuous, that is, for each $(c_0', x') \in \Re \times X$, the sets $\{(c_0, x) \in \Re \times X : (c_0, x) \succeq (c_0', x')\}$ and $\{(c_0, x) \in \Re \times X : (c_0', x') \succeq (c_0, x)\}$ are τ-closed;
- $\succeq$ is strictly increasing, that is, if $(c_0', x') \in \Re_+ \times X_+$ and either $c_0' > 0$ or $x' > 0$, then we have $(c_0, x) + (c_0', x') \succ (c_0, x)$ for every $(c_0, x) \in \Re \times X$, where $\succ$ denotes the strict preference relation induced by $\succeq$.

Denote the set of all such preference relations on $\Re \times X$ by **A**.

For ease of discussion, assume that there are N limited-liability securities traded on the market with payoffs $x_n \in X_+$ and prices P_n, for $n \in N$, where N stands for both the number and the set of securities. The prices P_n are taken from some equilibrium (to be defined later).

Given the N traded securities, a payoff $x \in X$ is said to be *feasible* if x can be obtained through some portfolio satisfying any given trading constraints, transaction costs and/or other regulations. Let M be the set of all feasible payoffs, and $\phi : M \to \Re$ the *cost functional* such that $\phi(x)$, for $x \in M$, determines the minimum amount that any investor has to pay to obtain x.

As noted before, we distinguish between a *cost system* and a *price system.* A cost system, denoted by the pair (M, ϕ), determines what an investor has to pay to achieve a feasible payoff x and what he can receive from selling x [i.e., $-\phi(-x)$ if $-x$ is also in M], whereas a price system, denoted by (X, π), determines what prices security issuers can quote for new and existing securities, where $\pi(\cdot) : X \to \Re$. In frictionless economies, a cost system is a price system (at least over the marketed payoff span) and vise versa [e.g., Arrow (1964), Harrison and Kreps (1979), and Kreps (1981)]. However, as shown later, in frictional economies this is no longer true.

Assumption 1. *In the economy under consideration, the cost system (M, ϕ) is convex, that is, both M and ϕ are convex. Further, to avoid trivial cases, there is at least one $x \in M$ such that $Pr(x > 0) = 1$.*

The convexity of M imposes some structure on feasible asset holdings while that of ϕ puts certain restrictions on security prices and transaction technologies. However, the class of convex functionals is abundantly rich and it contains both linear and sublinear functionals as special cases. For this reason, many types of economy possess a convex cost system. The following are some interesting examples:

Example 1. *Perfect markets economies.* This is the case examined in, among others, Harrison and Kreps (1979). The set M is a subspace in X:

$$M = \{x \in X : \exists \alpha \in \Re^N \text{ s.t. } x = \sum_{n \in N} \alpha_n x_n\},$$

and the cost functional ϕ is linear:

$$\phi(x) = \sum_{n \in N} \alpha_n P_n,$$

for any $\alpha \in \Re^N$ such that $x = \sum_{n=1}^N \alpha_n x_n$. Then, (M, ϕ) is convex trivially.

Example 2. *Economies with proportional transaction costs.* Jouini and Kallal (1992, 1995) examine this case, for instance. To describe this type of economy, let P_n and p_n be the prices at which investors can respectively buy and sell security x_n. Further, let α_n and θ_n be respectively a long and a short position in x_n. Clearly, the set of feasible payoffs, M, is still a subspace as given in Example 1. The cost functional is, however, as follows:

$$\begin{aligned} \phi(x) &= \min_{\alpha_n \geq 0, \theta_n \geq 0:\, n \in N} \sum_{n \in N} (\alpha_n P_n - \theta_n p_n) \\ \text{s.t.} & \quad \sum_{n \in N} (\alpha_n - \theta_n) x_n \geq x, \end{aligned}$$

for each $x \in M$. It is apparent that ϕ is convex. Moreover, as can be checked, ϕ is sublinear.

Example 3. *Economies with lower bounds on asset holdings.* Let $\ell_n \in \Re$ be the lower bound on positions in security x_n. In this case, $M = \{x = \sum_{n \in N} \alpha_n x_n : \alpha_n \geq \ell_n,\ \forall n\}$, which is convex, and

$$\phi(x) = \min_{\alpha_n \geq \ell_n:\, n \in N} \sum_{n \in N} \alpha_n P_n \tag{1}$$

$$\text{s.t.} \quad \sum_{n \in N} \alpha_n x_n \geq x, \tag{2}$$

for each $x \in M$. Again, ϕ is convex. A particular case in this class of economies is when no short sales are allowed: $\ell_n = 0$ for each n. In that case, one can verify that ϕ becomes a sublinear, instead of just convex, cost functional and M is a convex cone.

Example 4. *Economies with convex holding constraints*, that is, $a_n \leq \alpha_n \leq b_n$, for some $a_n, b_n \in \Re$ such that $a_n < b_n$ and for each n. Then,

$$M = \left\{ x = \sum_{n \in N} \alpha_n x_n : \ a_n \leq \alpha_n \leq b_n, \ \forall n \in N \right\},$$

and

$$\begin{aligned} \phi(x) \ &= \min_{a_n \leq \alpha_n \leq b_n:\, n \in N} \sum_{n \in N} \alpha_n P_n \\ \text{s.t.} \quad & \sum_{n \in N} \alpha_n x_n \geq x, \end{aligned}$$

for each $x \in M$. Both M and ϕ are still convex, but in general ϕ cannot be sublinear. In a continuous-time framework, Cvitanic and Karatzas (1993) examine the contingent-claims hedging problem with such convex constraints.

Example 5. *Economies with solvency constraints.* In a continuous-time context, Dybvig and Huang (1988) and Hindy (1995) study this type of economy. Given the single-period framework here, solvency constraints can be modelled by requiring each feasible portfolio to generate a non-negative future payoff. That is,

$$M = \left\{ x \in X_+ : \ x = \sum_{n \in N} \alpha_n x_n, \ \ \text{for some} \ \ \alpha_n \in \Re \ \ \text{and each} \ \ n \right\},$$

where any large short position in a security is clearly allowed – so long as the resulting portfolio payoff is non-negative almost surely. In this case, M is a convex cone in X_+, since (i) $\lambda x \in M$ for any $x \in M$ and every $\lambda \in \Re_+$ and (ii) $\theta x + (1-\theta) y \in M$ for each $\theta \in [0,1]$ and $x, y \in M$. The cost functional is, for each $x \in M$,

$$\begin{aligned} \phi(x) \ &= \min_{\alpha \in \Re^N} \sum_{n \in N} \alpha_n P_n \\ \text{s.t.} \quad & \sum_{n \in N} \alpha_n x_n \geq x, \end{aligned}$$

which is apparently sublinear because the portfolio weights are not directly constrained. It is, however, worth mentioning that ϕ in general will not be linear due to the solvency constraint and the fact that M is not a subspace.

Example 6. *Economies with convex transaction costs.* Suppose that for a portfolio $\alpha \in \Re^N$, $f(\alpha_1, \ldots, \alpha_N)$ determines its total cost, including the security purchasing prices, and that there is no constraint on asset holdings. Assume that $f(\cdot, \ldots, \cdot)$ is strictly increasing and convex in portfolio weights α. In this case, M is still the subspace spanned by the x_n's, while the cost functional is

$$\begin{aligned} \phi(x) &= \min_{\alpha \in \Re^N} f(\alpha_1, \ldots, \alpha_N) \\ \text{s.t.} & \quad \sum_{n \in N} \alpha_n x_n \geq x, \end{aligned}$$

for each $x \in M$. Following standard steps, one can verify that ϕ so defined is convex.

Example 7. *Economies with taxes, transaction costs and holding constraints.* As in Example 6, let the increasing and convex function $f(\alpha)$ be the total cost of portfolio α, and ℓ_n the lower bound on positions in security n. In addition, the after-tax future payoff of any portfolio α is assumed to be given by some function $h(\cdot) : \Re^N \to X$. Then, the set of feasible after-tax payoffs is

$$M = \{x = h(\alpha) : \ \alpha_n \geq \ell_n \ \ \forall n\}.$$

Assume the after-tax payoff function $h(\cdot)$ is such that the set M is convex. For instance, when $h(\alpha)$ is sublinear or linear in portfolio vector α, one can show that M is convex. At the same time, the minimum hedging cost functional is

$$\begin{aligned} \phi(x) &= \min_{\alpha_n \geq \ell_n :\, n \in N} \quad f(\alpha) \\ \text{s.t.} & \quad h(\alpha) \geq x, \end{aligned}$$

for each $x \in M$. This ϕ is convex. Thus, economies with taxation can also result in a convex cost system. For other features of economies with taxation, refer to Prisman (1986) and Ross (1987), who characterize arbitrage-free asset prices under taxation but no transaction costs. Of course, caution should be applied in generalizing their results to the type of economies assumed in this example because, among other things, their discussions are cast on a set-up with only finitely many states of nature.

The above examples are by no means exhaustive. For instance, any combination of these cases will give rise to a convex cost system. Realizing the generality of Assumption 1, we turn to characterizing such a cost system (M, ϕ).

3. VIABLE COST SYSTEMS

Definition 3.1. A convex cost system (M, ϕ) is **viable** if there exists some $\succeq \in \mathbf{A}$ and $(c_0^*, x^*) \in \Re \times M$ such that

- $c_0^* + \phi(x^*) \leq 0$;
- $(c_0^*, x^*) \succeq (c_0, x)$ for all net trade pairs $(c_0, x) \in \Re \times M$ such that $c_0 + \phi(x) \leq 0$.

This criterion makes sense because, if (M, π) is to be embedded in an economic equilibrium, it is only natural to require that at least some investor from the class $\mathbf{A}$, when subject to the budget constraint $c_0 + \phi(x) \leq 0$, be able to achieve an optimal trade. According to Harrison and Kreps (1979) and Kreps (1981), this criterion may also be sufficient for (M, ϕ) to be an *equilibrium cost system.* To see this, suppose such $\succeq$ and (c_0^*, x^*) exist and define $\succeq'$ by

$$(c_0, x) \succeq' (c_0', x') \text{ if } (c_0 + c_0^*, x + x^*) \succeq (c_0' + c_0^*, x + x^*).$$

Clearly, $\succeq' \in \mathbf{A}$ and the net trade $(0, 0)$ is optimal for all agents with both $\succeq'$ and the budget constraint in the definition. Therefore, if all investors are from the class $\mathbf{A}$, (M, ϕ) will be an equilibrium cost system.

Theorem 1. *Under Assumption 1, a convex cost system (M, ϕ) is viable if and only if there is a strictly positive, continuous linear functional ψ on X such that $\psi\,|_M \leq \phi$, where $\cdot\,|_M$ means "the restriction to M."*

Proof. To prove sufficiency, suppose there is such a ψ. Define some $\succeq$ on $\Re \times X$ by

$$\begin{aligned} (c_0, x) \succeq (c_0', x') &\quad \text{if } c_0 + \psi(x) \geq c_0' + \psi(x') \\ (c_0, x) \succ (c_0', x') &\quad \text{if } c_0 + \psi(x) > c_0' + \psi(x'). \end{aligned}$$

As can be checked, $\succeq$ is in the class $\mathbf{A}$. Furthermore, among all net trade vectors $(c_0, x) \in \Re \times M$ such that $c_0 + \phi(x) \leq 0$, the no trade vector $(0, 0)$ is optimal for all investors with $\succeq$ since for each such trade vector, $c_0 + \psi(x) \leq c_0 + \phi(x) \leq 0 = 0 + \psi(0)$. Thus, (M, ϕ) is viable.

To establish necessity, assume without loss of generality that, subject to the budget constraint $c_0 + \phi(x) \leq 0$, the pair $(c_0^*, x^*) = (0, 0)$ is optimal for some investor with $\succeq \in \mathbf{A}$. Define two sets:

$$\begin{aligned} G &\equiv \{(c_0, x) \in \Re \times X : (c_0, x) \succ (0,0)\} \\ H &\equiv \{(c_0, x) \in \Re \times M : c_0 + \phi(x) \leq 0\}. \end{aligned}$$

Then, G and H are disjoint because of the optimality of $(0,0)$ over all the pairs in H, and both sets are convex (G is so because of the convexity of $\succeq$ and H is so because both M and ϕ are convex). The continuity of $\succeq$ implies that G is also open in the topology τ. By a separating hyperplane theorem [Berberian (1974, theorem 30.6)], there is a non-trivial τ-continuous linear functional f on $\Re \times X$ such that $f(c_0, x) > 0$ for each $(c_0, x) \in G$ and $f(c_0, x) \leq 0$ for $(c_0, x) \in H$.

Since f is τ-continuous and $\succeq$ is strictly increasing, we have $f(0,0) = 0$. In addition, $(1,0) \succ (0,0)$ and hence $(1,0) \in G$. So $f(1,0) > 0$. Renormalize f such that $f(1,0) = 1$, and define ψ on X by $f(c_0, x) = c_0 + \psi(x)$. It is then straightforward that ψ is continuous (in the L^p norm topology) and linear. Furthermore, for any $x \in X_+$ such that $x \neq 0$, we have $(0,x) \succ (0,0)$ and $(0,x) \in G$, which means $f(0,x) = 0 + \psi(x) > 0$. That is, ψ is strictly positive. To show that $\psi|_M \leq \phi$, note that for any $x \in M$, we have $(-\phi(x), x) \in H$ and $f(-\phi(x), x) = -\phi(x) + \psi(x) \leq 0$. So $\psi(x) \leq \phi(x)$. ▮

A convex cost system is therefore viable if and only if there is a strictly positive, continuous linear functional supporting the cost functional ϕ. Theorem 1 applies to any convex cost system. In this sense, it represents a complete characterization of viable cost systems. Note that for viability to imply a $\psi \in \Psi$ lying below ϕ, it is necessary that both M and ϕ be convex. Otherwise, the proof of Theorem 1 would not follow through (the separating hyperplane theorem could not be used in that case).

COROLLARY 1. *Suppose that Assumption 1 holds and that $X = L^p$ for some p, $1 \leq p < \infty$. Then, a convex cost system (M, ϕ) is viable if and only if there is some stochastic discount factor $d \in L^q$, for q satisfying $\frac{1}{p} + \frac{1}{q} = 1$, such that (i) $d > 0$ almost surely and (ii) d satisfies*

$$E(d \cdot x) \leq \phi(x) \qquad \forall x \in M. \tag{3}$$

Proof. The norm dual of L^p is L^q, for q satisfying $\frac{1}{p} + \frac{1}{q} = 1$. By the Riesz representation theorem, each continuous linear functional ψ on L^p can be represented via some $d \in L^q$ as follows:

$$\psi(x) = E(d \cdot x) \qquad \forall x \in L^p. \tag{4}$$

Clearly, $d > 0$ almost surely if and only if ψ is strictly positive. Then, Theorem 1 gives the desired corollary. ▌

Remark 3.1. In Example 1, M is a subspace and ϕ is linear. Suppose (M, ϕ) is viable. By Theorem 1, there is $\psi \in \Psi$ such that, for each $x \in M$, $\psi(x) \le \phi(x)$ and $\psi(-x) \le \phi(-x)$, which means $\psi(x) = \phi(x)$ because of the linearity of both ψ and ϕ. Thus, Harrison and Kreps (1979, theorem 1) is a special case of Theorem 1.

Remark 3.2. In Example 2, M is a subspace but ϕ is sublinear. Granted that (M, ϕ) is viable, Theorem 1 implies that there is some $\psi \in \Psi$ such that, for each $x \in M$, $\psi(x) \le \phi(x)$ and $\psi(-x) \le \phi(-x)$, which yields: $-\phi(-x) \le \psi(x) \le \phi(x)$. This is the result in Jouini and Kallal (1992, theorem 2.1).

Remark 3.3. Recall Example 3 and let $X = L^p$, for any p satisfying $1 \le p < \infty$. First, when $\ell_n = 0$ for each security n and no short-selling is allowed, the set M is a convex cone. Further, since $x_n \in X_+$ for each n, M is a convex cone in X_+. Provided that (M, ϕ) is viable, Corollary 1 implies that there is some $d \in L^q_+$ such that $d > 0$ almost surely and $E(x \cdot d) \le \phi(x)$ for each $x \in M$, where $\frac{1}{q} + \frac{1}{p} = 1$. Note that $-x \notin M$, whenever $x \in M$. Thus, each such d is "bounded above" by ϕ but not below (except that $d > 0$ almost surely), which means

$$0 \le E(x \cdot d) \le \phi(x) \qquad\qquad \forall x \in M \subseteq X_+.$$

Comparing this with the results in Remarks 1 and 2 reveals that *the more severe the trading frictions, the fewer restrictions viability puts on the admissible stochastic discount factor d.* This statement is true because, on the one hand, no- short-selling is more severe than the proportional transaction costs in Remark 2, which is in turn more severe than the absence of trading frictions assumed in Remark 1; On the other hand, d has to satisfy, for each $x \in M$: $E(x \cdot d) = \phi(x)$ in Remark 1; $-\phi(-x) \le E(x \cdot d) \le \phi(x)$ in Remark 2; and $0 \le E(x \cdot d) \le \phi(x)$ here in Remark 3. This observation explains why, for instance, Luttmer (1992) can lower the Hansen-Jagannathan (1991) bounds on stochastic discount factors by assuming the existence of transaction costs. For the same reason, we can expect the Hansen-Jagannathan bounds to be even lower by assuming away short selling completely.

Next, suppose $\ell_n < 0$ for at least some securities n (i.e., there are bounds on short positions). In this case, as mentioned before, M is no longer a convex cone. Especially, for some payoffs $x \in M$, we have $-x \in M$. Again, by Corollary 1, there is some $d \in L^q$ such that $d > 0$ almost surely and $E(x \cdot d) \leq \phi(x)$ for each $x \in M$. Then, for those payoffs $x \in M$ such that $-x \notin M$, the restriction on d is simply $E(x \cdot d) \leq \phi(x)$ and nothing more. The other payoffs $x \in M$ for which $-x \in M$, however, put a stronger restriction on d: $-\phi(-x) \leq E(x \cdot d) \leq \phi(x)$, which is similar to the restrictions discussed in Remark 2. In words, when $\ell_n < 0$ for some securities n, certain feasible payoffs play the role of "bounding" an admissible stochastic discount factor from only above while others bound it from both above and below. In this sense, limits on short positions are not as severe a friction type as complete prohibition of short positions, but more severe than the existence of transaction costs assumed in Example 2 and Remark 2. As a result, one should expect the Hansen-Jagannathan bounds implied by the short position limits $\ell_n < 0$ to be lower than those implied by proportional transaction costs [and empirically estimated by Luttmer (1992)], but higher than those implied by no-short-selling.

More generally, for *any type* of frictional economy that leads to a convex cost system (M, ϕ), Hansen-Jagannathan bounds can be constructed solely from the restriction: $E(x \cdot d) \leq \phi(x)$ for each $x \in M$. This means that, in order to use Hansen-Jagannathan bounds to test asset pricing models, it may not be necessary to know the exact types of frictions that the actual securities markets have – as long as one can identify ϕ and M. In this regard, it is worth remembering that the set $\{d \in L^q_+ : \ E(x \cdot d) \leq \phi(x) \ \ \forall x \in M\}$ is typically a proper subset of $\{d \in L^q_+ : \ E(x_n \cdot d) \leq P_n \ \ \forall n\}$. For further discussions on implementing Hansen-Jagannathan bound tests in a frictional markets context, the reader is referred to Cochrane and Hansen (1992), He and Modest (1995), and Luttmer (1992).

Remark 3.4. Recall Example 5, which differs from Example 1 only in that the set of feasible payoffs is, instead of a subspace, a convex cone contained in X_+. This difference is, however, sufficient to prevent us from drawing the same conclusion as in Remark 1. Under solvency constraints, the equivalent condition of viability is the existence of some $\psi \in \Psi$ such that $\psi\mid_M \leq \phi$, rather than the requirement that $\psi\mid_M = \phi$ (as in Remark 1).

Remark 3.5. The result in Corollary 1 might not hold when $X = L^\infty$, because the norm dual of L^∞ is not L^1 but the space of bounded additive measures on $(\Omega, \mathcal{F})$. If we endow L^∞ with the L^1-Mackey topology, however, L^1 becomes its topological dual [see Bewley (1972)]. In that case, each Mackey-continuous linear functional ψ on L^∞ can be represented by some $d \in L^1$ as in (4). Therefore, when $X = L^\infty$, the result in Corollary 1 can hold, except that L^∞ may have to be given a topology different from its norm topology.

Remark 3.6. We can also replace each stochastic discount factor d in Corollary 1 with an equivalent probability measure μ and an implicit riskfree rate r_0. Briefly, let each d in Corollary 1 be the Radon-Nikodym derivative of some probability measure μ on $(\Omega, \mathcal{F})$ with respect to Pr. Then, by the Radon-Nikodym theorem,

$$\psi(x) = E(x \cdot d) = \frac{E^\mu(x)}{1 + r_0},$$

for $\psi \in \Psi$, where $E^\mu(\cdot)$ is the expectation operator with respect to μ and r_0 is the implicit riskfree rate determined by $E(d) \equiv \frac{1}{1+r_0}$. Furthermore, since $d > 0$ almost surely, μ must be an equivalent probability measure of Pr. Then, from Corollary 1, a convex cost system (M, ϕ) is viable if and only if there are an equivalent probability measure μ of Pr and an implicit riskfree rate r_0 such that

$$\frac{E^\mu(x)}{1 + r_0} \le \phi(x) \quad \forall x \in M.$$

4. THE EXTENSION PROPERTY

Suppose that all bundles in X are marketed somewhere and (M, ϕ) is only part of a larger equilibrium. Then, if (M, ϕ) is to be embedded in the larger equilibrium, it is minimal that some "reasonable" extension of ϕ to all of X exist. In the frictionless case examined by Harrison and Kreps (1979), the minimal requirement is that ϕ have a strictly positive, continuous linear extension to all of X, which is clearly satisfied by the viability of (M, ϕ). Absent of frictions, the existence of such a linear extension seems a natural requirement. When (M, ϕ) is an arbitrary convex cost system, however, difficulty arises. The following definition nonetheless represents one generalization of the Harrison-Kreps extension property:

Definition 4.1. A convex cost system (M, ϕ) has the **extension property** if there exists a continuous, strictly positive extension of ϕ to all of X that shares the same properties with ϕ, that is, there is some $\upsilon(\cdot) : X \to \Re$ such that (i) υ is strictly positive on X; (ii) $\upsilon \mid_M = \phi$; and (iii) υ is linear if ϕ is linear, sublinear if ϕ is sublinear, and, otherwise, convex.

The first two conditions in Definition 1 do not need further explanation. The last condition should be natural because it would not make sense to require that the extension be linear when ϕ is strictly convex or vise versa.

THEOREM 2. *Under Assumption 1, (M, ϕ) satisfies the extension property if (M, ϕ) is viable.*

Proof. Suppose (M, ϕ) is viable. Then, by Theorem 1, there is a $\psi \in \Psi$ such that $\psi \mid_M \leq \phi$. Let $\mathcal{P} \equiv \{\psi \in \Psi : \psi \mid_M \leq \phi\}$. The rest of the proof is divided into three cases:

1. ϕ is linear. In this case, the result holds trivially.

2. ϕ is sublinear. Let M' be the convex cone generated by M: $M' \equiv \{\lambda x : \ \forall \lambda \in \Re_+ \text{ and } \forall x \in M\}$. Then, for each $x' \in M'$, there exist some $x \in M$ and $\lambda \in \Re_+$ such that $\lambda x = x'$, in which case we set $\phi'(x') = \lambda \phi(x)$, where $\phi' : M' \to \Re$. It is easy to see that $\phi' \mid_M = \phi$. To verify that ϕ' is sublinear, first note that, by design, $\phi'(\lambda x') = \lambda \phi'(x')$ for each $\lambda \in \Re_+$ and $x' \in M'$. Second, for any $x', y' \in M'$ such that $x' = \lambda_1 x$ and $y' = \lambda_2 y$ for some $\lambda_1, \lambda_2 \in \Re_+$ and $x, y \in M$, we have

$$\begin{aligned} \phi'(x' + y') &= \phi'\left((\lambda_1 + \lambda_2)(\frac{\lambda_1}{\lambda_1 + \lambda_2} x + \frac{\lambda_2}{\lambda_1 + \lambda_2} y)\right) \\ &= (\lambda_1 + \lambda_2)\phi\left(\frac{\lambda_1}{\lambda_1 + \lambda_2} x + \frac{\lambda_2}{\lambda_1 + \lambda_2} y\right) \\ &\leq \lambda_1 \phi(x) + \lambda_2 \phi(y) \\ &= \phi'(x') + \phi'(y'), \end{aligned}$$

where in the second step the following fact is used: $\frac{\lambda_1}{\lambda_1 + \lambda_2} x + \frac{\lambda_2}{\lambda_1 + \lambda_2} y \in M$ because of the convexity of M.

Now extend ϕ' to all of X by $\upsilon' : X \to \Re$:

$$\upsilon'(x) \equiv \begin{cases} \phi'(x) & \text{if } x \in M' \\ \sup_{\psi \in \mathcal{P}} \ \psi(x) & \text{otherwise.} \end{cases}$$

It holds by design that $\upsilon' \mid_{M'} = \phi'$ and υ' is strictly positive. Since each $\psi \in \mathcal{P}$ is strictly positive and linear, the supremum operator in the above definition is, as can be checked, sublinear. Then, the extension υ' must also be sublinear, which proves the desired result.

3. ϕ is convex. In this case, consider the extension $\upsilon : X \to \Re$:

$$\upsilon(x) \equiv \begin{cases} \phi(x) & \text{if } x \in M \\ \sup_{\psi \in \mathcal{P}} \ \psi(x) & \text{otherwise.} \end{cases}$$

It is true by design that $\upsilon\ |_M = \phi$ and υ is strictly positive. It is also straightforward to show that υ is convex. Thus, the extension property obtains. ∎

Can the extension property imply viability for a convex cost system? – Our answer to this question remains a partial one, which is stated in the next theorem:

THEOREM 3. *Let $X = L^p$, for any p satisfying $1 \le p < \infty$. Assume that the non-empty set of feasible payoffs, M, is convex and the cost functional ϕ is* **sublinear**. *Then, the cost system (M, ϕ) satisfies the extension property if and only if it is viable.*

Proof. Sufficiency follows from Theorem 2. To prove necessity, suppose that the extension property holds and that υ is a strictly positive sublinear extension of ϕ. First notice that $X = L^p$, for $1 \le p < \infty$, is a separable Banach space. So must be the product space $\Re \times X$. Next, define two sets:

$$\begin{aligned} S &\equiv \{(c_0, x) \in \Re_+ \times X_+ : (c_0, x) \neq (0,0)\} \\ T &\equiv \{(c_0, x) \in \Re \times X : c_0 + \upsilon(x) \le 0\}. \end{aligned}$$

Here, S is the positive cone of $\Re \times X$ with the origin deleted and hence convex. The set T is also a non-empty convex cone because υ is sublinear, with $(0,0) \in T$. By assumption, υ is strictly positive on X, which means that S and T must be disjoint and that S must have empty intersection with the τ-closure of $(T - S)$. Then, by the separating hyperplane theorem for convex cones [Clark (1993, theorem 5)], there exists a nontrivial τ- continuous linear functional h on $\Re \times X$ such that $h(c_0, x) > 0$ for each $(c_0, x) \in S$ and $h(c_0, x) \le 0$ for each $(c_0, x) \in T$. That is, there are $\lambda_0 \in \Re$ and a continuous linear functional ψ on X such that $h(c_0, x) = \lambda_0 c_0 + \psi(x)$. Since $(1,0) \in S$ and $(0,x) \in S$ for $x \in X_+$ such that $x \neq 0$, it holds that $\lambda_0 > 0$ and $\psi(x) > 0$. Thus, $\psi \in \Psi$. Choose $\lambda_0 = 1$ so that $h(c_0, x) = c_0 + \psi(x)$. At the same time, for any $x > 0$, $(-\upsilon(x), x) \in T$, which means $-\upsilon(x) + \psi(x) \le 0$. Therefore, $\psi \le \upsilon$ and $\psi\ |_M \le \upsilon\ |_M = \phi$, implying (M, ϕ) is viable (according to Theorem 1). ∎

In proving Theorem 3, we relied on the fact that the epigraph of a sublinear functional on X is a convex cone. Then, the heart of the proof became finding a hyperplane to separate two convex cones, which is technically much easier to do than to separate two general convex sets. This is why we restricted attention to the class of sublinear cost functionals in Theorem 3. For two convex cones in a separable Banach space, the separating hyperplane theorem does not require either cone to have an interior [Clark (1993, theorem 5)]. In order for there to be a hyperplane separating two *general* convex sets, however, it is typically necessary that at least one of the two sets have interior points. In the case of the proof of Theorem 3, this means the positive cone of X should have a non-empty interior. Unfortunately, the common spaces L^p, for all p satisfying $1 \leq p < \infty$, do not meet this requirement.[4] For further discussion on this technical requirement, see Luenberger (1969, pages 133-134 and 219).

Remark 4.1. For the perfect-markets economies in Example 1, ϕ is linear and hence sublinear as well. For economies with proportional transaction costs (Example 2), with no-short-selling (Example 3), or with solvency constraints (Example 5), their corresponding cost functionals are all sublinear. Thus, for these types of economy, viability is equivalent to the extension property.

5. FREE LUNCHES

Another related criterion on a cost system is the absence of arbitrage opportunities or, more generally, of free lunches. Indeed, for (M, ϕ) to be a reasonable securities market model, it is natural that no one can obtain something for nothing, either now or in the future. Formally, a free lunch is defined below:

DEFINITION 5.1. A convex cost system (M, ϕ) admits free lunches if there exists a sequence, $\{(c_k, m_k, x_k) : k = 1, 2, \ldots\} \subseteq \Re \times M \times X$, such that (i) $\{(c_k, x_k) : k = 1, 2, \ldots\}$ converges to some $(c^*, x^*) \in \Re_+ \times X_+$, for $(c^*, x^*) \neq (0, 0)$; (ii) $m_k \geq x_k$ for each k; and (iii) $\lim\ \inf_k[c_k + \phi(m_k)] \leq 0$.

THEOREM 4. *Suppose that a cost system (M, ϕ) satisfying Assumption 1 is viable. Then, (M, ϕ) admits no free lunches.*

Proof. Suppose a convex cost system (M, ϕ) is viable. By Theorem 1, there is $\psi \in \Psi$ such that $\psi\ |_M \leq \phi$. Consider a sequence, $\{(c_k, m_k, x_k) : k = 1, 2, \ldots\} \subseteq \Re \times M \times X$, such that (i) $\{(c_k, x_k) : k = 1, 2, \ldots\}$ converges to some $(c^*, x^*) \in \Re_+ \times X_+$, for $(c^*, x^*) \neq (0, 0)$ and (ii) $m_k \geq x_k$ for each

[4] To get around this problem, one can assume $X = L^\infty$, as we will in Section 6, because L^∞_+ has an interior. However, in that case, we need S to be open in order to obtain *strict separation* between S and T in the proof of Theorem 3 [see Berberian (1974, theorem 30.6)]. Alternatively, one could assume X to be the space of real-valued continuous functions defined on the state space Ω, taking it as given that Ω is some closed interval on the real line. That, however, would not fit the purpose of modelling an economy with uncertainty. See Mas- Collel (1986) for other uses of such a modelling structure.

k. Then, for each k, $c_k + \phi(m_k) \geq c_k + \psi(m_k) \geq c_k + \psi(x_k)$. By continuity of ψ, we have: $\lim\ \inf_k[c_k+\phi(m_k)] \geq \lim\ \inf_k[c_k+\psi(x_k)] = c^*+\psi(x^*) > 0$, because $(c^*, x^*) \geq (0,0)$ and $(c^*, x^*) \neq (0,0)$. Thus, there cannot be free lunches. ▌

The absence of free lunches is therefore necessary for (M, ϕ) to be viable. The following result states that for certain securities markets, it is also sufficient.

THEOREM 5. *Let $X = L^p$ for some p satisfying $1 \leq p < \infty$, and suppose that M is a convex cone and ϕ is sublinear. Further, assume there is some $x \in M$ such that $Pr(x > 0) = 1$. Then, (M, ϕ) is viable if and only if it admits no free lunches.*

Proof. Necessity follows from Theorem 4. To prove sufficiency, suppose no free lunches exist. Consider again the following two sets:

$$\begin{aligned} S &= \{(c, x) \in \Re_+ \times X_+ :\ (c, x) \neq (0, 0)\} \\ T' &\equiv \{(c, x) \in \Re \times M :\ c + \phi(x) \leq 0\}, \end{aligned}$$

where S is the same non-empty, convex positive cone used in the proof of Theorem 3. By assumption, M is a non-empty convex cone and ϕ is sublinear, which means, as can be easily verified, that T' is also a non-empty convex cone. The absence of free lunches implies that S must have an empty intersection with the τ-closure of $(T' - S)$. Note that X is a separable Banach space. Then, by the separating hyperplane theorem [Clark (1993, theorem 5)], there is some τ-continuous linear functional f on $\Re \times X$ such that (i) $f(c, x) > 0$ for each $(c, x) \in S$ and (ii) $f(c, m) \leq 0$ for each $(c, x) \in T'$. Using the same steps as in the proof of Theorem 3, we can verify that there is some $\psi \in \Psi$ such that $\psi\ |_M \leq\ \phi$. Thus, (M, ϕ) is viable. ▌

Theorem 5, which is due to Jouini and Kallal (1992, theorem 2.2), says that if M is a convex cone and ϕ is sublinear, viability is then equivalent to the absence of free lunches. When M is a general convex set or ϕ is an arbitrary convex functional, we don't yet know whether the absence of free lunches is also sufficient for the viability of (M, ϕ). Again, the difficulty lies in the fact that it is generally extremely hard to find a hyperplane strictly separating the positive cone S from some arbitrary convex set T', where S and T' are defined as in the above proof. Also see the brief discussion following Theorem 3.

Remark 5.1. By Theorems 3 and 5, the three criteria – viability, absence of free lunches, and the extension property – are equivalent at least for the following types of economy: perfect-markets economies (Example 1), economies with proportional transaction costs (Example 2), economies with no-short-selling (Example 3), economies with solvency constraints (Example 5), and any combination thereof.

6. VIABLE COSTS AND EQUILIBRIUM PRICES: AN EXAMPLE

In this section, we explicitly consider the type of economy specified in Example 3 and use the Lagrange duality theorem to characterize the hedging cost problem in (1). The purpose of this exercise is two-fold. First, it is to show that the replication-based hedging cost for a given payoff x, $\phi(x)$, is generally different from the equilibrium price of x. Second, we demonstrate the technical conditions under which a hedging cost problem can be characterized in infinite dimensional spaces. For a similar analysis of the hedging cost problem in a finite state space setup, see Chen (1995).

6.1. The Cost Functional

Suppose $X = L^\infty$ and denote its dual by ba, where ba is the space of bounded additive measures on $(\Omega, \mathcal{F})$ that are absolutely continuous with respect to Pr. Let $\ell \equiv (\ell_1, \ldots, \ell_N)'$ be the vector of lower bounds on positions in the N traded securities. Rewrite the hedging cost problem below:

$$\phi(x) = \inf_{\alpha \geq \ell} \sum_{n \in N} \alpha_n \cdot P_n \tag{5}$$

$$\text{s.t.} \quad \sum_{n \in N} \alpha_n \cdot x_n \geq x, \tag{6}$$

for any $x \in M$. The constraint in (6) does not require an exact replication state-by-state because other portfolios that satisfy the constraint may cost less than an exact replication [e.g., Bensaid *et al* (1992)].

THEOREM 6. *For every $x \in M$, there is a solution to (1) if and only if the set D is non-empty, where D contains all positive continuous linear functionals on X, ψ, satisfying*

$$\psi(x_n) \leq P_n \qquad \forall n \in N, \tag{7}$$

and there exists a solution to

$$\phi(x) = \sup_{\psi \in D} \left\{ \psi(x) + \sum_{n \in N} \ell_n [P_n - \psi(x_n)] \right\}. \tag{8}$$

Proof. For any $x \in M$, define $g(\alpha, x) \equiv \sum_{n\in N} \alpha_n \cdot x_n - x$. Fix any $x \in M$ and its corresponding $\alpha \in \Re^N$ such that $\alpha \geq \ell$ and $x = \sum_{n\in N} \alpha_n x_n$ (this α exists since $x \in M$). Note that by Assumption 1, there is some $x' \in M$ such that $Pr(x' > 0) = 1$. Pick any such x' and let α' be such that (i) $\alpha' \geq \ell$ and (ii) $x' = \sum_{n\in N} \alpha'_n x_n$. Then, $g(\alpha + \alpha', x) = x'$ and, since $Pr(x' > 0) = 1$, $g(\alpha + \alpha', x)$ is an interior point of X_+. Note that the positive cone $X_+ = L^\infty_+$ has a non-empty interior. Thus, the space X and the constraint $g(\alpha, x) \geq 0$ satisfy the technical conditions for the Lagrange duality theorem [Luenberger (1969, p. 224)].

Define for any $\mu \in ba$ the dual functional of (5) as

$$\delta(\mu) \equiv \inf_{\alpha\geq\ell} \left\{ \sum_{n\in N} \alpha_n P_n + \int_\Omega (x - \sum_{n\in N} \alpha_n \cdot x_n) d\mu \right\}$$

or, equivalently,

$$\delta(\mu) = \inf_{\theta\geq 0} \left\{ \sum_{n\in N} \theta_n [P_n - \int_\Omega x_n d\mu] + \int_\Omega x d\mu + \sum_{n\in N} \ell_n [P_n - \int_\Omega x_n d\mu] \right\}. \tag{9}$$

By the Kuhn-Tucker condition, any solution θ^* to (9) must satisfy

$$\sum_{n\in N} \theta^*_n \cdot \{P_n - \int_\Omega x_n d\mu\} = 0, \tag{10}$$

for each $\mu \in ba$ satisfying

$$\int_\Omega x_n d\mu \leq P_n \qquad \forall n \in N. \tag{11}$$

For those μ violating (11), no solution to (9) exists. In other words, a necessary and sufficient condition for the existence of solution to (9) is that there be some μ satisfying (11). Since ba is the dual of X and hence each $\mu \in ba_+$ defines a positive continuous linear functional ψ on X, this means that, in order for there to exist a solution to (9), it is necessary and sufficient that the set D is non-empty, where $ba_+ \equiv \{\mu \in ba : \mu \geq 0\}$.

For any μ satisfying (11), the solution to (9) yields

$$\delta(\mu) = \int_\Omega x d\mu + \sum_{n\in N} \ell_n [P_n - \int_\Omega x_n d\mu]. \tag{12}$$

Now, suppose there is a solution to (5) with $\phi(x) > -\infty$. By the Lagrange duality theorem,

$$\phi(x) = \sup_{\mu \in ba_+} \delta(\mu), \tag{13}$$

has a solution, which means that there is some $\mu \in ba_+$ satisfying (11) and D is non-empty. Equation (13), together with (11) and (12), gives rise to

$$\begin{aligned} \phi(x) &= \sup \left\{ \int_\Omega x d\mu + \sum_{n \in N} \ell_n [P_n - \int_\Omega x_n d\mu] : \ \mu \in ba_+ \text{ and } \mu \text{ satisfies (11)} \right\}, \\ &= \sup_{\psi \in D} \left\{ \psi(x) + \sum_{n \in N} \ell_n [P_n - \psi(x_n)] \right\}. \end{aligned}$$

This proves (8).

To prove sufficiency, assume there is some $\psi \in D$. Then, by the duality relation between X and ba, there is some $\mu \in ba_+$ representing ψ such that μ satisfies (11), implying a solution to the dual problem in (13) exists, with $\phi(x) = \sup\{\int_\Omega x d\mu + \sum_{n \in N} \ell_n [P_n - \int_\Omega x_n d\mu] : \ \mu \in ba_+$ and μ satisfies (11)$\} > -\infty$. By the Lagrange duality theorem, a solution to the primal in (5) also exists. ▮

Note that the use of the Lagrange duality theorem in the above proof requires the positive cone of X to have interior points. As mentioned before, this requirement is not met by any L^p, for $1 \leq p < \infty$, except by L^∞. This is why we started with $X = L^\infty$. Clearly, this technical condition is quite troublesome, because it means that in almost all L^p spaces, we cannot apply the duality theorem to interpret the hedging cost problems specified in Examples 2 through 7. Also note that in a relative sense, the viability criterion on (M, ϕ) is indeed strong because it is sufficient to induce some "duality" restriction on ϕ (Theorem 1 and Corollary 1).

Remark 6.1. Suppose (M, ϕ) is viable. Then, the set D contains some ψ such that $\psi \in \Psi$. To see this, note that by design, $\phi(x_n) \leq P_n$ for each security n. By Theorem 1, this means there is $\psi \in \Psi$ such that $\psi(x_n) \leq \phi(x_n) \leq P_n$ for each n. Thus, $\psi \in D$.

Notice that even when (M, ϕ) is viable, it may not hold that $P_n = \phi(x_n)$ for any n. The lemma below specifies one demanding condition under which this equality holds:

Lemma 1. *Assume that each investor from* **A** *solves the hedging problem in (5) for some $x \in M$ and that (M, ϕ) is viable. Suppose there is some* *investor whose position in every security n exceeds the lower bound ℓ_n. Then $P_n = \phi(x_n)$ for every n.*

Proof. Suppose some investor i who solves the hedging problem in (5) for some $x^i \in M$ finds it optimal to hold α_n^i shares of security n such that $\alpha_n^i > \ell_n$. Let $\mu^i \in ba_+$ be the solution to the dual problem in (8), when x is replaced by x^i. Rewrite the Kuhn-Tucker condition in (10) as follows

$$\sum_{n \in N} (\alpha_n^i - \ell_n)(P_n - \int_\Omega x_n d\mu^i) = 0.$$

Since $\alpha_n^i > \ell_n$ and $P_n \geq \int_\Omega x_n d\mu^i$ for every $n \in N$ (see the proof of Theorem 6), it must hold that $P_n = \int_\Omega x_n d\mu^i$ for each n. Let $\psi' \in D$ be the positive continuous linear functional defined by μ^i. So, $P_n = \psi'(x_n)$. Then, $P_n \geq \phi(x_n) = \sup_{\psi \in D} \{\psi(x_n) + \sum_{j \in N} \ell_j [P_j - \psi(x_j)]\} \geq \psi'(x_n) = P_n$. ▌

6.2. The Equilibrium Price Functional

Suppose (M, ϕ) is viable. Given the characterization of ϕ in Theorem 6, we are ready to compare it with an equilibrium price functional. To define a specific equilibrium, assume that in the economy under consideration, the normalized supply of each security n is one unit and there are I investors. Let the initial endowment of investor i be given by some time-0 portfolio, $\alpha_0^i \in \Re_+^N$, and his preferences represented by the time-separable expected utility: $\sum_{t=0,1} E[U^i(c_t, t)]$, for a consumption plan $c \equiv (c_0, c_1) \in \Re \times X$, where, for $t = 0, 1$, $U^i(\cdot, t) : \Re \to \Re$ is strictly increasing, concave and twice continuously differentiable. At time 0, investor i is assumed to solve

$$\max_{c_0, \alpha \geq \ell} \left\{ U^i(c_0, 0) + EU^i(\sum_{n \in N} \alpha_n \cdot x_n, 1) \right\} \tag{14}$$

$$\text{s.t.} \quad \sum_{n \in N} \alpha_{n,0}^i \cdot P_n = c_0 + \sum_{n \in N} \alpha_n \cdot P_n. \tag{15}$$

Let (c^i, α^i) be investor i's *optimal consumption-portfolio plan* that solves (14) (when it exists).

For the frictional exchange economy, define an **equilibrium** to be a collection:

$$\mathcal{E} \equiv \left\{ [(x_n, P_n) : n \in N], [(c^i, \alpha^i) : i \in I] \right\},$$

such that (i) for each investor i, the plan (c^i, α^i) is optimal given (x_n, P_n) for $n \in N$ and initial endowment α_0^i; and (ii) the securities market at time

0 as well as the goods market at time 0 and in each non-zero probability event at time 1 clear. On the existence and efficiency characterizations of such an equilibrium, refer to Allen and Gale (1988) and Hart (1979). Here, we take the existence of equilibrium as given.

The above definition does not yet involve a price functional that firms and innovators can use to quote prices for new securities. To incorporate a price functional as part of the equilibrium condition [as in Allen and Gale (1988)], consider the Kuhn-Tucker condition for investor i's problem in (14):

$$P_n \ \geq \ E\left(m^i \cdot x_n\right) \qquad \forall n \in N, \tag{16}$$

with equality whenever $\alpha_n^i > \ell_n$, where

$$m^i \equiv \frac{U_c^i(c_1^i, 1)}{U_c^i(c_0^i, 0)}, \tag{17}$$

is the intertemporal marginal rate of substitution (IMRS) between time-0 and time-1 consumption. Since $m^i \in L^1(\Omega, \mathcal{F}, Pr)$, setting

$$\psi^i(x) \equiv E(x \cdot m^i) \qquad \forall x \in X,$$

we obtain $\psi^i \in \Psi$. Economically, $\psi^i(x)$ stands for how much investor i would be willing to pay in order to hold at the margin a small quantity of the security x. In other words, $\psi^i(x)$ is investor i's *marginal valuation*.

Given both the differences in marginal valuations ψ^i among investors and the trading frictions, *the price that firms and innovators expect to receive upon issuing a security should be the maximum amount that any investor would be willing to pay in order to hold a small quantity at the margin.* This is what Allen and Gale (1988) and Hart (1979) refer to as the *rational conjecture condition*. Formally, this condition defines the *equilibrium price functional* $\pi : X \rightarrow \Re$:

$$\pi(x) \equiv \max_{i \in I} \ \psi^i(x) \qquad \forall x \in X, \tag{18}$$

where the maximum exists since I is finite. Note that π is sublinear and strictly positive on X, because each ψ^i is strictly positive and linear. Assuming that for each security n there is at least one investor who holds more than the lower bound ℓ_n, we have, by the Kuhn-Tucker condition, $\pi(x_n) = P_n$ for $\forall n \in N$. For more discussion on the rational conjecture condition, see Allen and Gale (1988) and Hart (1979).

6.3. Relations between a Viable Cost System and an Equilibrium Price System

Fix (X, π), which we refer to as the *equilibrium price system.* Suppose (M, ϕ) is the viable cost system embedded in (X, π). Can there be cases in which (X, π) is an extension of (M, ϕ)? – An answer to this question is important because it allows us to see the effectiveness of the "pricing by arbitrage" approach in frictional economies. For instance, if the answer were yes, one could then use ϕ to price any marketed or feasible payoff in M and, in particular, it would not make a difference whether a security issuer uses ϕ or π (at least over M).

THEOREM 7. *Suppose that the economy is in equilibrium and that for each security $n \in N$ there is at least one investor who holds more than the lower bound ℓ_n. Then, the following statements are true:*

1. When the frictions are such that $\ell_n \geq 0$ for each n, which includes no-short-selling as a special case, it holds that

$$\phi(x) \geq \pi(x) \qquad \forall x \in M. \tag{19}$$

In this case, $\phi(x_n) = \pi(x_n) = P_n$ for each traded security $n \in N$.

2. Under more general frictions, $\phi(x) \geq \pi(x)$ for certain payoffs $x \in M$, but there can exist other payoffs $y \in M$ for which $\phi(y) < \pi(y)$. Furthermore, for each traded security n, $\phi(x_n) \leq \pi(x_n) = P_n$.

Proof. The proof of this result is analogous to the proof of Proposition 3 in Chen (1995). For completeness, adopt his proof here. Let $\Delta \equiv \{\psi^i : i \in I\}$. By definition, $\Delta \subseteq D$ in equilibrium. First, observe that for any $n \in N$, $\phi(x_n) \leq P_n$ by definition of the hedging cost problem in (5). Since $\pi(x_n) = P_n$, it holds that $\phi(x_n) \leq \pi(x_n)$.

Next, consider the case in which $\ell_n \geq 0$ for each n. By equation (8), we have for every $x \in M$:

$$\begin{aligned} \phi(x) &\geq \max_{\psi \in D} \psi(x) \\ &\geq \max_{\psi \in \Delta} \psi(x) = \pi(x). \end{aligned}$$

For each traded security n, the above derivations show that $\phi(x_n) \geq \pi(x_n)$ and $\phi(x_n) \leq \pi(x_n)$, meaning $\phi(x_n) = \pi(x_n) = P_n$. This proves the first statement.

To prove the second statement, suppose there is at least one security n for which $\ell_n < 0$. Pick some $x \in M$ for which the solution to (1) is α^* such that $\alpha_n^* > 0$ for every n (clearly such x exist). For this x, $\phi(x) = \sum_{n=1}^{N} \alpha_n^* P_n$.

Since $P_n \geq \psi^i(x_n)$ for each n, we have for every $i \in I$: $\sum_{n\in N} \alpha_n^* P_n \geq \psi^i \left[\sum_{n\in N} \alpha_n^* x_n\right]$, which implies $\phi(x) \geq \pi(\sum_{n=1}^N \alpha_n^* x_n) \geq \pi(x)$.

There can be payoffs $x \in M$ for which the solution to (5) is such that $\alpha_n < 0$ for some n. In such cases, we can have $\phi(x) < \pi(x)$ or $\phi(x) \geq \pi(x)$, both of which are possible. For example, we have shown that $\phi(x_n) \leq P_n = \pi(x_n)$ for each n, where strict inequality can arise in equilibrium (for instance, the condition in Lemma 1 is not satisfied here). To see this, suppose that for some security n',

$$\phi(x_{n'}) < \pi(x_{n'}) = \max_{i\in I} \ \psi^i(x_{n'}),$$

that is, there is at least some investor i' such that

$$\phi(x_{n'}) < \psi^{i'}(x_{n'}) = \max_{i\in I} \ \psi^i(x_{n'});$$

further, assume the solution α' to (5), when the x in (5) is replaced by this $x_{n'}$, is such that $\alpha'_{n"} = \ell_{n"} < 0$ for some $n"$. Then, if the existing optimal consumption-portfolio plan $(c^{i'}, \alpha^{i'})$ of investor i' is also such that $\alpha^{i'}_{n"} = \ell_{n"}$ and $\alpha^{i'}_{n'} > \ell_{n'}$, investor i' may not be able to use portfolio α' to substitute for security n' because that may lead to a short position in $n"$ exceeding the lower bound $\ell_{n"}$. Thus, in some economies, it can occur that in equilibrium $\phi(x_n) < P_n$ for some n. ▌

Theorem 7 has important implications for the characterization of frictional economies. First, the equilibrium price system (X, π) in general cannot be an extension of the viable cost system (M, ϕ). Since $\phi(x)$ gives the "arbitrage price" of x for each $x \in M$, in most cases the arbitrage price is not an achievable equilibrium price (even as x is redundant!). In contrast, it is known that in frictionless economies the arbitrage price is the same as the equilibrium price, at least over the marketed payoff span [e.g., Arrow (1964), Harrison and Kreps (1919) and Ross (1978)]. Next, in view of $\pi\mid_M \neq \phi$, (M, ϕ) cannot be a price system to be used by security issuers. This explains why we chose to refer to (M, ϕ) as a cost system.

Theorem 7 also has implications for the economic role of financial innovation. For instance, suppose $\ell_n \geq 0$ for each n. Then, $\phi(x) \geq \pi(x)$ for each $x \in M$. Now, take any $y \in M$ such that $y \neq x_n$ for any $n \in N$. Clearly, $\phi(y)$ is the existing minimum portfolio cost for y and $\pi(y)$, which may be lower than $\phi(y)$, is the price that would prevail if a market would be opened for y. This means that opening a market for y can reduce the cost for y. For further discussion on financial innovation in frictional economies, see Allen and Gale (1988) and Chen (1995).

7. CONCLUDING REMARKS

In this paper, we have provided a general characterization of a viable convex cost system. The main conclusion is that a convex cost system is viable if and only if there is a strictly positive continuous linear functional lying below the cost functional (over the set of feasible payoffs). This result is quite general since many types of frictional or frictionless securities markets can be described by a convex cost system. For some subclasses of frictional markets, we were also able to show that viability is equivalent to the generalized extension property which is in turn equivalent to the absence of free lunches.

There are still some open issues to be resolved in future research. For instance, can the extension property also be sufficient to guarantee the viability of a general convex cost system? Can the absence of free lunches imply viability for a general convex cost system? — Any answer to these two related questions depends on whether one can weaken the technical conditions for the existence of a strictly separating hyperplane in infinite dimensional spaces. Our conjecture is that when $X = L^p$, for $1 \leq p < \infty$, the answer to the above two questions is likely to be negative.

The discussion in this paper was cast on a single-period framework where securities trading takes place only once. It is straightforward to extend the discussion to a framework in which trading occurs at discrete time points and in which asset prices and dividends follow discrete-time processes. In that regard, the set-up in Hansen and Richard (1987) should be helpful. Extension of our discussion to a continuous-time framework, however, may not be simple. Hindy (1995) and Jouini and Kallal (1992, 1995) study viable price processes by assuming specific types of friction. For more general convex cost systems (M, ϕ), where M is presumably the convex set of feasible dividend/price processes and ϕ maps each process in M to a minimum hedging cost process, we expect that the viability of (M, ϕ) will be equivalent to the existence of some equivalent martingale measure with respect to which there is a discounted martingale process lying below $\phi(x)$ for each process $x \in M$ (see Remark 6). We leave this interesting extension for future research.

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