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A Stochastic Theory of Limit Order Transactions in Securities Markets

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The subject of this research paper is the same as the focus of criticisms in a recently released SEC report: namely, failures to display and execute limit orders in securities markets. Based on a statistical sample, the SEC study found frequent violations of limit order display rules. How pervasive are these kinds of violations in the market ?The theory in the paper addresses the question by identifying the probability density functions governing the display and execution of limit orders in properly functioning markets. The paper demonstrates that two distinct stochastic processes are sufficient to completely describe the execution of limit orders in markets: a conditional Binomial distribution compounded with a conditional Poisson distribution. These distributions permit rigorous tests of the statistical significance of the SEC sample findings. © 2002 Peking University Press

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1. INTRODUCTION

The subject of this research paper is the same as the focus of criticisms in a recently released SEC report: namely, failures to display and execute limit orders in securities markets. Based on a statistical sample, the SEC study found frequent violations of limit order display rules. There is a serious and still unanswered question as to how pervasive these violations are.

Limit orders account for about two-thirds of all system orders on both the NYSE and the NASDAQ. If a significant number of them are not executed when they should be, investors are harmed because the brokerages

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improperly extract from them what amounts to a monopoly rent effected by impacted information.

The SEC study suggested that failures to display and execute limit orders can be ascribed to improper conduct by brokers: viz, venality, inattentiveness, laziness , technical incompetence, or some combination . However, imputations of broker misconduct are based on an implicit assumption that properly functioning markets are perfectly functioning markets; i.e. that all limit orders are immediately and accurately displayed, and all are executed unless cancelled by the customer. That presumption may be specious.

The theory in the paper addresses the question of market efficiency by identifying the probability density functions governing the display and execution of limit orders in properly functioning markets. This research paper develops a stochastic model of limit order executions in securities markets that allows the SEC (or any interested researcher) to carry out a test of the hypothesis that the statistical incidence of the failure to display and execute limit orders in a sample is a significant departure from a randomly determined outcome.

The theory in the paper embodies the defining financial characteristics of limit orders in capital markets. The paper demonstrates that two distinct stochastic processes are sufficient to completely describe the execution of limit orders in markets: a conditional Binomial distribution compounded with a conditional Poisson distribution.

The main conclusions of the paper consist of three propositions that establish the statistical properties of limit order executions. These propositions imply, inter alia, that the expected number of limit order executions is shown to be a random function with a systematic component reflecting the unit selling expenses of the brokerages effecting the transactions.

2. THE DIMENSIONS AND THE CONSEQUENCES OF THE FAILURE TO EXECUTE LIMIT ORDERS

With the explosive growth of securities trading on electronic networks comes fragmentation, or splitting of trades among different exchanges, dealers and networks. There are at least two obvious market consequences to fragmentation: (a) investors will get more choices of where to trade stocks and (b) some market participants (i.e. institutional investors and transacting firms) will offer better prices than other firms in ways that might not be discovered by individual investors.[1] Current rules of the Securities and Exchange Commission (hereafter "the SEC") require brokers to provide to their customers the best price available anywhere. But there is some

[1]For some recent research on the so-called "fragmentation literature" see Cohen and Conroy (1990), Davis and Lightfoot (1994) and Wood and McInish (1992)

recent evidence to suggest that the rule is frequently violated, sometimes knowingly, sometimes not.

A report issued by the SEC on May 4, 2000 studied practices at a sampling of the nation's stock and options exchanges and brokerage-firm trading desks in 1999 [15] The Report found, inter alia, that brokerage firms are routinely flouting securities rules intended to ensure that investors receive the best possible prices on their trades.

The subject of study in this paper is the same as the focus of the criticisms found in the SEC Report: namely, the treatment of investors who place orders to buy or sell a stock or an option at a specified price, known in the trade as a limit order. The management and execution of limit order postings by brokerages is of great significance to market dynamics because they account for about two-thirds of all system orders on both the NYSE and the NASDAQ Atkinson (1999).Most quotes on the NYSE are set by limit orders Chung (1999).

If an investor places say, a bid limit order, he expects that his price will be posted in the overall market. If this limit order is slightly below the prevailing market price for a stock he may nevertheless attract a seller willing to effect a trade who otherwise might not have come forward. The buyer's limit order, therefore, affords him a possibility of paying a lower price for his shares than he would have paid had he been willing to buy at the prevailing price. There exists statistical evidence that spreads appear to be narrowest when set by limit orders Chung (1999).

A brokerage that fails to promptly display a limit order makes it increasingly likely that the investor will miss his opportunity to buy or sell because the market in the stock has moved. To the extent that these sorts of transactions occur, the proper and prompt display of limit orders can lead to an improvement in the efficiency of stock transactions.[2] In addition, limit orders supply additional liquidity to the market. Kavajecz (1999)

Traditionally, competition from limit orders on Nasdaq was considered unnecessary because competitive spreads were assumed to prevail when multiple dealers competed for orders on the basis of price. However, Dutta and Madhavan (1997) and Kandel and Marx (1997) showed theoretically that discrete price increments and access to alternative methods of securing order flow could result in bid-ask spreads that exceeded competitive levels.

[2]Individual investors can buy or sell without a broker through a private online trading firms such as Datek Securities Corp., which provides access to NASDAQ's computerized Small Order Execution System (known as "SOES") or the NYSE's SuperDOT system. Nevertheless individual investors are still faced with the problem of mismanagement of their limit orders because these systems rely on brokers' terminals being linked to the computerized exchange systems of the NYSE or NASDAQ. The individual investors using these on-line trading systems face the same exchange markets which are essentially broker-organized marketplaces that centralize the trading and settlements of payments. See Choi (1997)

Abuse of customer limit orders was at the heart of the investigations into the NASDAQ market conducted by the SEC and the Justice Department in the mid-1990s. Professional traders on the NASDAQ were found to have routinely failed to display customer limit orders at prices that were better than the prevailing market price. The failure of these broker/dealers to post the limit price information enabled them to exploit this "impacted information" by trading against it to enjoy super-normal profits.[3]

A common way in which broker/dealers exploited the impacted information was to artificially intervene as market makers and specialists, often widening the spread between bid and ask prices irrespective of market conditions.[4] If a dealer gets a limit order to buy shares at a price slightly below the prevailing market price, he can sell shares to the customer at a profit, precluding the sale by another investor who would be willing to sell his shares for a slightly higher price. Regulators believe that broker/dealers would be far less likely engage in this kind of conduct if limit orders had been accurately and promptly posted in the market.

In the wake of the SEC investigation in the mid-1990s, the Commission enacted rules in 1996 requiring that stock exchange specialists and NASDAQ market makers display customer limit orders that improve the prevailing market price within 30 seconds of having received them. But the SEC Report issued on May 4 indicated that four years after they were introduced, these regulations are consistently violated by brokerages and exchanges.

The Commission did not identify the brokerages and exchanges that it sampled, but the Report stated that at three large NASDAQ trading firms the Commission found violations of limit order display rules in 46 percent, 59 percent and 92 percent of trades sampled.

There are explanations for the mismanagement of limit order postings other than the venality of broker/dealers. Failures to properly display limit orders might be caused by, for example, technological inadequacies (i.e. hardware and/or software defects), inattentiveness of broker/dealers, or simple incompetence of brokerage employees who manage the postings at web sites. This paper is not concerned with ascribing fault. The objective of the paper is the development of a realistic theory of how the failure to display and execute limit orders will affect the volume of transactions in securities markets in two kinds of scenarios: (a) a marketplace where broker/dealers are homogeneous and (b) a market where the broker/dealers are differentiated.

[3]Williamson (1976) defines the condition of "impacted information" as present when important information " is known to one or more [of the] parties [involved] but cannot be costlessly discerned by or displayed for the others."

[4]See Morgenson (1996)

The main focus of this paper is an extension of one of the conclusions expressed by Glosten (1994, p. 1152) in his 1994 paper. He wrote:

"After setting up a reasonably general model of investor behavior, the article develops some characteristics of the equilibrium in an electronic market where there are a larger number of limit order submitters. It is shown that the equilibirum involves an upper (lower) tail' conditional expectation in the determination of offers (bids.)"

The theory in this paper is based, in part, on the proposition that the volume of limit order postings and executions can be characterized as a random process generated, in part, by investor postings. The main accomplishment of this paper is the derivation of a well-defined statistical distribution that is amenable to hypothesis testing as respects the sample incidence of the failure to display and execute limit orders. Without a basis for statistical hypothesis testing the SEC will be unable to distinguish random (and statistically insignificant) failures to execute from correctable systemic failures to display and execute.

3. TRANSACTION VOLUME IN A HOMOGENEOUS MARKET

Consider an investor who wants to purchase a stock at a share price not exceeding P_B. Hereafter I will refer to P_B as the limit price. I assume that the investor has established a trading account at an on-line brokerage. The brokerage hosting the account is required to carry out a search for the best price; namely the lowest possible price not exceeding the limit price. The first question to be addressed is: how will the volume of securities transactions in a homogeneous market be distributed among the brokerages?

The description of the market as "homogeneous" does not refer to investors' attitudes towards risk. The characterization is employed in this context to describe the attributes of the market in which the transactions are effected. These are adumbrated below.

(a) All investors who purchase stock enter limit price orders. These are the bid prices.

(b) Investors pay the same transaction fees to the brokerages that host their accounts.

(c) Investors regard the brokerages as functionally indistinguishable.

(d) Each brokerage employs a search engine to determine the asking prices for stock at brokers' websites. The search engine may be proprietary to the brokerage hosting the investor's account, or it may be a generalized webcrawler.

(e) Each hit on a broker's website results in a binomial event; either the limit order is executed or the parties disengage. The buyers and the sellers do not participate in a price negotiation.

(f) Buyers do not communicate among themselves or with sellers as to the array of bid and ask prices posted in the limit order book.

Some of these assumptions will be relaxed in following sections of the paper. At this point I am focusing on the question of whether the distribution of limit order executions can be derived as a function with estimable parameters.

If the limit price is "better" than the current market price, such orders are generally held in the specialist's book until the price moves to the designated level, if ever. Since limit orders are price-contingent, however, their representation by a simple stochastic process is problematic.[5] The main contribution of this paper is the derivation of a stochastic processes with empirically verifiable parameters.

Let us assume that there are n linked brokerages in the market. The array of asking prices for a specific stock that are posted in the network is represented as the vector $\{P_1, P_2, \cdots, P_n\}$. If a brokerage is not a participant in the market for a stock, its posted asking price is infinitely large. The multivariate probability density function governing the distribution of the vector of asking prices is symbolized by $F(P_1, P_2, \cdots, P_n)$.[6]

Investors arrive sequentially in the market with an opportunity to trade. Investors can place an order to buy or an order to sell one unit of stock at a price chosen from the set $\{P_i \,|\, i = 1, 2, \cdots, n\}$.[7] However, the assumptions in this paper characterizing the market transactions differ from the conventional assumption found in the literature.

Virtually every author who has written on the subject of limit order transactions has assumed, explicitly or implicitly, that all limit orders are speedily and accurately displayed and either executed or cancelled by the customer. A typical statement is the following:

"Once a [limit order] has been submitted, it either may trade immediately, or it may enter the queue of unfilled orders, referred to as the limit order book. Upon entry into the limit order book, one of two things eventually occur. The order may be executed or it may be cancelled." Hollifield [8, ,p. 7]

Notice that this way of describing the market leaves no room for the possibility that the brokerage may fail to display a limit order; the assumption reproduced above assures that every limit order is acted upon in a dispositive way. In this paper there is another possibility considered, namely that

[5] See, for example, O'Hara (1995, p.37)

[6] I assume that F is a stationary distribution.

[7] This assumption is the same as found in the paper by Hollifield, et. al.

a failure to act on a limit order may be a consequence of failure to display it.

Suppose we focus attention on the likelihood that brokerage 1 will fill an investor's purchase order at or below the limit price posted by the investor. In a perfectly functioning and frictionless market where all limit orders are immediately and accurately displayed, brokerage 1 will execute the transaction if and only if the asking price that it posts on the electronic trading floor is less than the price of every other brokerage in the market, and if its posted asking price is less than or equal to the limit order bid price offered by the counterparty firm where the investor has his account.

The SEC Report has established that in the real marketplace the investors and the transacting firms may not know all of the elements of the set of limit prices, perhaps because not all of them are displayed adequately, or perhaps not displayed at all. Search engines may be inefficient or defective; dealers may individually engage in improper conduct; there may be collusion among the market makers to subvert an effective search for the best price. Venality, carelessness and simple technological incompetence will introduce sand into the gears of the market. The "sand" may be manifested as failure to adequately display some of the limit orders. The practical implication of these possibilities is that the brokerage that is actually offering the lowest asking price will not necessarily be found by the search engine employed by the brokerage hosting the investor's limit order.

Brokerage 1 cannot be assured that it will execute the sale at or below the limit order even if its asking price is the best from the buyer's point of view. In a fragmented market the brokerage will recognize that its transactional volume and its fee revenues are determined probabilistically.

The probability that firm 1 will fill the investor's limit order can be represented as follows:

$$
\begin{aligned}
& Prob[\text{brokerage 1 fills the limit order bid at asking price} P_1] \\
= \; & Prob[P_2 > P_1, P_3 > P_1, \cdots, P_n > P_1, P_B > P_1]
\end{aligned}
$$

Brokerage 1 cannot know the ex ante bid price of any particular investor. However, the brokerage does know that there is a population of investors in the market who are interested in purchasing the stock. In this paper I adopt the paradigm of investor behavior adumbrated by Glosten (1994, p1131):

"Bids and offers are submitted without knowing what the next arriving order will be. The next trader to come to market chooses the trade based on his or her privately known but generally unobservable characteristics preferences, information, portfolio position, etc."

From the point of view of a brokerage, the posted limit order bid prices are drawn from a probability distribution. The firm posts an asking price in its electronic book. From the firm's point of view, the limit order bid prices are assumed to be drawn from a probability distribution.[8]

I assume that the firm estimates that the distribution of the limit order bid prices can be represented by a c.d.f. symbolized by $G(P)$. This distribution governs the likelihood of the limit order bid prices posted by the population of investors at an arbitrary point in time.[9] A brokerage can calculate the probability that it will execute a sale to an investor randomly selected from that population. That probability is symbolized by $\pi(P_1)$ where :

$$\begin{aligned} &Prob[\text{brokerage 1 will execute a limit order of a random investor at price } P_1] \\ &= \left[\int_{P_1}^{\infty} \cdots \int_{P_1}^{\infty} F(P_1, x_2, x_3, \cdots, x_n) dx_2 dx_3 \cdots dx_n\right] \int_{P_1}^{\infty} dG(x) = \pi(P_1) \qquad (1) \end{aligned}$$

The symbol $\pi(P_1)$ represents the probability that brokerage 1 will execute a randomly selected buyer's limit order at a posted asking price equal to P_1. I will refer to $\pi(P_1)$ as the transaction probability . It is an obvious mathematical property of (1) that $\frac{d\pi(P_1)}{dP_1} \le 0$.

For a fixed value of the posted asking price P_1, the brokerage can calculate the probability that S executions will be effected at or below investor limit orders if its website posting is hit a total of H times. This is a conditional binomial distribution in which $\pi(P_1)$ is the parameter. The expression is given in equation (2) below.

$$\begin{aligned} &Prob[\text{brokerage 1 will effect } S \text{ limit order sales at asking price } P_1 | H \text{ hits}] \\ &= \binom{H}{S} [\pi(P_1)]^S [1 - \pi(P_1)]^{H-S} = b(S, H; \pi(P_1)) \qquad (2) \end{aligned}$$

Hereafter I will suppress the subscript 1; it being understood that the mathematical expressions refer to an arbitrarily identified brokerage participating in the market. The probability distribution symbolized by $b(S, H; \pi)$ in (2) is amenable to applications only if the parameter $\pi(P)$ is estimable. We turn now to this problem.

[8]The assumption in this paper is more general and, arguably, more realistic than the assumption found in O'Hara (1995, p.39). In that paper it is assumed that the limit orders to buy from a dealer are assumed to be linear functions of the price. In most of Glosten's paper (1994, p.1138) he assumes that the set of allowable prices is a continuum, he observes that the more realistic case is where prices are restricted to a discrete set.

[9]I assume that G is a stationary distribution.

4. ESTIMATION OF TRANSACTION PROBABILITIES IN A HOMOGENEOUS MARKET

In order to be of practical usefulness to the brokerage, the estimation of the transaction probability at a fixed asking price should require only observable empirical data. An estimator with optimal properties is identified by PROPOSITION 1.

PROPOSITION 1. *If the asking price posted by the brokerage on the website is fixed, and the site statistics consist of H hits and S transactions at the posted price in a fixed time period, an unbiased estimator of the transaction probability at the posted price is given by:*

$$\overline{\pi}(P) = \frac{S}{H}.$$

Proof. Statistical estimation of the transaction probability in a homogeneous population of investors can be accomplished by exploiting the properties of an indicator function.[10] The indicator function is a binary-valued random variable defined in such a way that its domain is the asking price posted on the website of the brokerage.

Suppose the limit order asking price posted on the website is P. The website records H hits in a defined period of time. Each hit constitutes a limit price bid order. The hits are enumerated as: $j = 1, 2, \cdots, H$. In each of those hits the indicator function is triggered. The indicator function for an arbitrary hit j at the posted price is P is symbolized by $I_j(P)$. It is defined as follows:

$$I_j(P) = \begin{cases} 1 & \text{if hit } j \text{ results in an execution at price } P \\ 0 & \text{otherwise} \end{cases}$$

Defined in this way we see that $\pi(P) = Prob[I_j = 1] = E[I_j]$. This equality among the parameters allows us to calculate $\pi(P)$ using observable data collected from website statistics.[11]

Suppose that of the H hits on the site, a subset S results in a sale of the stock at the posted asking price where the set $S = \{1, 2, \cdots, H\}$ This defines S as a set of integer-valued random variables on the domain of non-negative integers $[0, H]$ The likelihood of an element of S in the sample space H (regardless of runs) can be calculated as $b(S, H; \pi)$ in (2).

The m.l.e. of π is symbolized by $\overline{\pi}$. It is a textbook exercise to derive $\overline{\pi}$ as the unique non-trivial solution to the equation $\frac{d \ln b(H,S;\pi)}{d\pi} = 0$. That

[10] For a discussion of the application of the Indicator Function see Loeve (1963).

[11] Virtually any website can be designed to include a "hitometer." This is basically a tallying device that records the number of times that a website has been visited.

solution is:

$$\overline{\pi} = \frac{S}{H} \tag{3}$$

The m.l.e. of π in (3) is the fraction of hits that result in sales at the posted price. This is obviously an observable datum. Moreover, it is easy to demonstrate that $\overline{\pi}$ is an unbiased estimator of π.

According to the definition of the indicator function,

$$\sum_{j=1}^{H} I_j = S_H \tag{4}$$

The summation in (4) implies the following:

$$E[S_H] = E\left[\sum_{j=1}^{H} I_j\right] = \sum_{j=1}^{H} E[I_j] = \sum_{j=1}^{H} \pi(P) = H\pi(P) \tag{5}$$

In an arbitrary period of time the number of hits (H) is assumed to be a constant. It follows from (5) that the m.l.e. of π is unbiased; i.e.

$$E[\overline{\pi}] = E\left[\frac{S}{H}\right] = \frac{E[S]}{H} = \pi(P) \tag{6}$$

This completes the proof of PROPOSITION 1. ▮

5. TRANSACTION VOLUME IN A HETEROGENEOUS MARKET

We can modify some of the assumptions stated in Section (1) to reflect different transaction fees observed among brokerages in the market. A common pricing practice among brokerages consists of establishing a graduated commission fee structure that varies directly with the size of the transaction. In other words, brokerages engage in fee discrimination among investors based on purchase orders of different sizes. At the small end are the fees applying to the so-called SOES purchases; i.e. an acronym meaning "Small Order Execution System." At the other extreme are the fees for the large block trading institutions.

Specifically, suppose that the number of investors in the market is N. Suppose that this population is partitioned into m subsets. The subsets are defined in such a way that each subset contains only investors who execute transactions of approximately the same size. Let N_i represent the number of investors in subset i $(i = 1, 2, \cdots, m)$ and let Q_i represent the transactional volume defining subset i $(i = 1, 2, \cdots, m.)$ According to the definitions of the symbols, $N = \sum_{i=1}^{m} N_i$.

Fee discrimination practiced by brokers is manifested as a monotone array of asking prices applying to investors in the different subsets. For the sake of simplicity I assume that all the brokerages partition the set of investors into the same subsets. However, the brokerages do not necessarily post identical asking prices distributed among the subsets of investors.

Suppose that brokerage 1 establishes the array of asking prices as the set $\{P_1(1), P_1(2), \cdots, P_1(m)\}$. Each element of the array represents the asking price that the brokerage applies to the bidders in each subset of the partitioned set of investors.

The distributions F_i $(i = 1, 2, \cdots, m)$ and G_i $(i = 1, 2, \cdots, m)$ are likewise defined over the partitioned set of investors. F_i represents the multivariate probability density function of the asking prices of the brokerages for all investors in subset i. Likewise, G_i represents the cdf of the bid price limit orders for investors in subset i.

The definitions allow us to formulate the mathematical expression for calculating the probability that brokerage 1 will effect a transaction with an investor in subset i. This is displayed as equation (7).

$$\begin{aligned} & Prob[\text{brokerage 1 will execute a limit order of a random investor in subset } i] \\ = & \left[\int_{P_2(i)}^{\infty} \cdots \int_{P_n(i)}^{\infty} F_i[P_1(i), x_2, x_3, \cdots, x_n] dx_2 \cdots dx_n\right] \int_{P_1(i)}^{\infty} dG_i(x) dx \\ \equiv & \ \pi_i(1) \end{aligned} \tag{7}$$

The mathematical implication of (7) is that the set of asking prices posted by brokerage 1 generates a one-to-one mapping onto the set of transaction probabilities for that brokerage; i.e.

$$\{P_1(1), P_1(2), \cdots, P_1(m)\} \mapsto \{\pi_1(1), \pi_1(2), \cdots, \pi_1(m)\}$$

This mapping allows us to write an expression that calculates the probability of the total transaction volume effected by an arbitrary brokerage pursuant to its schedule of posted asking prices. The transactional probabilities are conditional insofar as they are conditioned on the schedule of posted asking prices.

The subscript designating the specific brokerage 1 will be discontinued. Hereafter the conditional probability distribution governing behavior of the investors in each subset will be symbolized by $\{\pi(i) \mid i = 1, 2, \cdots, m\}$.

Let S_i represent the number of investors in subset i who execute purchases at the posted asking price $P(i)$ applicable to their subset. Then the total volume of executions effected by the brokerage is simply $S =$

$\sum_{i=1}^{m} S_i Q_i$. Thus, if we can determine the probability distribution governing the transaction volume in the investor subsets, it is a trivial matter to calculate the sales volume.

The probability distribution governing S_i for each subset of investors is a binomial distribution:

$$b(S_i, N_i; \pi(i)) = \binom{N_i}{S_i} [\pi(i)]^{S_i} [1 - \pi(i)]^{N_i - S_i}$$

I assume that the investors in each subset behave independently of their counterparts in all the other subsets. Then the joint probability distribution for a given number of investors in each subset as well as a given number of transactions in each subset can be expressed as the product of the independent distributions:

$$\prod_{i=1}^{m} b(S_i, N_i; \pi(i)) \tag{8}$$

The brokerage recognizes that the expression in (9) represents a probability predicated on fixed values of parameters that are themselves random variables. In other words, the subsets that constitute the partition of the investor population are governed by a multinomial distribution. Suppose the symbol θ_i $(i = 1, 2, \cdots, m)$ represents the probability that an investor is in subset i. It is assumed that numerical values of the elements in the discrete distribution $\{\theta_i\}$ are fixed.

We can express the joint probability distribution governing the total sales volume of limit order executions by the brokerage as :

$$Prob[S_1, S_2, \cdots, S_m] = \left[\frac{N!}{N_1! N_2! \cdots N_m!} \prod_{i=1}^{m} \theta_i^{N_i} \right] \prod_{i=1}^{m} b(S_i, N_i; \pi_i) \tag{9}$$

Given empirical data for the elements of the partitions $\{N_i\}, \{S_i\}$ and $\{\theta_i\}$, it is not difficult to show that the maximum likelihood estimates of the set of conditional probabilities $\{\pi(i) \mid i = 1, 2, \cdots, m\}$ are the counterparts of the estimates in the homogeneous market; i.e. $\overline{\pi}(i) = \frac{S_i}{N_i}$. It follows that the conditional expected volume of transactions for a brokerage website can be calculated as:

$$E[\text{TOTAL TRANSACTIONS AT A WEBSITE}] = \sum_{i=1}^{m} \overline{\pi}(i) N_i Q_i \tag{10}$$

The elements of the set of conditional probabilities $\{\pi(i) \mid i = 1, 2, \cdots, m\}$ are directly estimable from statistical data recording sales.

6. LIMIT ORDER EXECUTION VOLUME IN A MARKET CONSISTING OF DIFFERENTIATED BROKERAGES

Suppose now we relax the assumption that the brokerages are functionally indistinguishable from the point of view of investors. I assume that some (or all) of the brokerages allocate a portion of their operating expenses to underwrite their marketing activities. These marketing activities are intended to persuade investors that the on-line brokerage is selling a "product" that is demonstrably superior to the counterpart products of its competitors.

Inasmuch as the core business activities of the on-line brokerages consists of effecting transactions in securities, it makes no sense to try to persuade investors that the securities themselves differ. The product that is differentiated by the marketing expenses consists of the array of amenities and services offered to investors who establish trading accounts at the website. The individual brokerage's decision variables are two: (a) the bid/ask price posted for stock transactions, incorporating the commission and fees and (b) the marketing cost that the brokerage will incur in a fixed time period.

In the derivation of PROPOSITION 1, I assumed, inter alia, that the traffic to a website (i.e. the number of hits experienced by a website) was fixed for a given period of time. However, the marketing costs incurred by a brokerage are intended to differentiate its website in appealing ways and thereby increase the traffic to it. To the extent that marketing expenditures are successful in this respect , the number of hits experienced by a brokerage in a fixed period of time can be characterized as if it were a random variable. An immediate consequence of this characterization is that H is no longer a constant per unit time; it is determined (at least in part) by the marketing expenditures associated with the brokerage.

There are some recent anecdotal reports that suggest the lengths to which the competing brokerages will go to attract investors to their web sites. Consider the actions of Ameritrade Holdings which is the sixth largest brokerage company. A few years ago the chairman of Ameritrade said that the price war among online brokerage companies might eventually drive trading commissions to less than zero. Like its rival, the E*Trade Group, Ameritrade has allocated huge amounts of money into a national advertising campaign for year 2000, wiping out virtually all of its operating profits.[12] At this writing American Express is the only big financial company that is giving away stock trades, i.e. zero commission.[13] Brokerage companies

[12] Ameritrade, which earned $3 million in the quarter ending March 2000, planned to spend $200 million on advertising for the year 2000. The CEO of Ameritrade is reported to have commented: " If you turn off your advertising, you'll die. Those big budgets will always be there." The New York Times, April 22, 2000, page C2.

[13] The offer by American Express is viewed as a lure to attract wealthy investors to use the company's other services; i.e. a loss-leader. Ibid.

are thought to be taking these extreme measures to hold the most active stock traders because they are so much more profitable than occasional traders.[14]

Suppose we assume that each of the n brokerages incurs a selling cost per unit of time. The array of unit costs is symbolized by $C = \{c_1, c_2, \cdots, c_n\}$ If we select an arbitrary brokerage with selling cost c, we can define a function $\lambda(c)$ that returns the expected number of hits to a brokerage site per unit time as a function of the selling expenses of the brokerage in that unit of time.

This model of the broker/dealer website is an application of the law of rare events. We have a sub-market environment (i.e. the broker/dealer's website) where there are many binomial trials with small probability of success and where the expected number of successes is constant per unit of time.

If the decisions of individual investors are independent and the population of investors is large relative to the number of brokerages, the distribution governing the number of hits at an arbitrary brokerage with selling expenses c can be described by a Poisson process with $\lambda(c)$ as the characteristic parameter.[15] I assume that the function λ satisfies the following conditions:

(i) $0 < \left(\dfrac{d\lambda}{dc}\right)_{c \geq 0} < \infty$

(ii) $0 < \lim_{c \to 0} \lambda(c) = \beta < \infty$

(iii) $\beta < \lim_{c \to \infty} \lambda(c) = \gamma < \infty$

(iv) $0 < \lim_{c \to \infty} \dfrac{d\lambda}{dc} = 0$

These four conditions are merely mathematical representations of plausible economic implications of the characteristic parameter. Condition (i) states that a marginal increase in the selling expenses of a brokerage will cause an increase the expected number of hits to its website.

Condition (ii) is more problematical; it states that as the selling expenses of a brokerage become infinitely small, the expected investor activity at its website will approach a non-zero minimum. The rationale for this assumption is found in the actual behavior of firms in the market. For example, the website called freetrade.com will not advertise or promote itself at all; the only announcement of its existence was made on Ameritrade's website.

[14]An analyst with Bear Stearns has estimated that "semiprofessional" stock traders constitute less than 1 percent of the customers at online brokerages but account for about 75 percent of all trades. Ibid.

[15]For a general discussion of the characteristics of Poisson-driven processes in securities markets see Merton (1990, p. 312-315)

Condition (iii) states that as the selling expenses of a firm increase without limit, the expected number of visits to its website will reach maximal capacity.

Condition (iv) is merely a statement of the law of diminishing marginal returns applied to the firm's selling expenses: viz marginal increases in the unit selling expenses of a brokerage have no effect on the expected number of hits at its website as its selling expenses become infinitely large.

A complete specification of the theory of limit order executions in a market where firm competition is manifested through selling expenses requires a modification of equation (1) to accommodate for the effect of those selling expenses. In Section 2 I established that the probability that a firm will execute a sale at a random investor's limit order price P is equal to the joint probability of two independent events: (a) the probability that the firm's asking price is less than the asking prices of all the other firms and (b) the probability that the firm's asking price is less than the limit order bid price posted by the investor. The joint probability was symbolized by $\pi(P)$. This transaction probability must satisfy the following five conditions:

(v) $-\infty < \left(\dfrac{d\pi}{dP}\right)_{P\geq 0} < 0$

(vi) $0 < \lim\limits_{P\to 0} \pi(P) = \varepsilon \leq 1$

(vii) $\lim\limits_{P\to\infty} \pi(P) = 0$

(viii) $\lim\limits_{P\to\infty} P\pi(P) = 0$

(ix) $\lim\limits_{P\to\infty} \dfrac{d\pi}{dP} = 0$

Condition (v) states that the probability of execution of a limit order bid is a decreasing function of the ask price. This is simply a mathematical property of equation (1).

Condition (vi) states that as the asking price becomes arbitrarily small, the probability of executing a limit order bid transaction approaches a maximum value not necessarily equal to one. This assumption reflects the findings of the SEC report adumbrated in Section I.

Condition (vii) states that as the asking price becomes arbitrarily large, the limiting probability that a firm will execute a limit order bid at the ask price is zero.

Condition (viii) states that as the asking price becomes arbitrarily large, the limit of the expected execution price is zero.

Condition (ix) states that as the asking price becomes arbitrarily large, the limiting value of a marginal change in the transaction probability of a limit bid order tends to zero. If all these conditions are satisfied we can derive the following proposition:[16]

PROPOSITION 2. *If a brokerage incurs selling expense of c dollars per unit time, and if the number of investor hits to its website in a unit of time is governed by a Poisson distribution with characteristic parameter* $\lambda(c)$*, the conditional probability distribution governing the number of limit order executions by that brokerage is given by :*

$$Prob[S \text{ limit orders filled at asking price } P|\text{selling cost } c] = \frac{[\Lambda(c,P)]^S}{S!}e^{-\Lambda(c,P)}$$

where $\Lambda(c,P) = \lambda(c)\pi(P)$.

PROPOSITION 2 establishes that the statistical incidence of the execution of limit orders in the ECN where brokerages are differentiated is governed by a Poisson distribution with mean $\Lambda(c,P)$. Assumptions (i) through (ix) allows us to derive the properties of the distribution. These are summarized in the following three-part proposition:

PROPOSITION 3.

(a) The expected number of limit order executions by a brokerage decreases as the asking price increases; the expected number of executions increases as the selling expenses increase, ceteris paribus. This proposition conforms to the behavior predicted by classical demand theory. It is derived from the properties of

$$\Lambda(c,P) : -\infty < \left(\frac{\partial \Lambda}{\partial P}\right)_{P\geq 0} < 0 \quad and \quad -0 < \left(\frac{\partial \Lambda}{\partial c}\right)_{c\geq 0} < \infty$$

(b) The expected number of limit order executions by a brokerage will become very large (although not infinite) as the asking price becomes indefinitely small, whereas the average number of limit order executions will approach zero as the asking price increases. These inferences are manifestations of the properties of $\Lambda(c,P)$ *symbolized as:*

$$0 < \lim_{P\to 0} \Lambda(c,P) = \varepsilon\lambda(c) < \infty \quad and \quad \lim_{P\to\infty} \Lambda(c,P) = 0$$

(c) As the selling expense approaches zero, the expected number of limit order executions will approach a constant. Likewise, as the selling expenses become arbitrarily large the expected number of limit order executions will

[16] The proof of PROPOSITION 2 is given in Appendix.

approach a larger constant. These inferences are consequences of the properties of $\Lambda(c, P)$ expressed as:

$$0 \lim_{c \to 0} \Lambda(c, P) = \beta\pi(P) < \infty$$

and

$$0 < \lim_{c \to \infty} \Lambda(c, P) = \gamma\pi(P) < \infty.$$

7. CONCLUDING REMARKS

The propositions derived in this paper are based, in part, on assumptions selected for their verisimilitude. To the extent that the statistical incidence of the display and execution of limit orders is governed by the stochastic processes identified in this paper, the sample incidence of the failures identified by the SEC Report are amenable to tests of statistical significance.

The practical policy implications of the theoretical results established in this paper are as yet inchoate. A prudent approach to policy making suggests that the SEC should proceed cautiously in imposing disciplinary sanctions pending the results of the study to be undertaken by the SEC's Office of Economic Analysis. That study should be designed in such a way as to enable the SEC to identify the parameters that are systematically related to execution failure. The propositions developed in this paper can be applied as templates for the estimation of the statistical parameters.

APPENDIX

Proof of PROPOSITION 2

The proof proceeds by recognizing that the distribution governing the execution of the limit orders can be written as the product of two conditional distributions.

$$\begin{aligned} & Prob[S \text{ sales at limit price } P | \text{selling cost } c] \qquad \text{(A.1)} \\ = & \sum_{H=0}^{\infty} Prob[\text{number of hits} = H | \text{selling cost } c] Prob[S \text{ sales at limit price } P | H \text{ hits}] \end{aligned}$$

The proposition assumes that the first term in the summand can be represented as a Poisson distribution. We can express the conditional probability distribution governing the number of hits in a unit of time as:

$$Prob[\text{number of hits} = H | \text{selling cost } c] = \frac{\lambda(c)^H}{H!} e^{-\lambda(c)} \qquad \text{(A.2)}$$

The second term in the summand can be expressed as equation (2). The product of these two distributions can be written as

$$Prob[S \text{ sales at limit price } P|\text{selling cost } c] \tag{A.3}$$
$$= \sum_{H=0}^{\infty} \frac{\lambda(c)^H}{H!} e^{-\lambda(c)} \begin{pmatrix} H \\ S \end{pmatrix} [\pi(P)]^S [1-\pi(P)]^{H-S}$$

The summation in (A.3) can be simplified to read

$$Prob[S \text{ sales at limit price } P|\text{selling cost } c] = \frac{e^{-\lambda}\pi^S}{S!} \sum_{H=0}^{\infty} \frac{\lambda^H (1-\pi)^{H-S}}{(H-S)!} \tag{A.4}$$

The right-hand side of equation (A.4) can be rewritten as

$$Prob[S \text{ sales at limit price } P|\text{selling cost } c] = \frac{e^{-\lambda}(\lambda\pi)^S}{S!} \sum_{H=0}^{\infty} \frac{[\lambda(1-\pi)]^{H-S}}{(H-S)!} \tag{A.5}$$

The summation in equation (A.5) has no meaning for values of $H < S$. Letting the index H start at S we have the result in (A.6):

$$\sum_{H=0}^{\infty} \frac{\lambda(1-\pi)]^{H-S}}{(H-S)!} = \sum_{H=S}^{\infty} \frac{[\lambda(1-\pi)]^{H-S}}{(H-S)!} = e^{\lambda(1\pi)} \tag{A.6}$$

Cancellation of the terms in the exponential functions in (A.5) and (A.6) immediately yields the probability function in PROPOSITION 2. ▌

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