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Nonparametric Inference Based on Sampled Minima

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A sampling plan that may find applications in economics, biomedical research, reliability and life testing consists of putting kn units on test in groups of k each and observing the minimum value of each group. Thus, the observations are sampled minima and they are to be used to do inference about the original distribution. In this investigation, we present properties of the empirical distribution and sample moments and show how these estimates can be used to do goodness of fit testing about the underlying distribution.

Key Words: Sampled minima; Empirical distribution; Sample moments; Mean square error; Goodness of fit tests; Asymptotic normality.
JEL Classification Numbers: C12, C14.

1. INTRODUCTION

In some economic, medical and reliability studies, a sampling plan that may be used is described as follows: Put on test a set of kn units in groups of k each. Observe only the minimum value of each group. Thus the sampled data are minima of random variables. Let $X_{11}, \dots, X_{1k}, \dots, X_{21}, \dots, X_{2k}, \dots, X_{n1}, \dots, X_{nk}$ be a set of nk independent identically distributed (iid) random variables (rv) from a distribution F. We only observe, however, $Z_i, i = 1, \dots, n$, where $Z_i = Z_{ik} = \min_{1 \le j \le k} \{X_{ij}\}$. Thus the distribution of Z_i is $G(x) = 1 - (1 - F(x))^k$. Set $\overline{F} = 1 - F$ and $\overline{G} = 1 - G$. Based on $Z_1, \dots, Z_n$ we would like to do (nonparametric) inference about F.

Note that when the data are lives, if we define the "hazard rate" of X by $h_F(x)$ and that of Z by $h_G(x)$, then these are related as follows:

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$$h_G(x) = kh_F(x), \quad x > 0 \tag{1}$$

where $h_F(x) = f(x)/\overline{F}(x)$, $\overline{F}(x) > 0$ and h_G is defined similarly. Thus hazard analysis of F can be done easily from the Z_i's.

In this investigation we discuss some points pertaining to the inference (estimation and hypothesis testing) about F using the data on $G, Z_1, \ldots, Z_n$. If one would like to estimate $F(x)$ one notices

$$F(x) = 1 - \left(\overline{G}(x)\right)^{1/k} \tag{2}$$

Thus the empirical estimation of F is given by

$$F_n(x) = 1 - \left(\overline{G}_n(x)\right)^{1/k}, \tag{3}$$

where $\overline{G}_n(x) = 1 - G_n(x)$ and $G_n(x) = \dfrac{1}{n}\sum_{i=1}^{n} I(Z_i \leq x)$. Basic properties of F_n such as consistency and asymptotic normality can easily be obtained from similar properties of G_n. We add to this by obtaining a representation of the mean square error of F_n. Next, the r-th moment of X, $\mu_\gamma = EX^\gamma = \int_{-\infty}^{\infty} x^\gamma dF(x)$ has the representation

$$\mu_\gamma = \frac{1}{k}\int_{-\infty}^{\infty} x^\gamma \left(\overline{G}(x)\right)^{-(k-1)/k} dG(x). \tag{4}$$

We can estimate μ_γ by:

$$\mu_{n,\gamma} = \frac{1}{k}\int_{-\infty}^{\infty} x^\gamma \left(\overline{G}_n(x)\right)^{-(k-1)/k} dG_n(x) = (kn)^{-1}\sum_{i=1}^{n} Z_i^\gamma \overline{G}_n^{-(k-1)/k}(Z_i). \tag{5}$$

The bias and mean square error of $\mu_{n,\gamma}$ will be discussed. Note also that the asymptotic behavior of $\mu_{n,\gamma}$ can be studied from the L-Statistics form:

$$\mu_{\gamma,n} = \left(\frac{n-1}{n}\right)^{(k-1)/k} (kn)^{-1}\sum_{i=1}^{n} Z_{(i)}[(n-i+1/n)]^{-(k-1)/k}, \tag{5*}$$

where $Z_{(i)}, \ldots, Z_{(n)}$ are the order statistics corresponding to $Z_1, \ldots, Z_n$.

Another important problem pertaining to F is to test the null hypothesis $H_0 : F = F_0$, F_0 is completely known or known up to a set of parameters against the alternative $H_1 : F \neq F_0$. In direct sampling (complete) case, the Cramer-vonMises statistic based on the L_2-norm between F and F_0 is highly popular and useful. In our situation, however, we shall propose the following norm:

$$\Delta(k) = \int_{-\infty}^{\infty} \left[\overline{F}^k(x) - \overline{F}_0^k(x)\right]^2 \psi(F_0(x))dF_0(x), \tag{6}$$

where ψ is a known function defined on $[0,1]$. If we were to plug in $\overline{F}_n(x)$ as our estimate of $\overline{F}$ in (6), then clearly $\sqrt{n}\Delta_{n,k} = o_p(1)$ and the asymptotic null distribution of $n\Delta_{n,k}$ is not easily obtainable as in the case for direct sampling (cf. Schorak and Wellner (1986) and D'Agostino and Stephens (1986) for details). Hence we utilize ideas developed in Ahmad (1993) to provide an alternative estimate $\hat{\Delta}_{n,k}(\gamma)$ (depending on a known constant $0 < \gamma \leq 1$) which is asymptotically normal both under H_0 (where it is also distribution-free) and under H_1 (thus we can also evaluate the power of the test with little efforts). More recently, however, interest in the L_p-norma for general $p \geq 2$, became subject of investigation and extending the Cramer-von Mises statistics to their case was studied (cf. Csorgo and Horvath (1993) and references therein). Thus the L_p-version of (6) is given by,

$$\Delta(k,p) = \int_{-\infty}^{\infty} \left[\overline{F}^k(x) - \overline{F}_0^k(x)\right]^p \psi(F_0(x))dF_0(x), \ p \geq 2. \tag{7}$$

We shall indicate how our test procedure can be extended to this case where p is taken to be any integer larger than 2.

In the remainder of this investigation, all integrals not carrying limits are to be taken over the entire real line. All proofs of results discussed here are given in the appendix when warranted.

2. ESTIMATION OF *F*

Let $Z_1, \ldots, Z_n$ be a random sample from a *df* $G(.)$ and we want to estimate the *df* $F(x)$ related to G by the relation $\overline{F} = \left(\overline{G}(x)\right)^{1/k}$ for all $x \in (-\infty, \infty)$, where $\overline{F} = 1 - F$ ($\overline{G} = 1 - G$). Let $G_n = \frac{1}{n}\sum_{i=1}^{n} I(Z_i \leq x)$ denote the empirical *df* of G. We estimate $F(x)$ by:

$$F_n(x) = 1 - \left(\overline{G}_n(x)\right)^{1/k}. \tag{8}$$

Using Part (1) of Rao (1973) p.385 we easily see that $F_n(x)$ is asymptotically normal with mean $F(x)$ and variance $\frac{1}{nk^2}\frac{1-\overline{F}^k(x)}{\overline{F}^{k-2}(x)}$ (provided that

$\overline{F}(x) > 0)$. Consistencies (Weak or Strong, pointwise or uniform) are easily obtainable from the inequality:

$$\begin{aligned}
|F_n(x) - F(x)| &= \left|\overline{F}_n(x) - \overline{F}\right| = \left|\overline{F}_n^k(x) - \overline{F}^k(x)\right| \left\{ \sum_{i=1}^{k} \overline{F}_n^i(x) \overline{F}^{k-i}(x) \right\}^{-1} \\
&= \left|\overline{G}_n(x) - \overline{G}(x)\right| \left\{ \sum_{i=1}^{n} \overline{G}_n^{1/k}(x) \overline{G}^{(k-1)/k}(x) \right\}^{-1} \\
&\leq k^{-1} \left|\overline{G}_n(x) - \overline{G}(x)\right| / \left(\overline{G}(x) + \varepsilon\right), \text{ for sufficiently large } n \\
&\leq (\varepsilon k)^{-1} \left|G_n(x) - G(x)\right|. \qquad (9)
\end{aligned}$$

The next theorem gives the bias and the mean square error of $F_n(x)$. The proof is given in the Appendix.

Theorem 1. *Up to the second order in n, we have*

$$Bias(F_n(x)) \simeq -\frac{(k-1)}{2nk^2} \frac{1-\overline{F}^k(x)}{\overline{F}^{k-1}(x)} - \frac{(k-1)(k-2)(k-3)}{8n^2k^4} \frac{\left(1-\overline{F}^k(x)\right)^2}{\overline{F}^{2k-1}(x)}, \qquad (10)$$

and

$$mse(F_n(x)) \simeq \frac{1}{nk^2} \frac{1-\overline{F}^k(x)}{\overline{F}^{k-2}(x)} - \frac{(k-1)(k-2)(2k+1)}{4n^2k^4} \frac{\left(1-\overline{F}^k(x)\right)^2}{\overline{F}^{2k-2}(x)}. \qquad (11)$$

The choice of k can be made to minimize the mean square error (11). In order to do that we propose the following global measure:

$$\delta_k = \int E(F_n(x) - F(x))^2 \overline{F}^{2k-2}(x) dF(x). \qquad (12)$$

Thus it is easy to see that up to the second order,

$$\delta \simeq \frac{1}{n} \left\{ \frac{1}{k(k+1)(2k+1)} - \frac{(k-1)(k-2)}{2nk^2(k+1)} \right\}. \qquad (13)$$

Another estimate of $F(x)$ that has received attention recently is the so-called "Kernel estimate"; cf. Azzalini (1981) or Lejeune and Sarda (1992). Let K be a known distribution function and $\{a_n\}$ be a sequence of positive reals such that (we shall write a for a_n henceforth). The Kernel estimate of $G(x)$ is defined by:

$$\hat{G}_n(x) = \frac{1}{n}\sum_{i=1}^{n} K\left(\frac{Z_i - x}{a}\right). \tag{14}$$

Hence, we estimate $F(x)$ by:

$$\hat{F}_n(x) = 1 - \left(\hat{\overline{G}}_n(x)\right)^{\frac{1}{k}}. \tag{15}$$

To compare $F_n(x)$ with $\hat{F}_n(x)$ one need a result as the following theorem.

THEOREM 2. *If F is twice differentiable, then up to the first order in n,*

$$Bias\left(\hat{F}_n(x)\right) \simeq \frac{a^2}{2k}\sigma_k^2\left(f'(x) - f^2(x)/\overline{F}(x)\right), \tag{16}$$

and

$$\begin{aligned} mse\left(\hat{F}_n(x)\right) \;\simeq\; & \frac{1-\overline{F}^k(x)}{nk^2\overline{F}^{k-2}(x)} - \frac{a}{nk^2}S(k)\frac{f(x)}{\overline{F}^{k-1}(x)} \\ & + \;\sigma_k^4 a^4 \frac{\left(kf'(x)\overline{F}(x) - k(k-1)f^2(x)\right)^2}{k^2\overline{F}^k(x)}, \end{aligned} \tag{17}$$

where $R(k) = \int k^2(u)du$ and $S(k) = \int uk(u)K(u)du$.

In order to find the optimal bandwidth a, one needs a global criterion for minimizing the mse $\left(\overline{F}_n(x)\right)$. So, we suggest using

$$\delta_{n,k}(a) = E\int \text{mse}\left(\hat{F}_n(x)\right)\overline{F}^k(x)dF(x). \tag{18}$$

Hence using (18) and integrating we get that

$$\delta_{n,k}(a) = \frac{1}{3nk(k+3)} - \frac{a}{nk}S(k)U(f) + \frac{a^4}{4}\sigma_k^4 V_k(f), \tag{19}$$

where $V_k(f) = \int \left(f'(x)\overline{F}(x) - (k-1)f^2(x)\right)^2 dF(x)$ and $U(f) = \int \overline{F}^2(x)f(x)dF(x)$, assumed finite. Therefore the value of a that minimizes (19) is equal to:

$$a_{0\ pt} = \left\{\frac{S(k)U(f)}{kn\sigma_k^4 V_k(f)}\right\}^{\frac{1}{3}}. \tag{20}$$

As is clear from (20), the optimal choice of a depends on the unknown functions f or F or both. Thus one is in need of a mechanism to provide data dependent choices of a. This we will do elsewhere. Note here that while the order of $a_{0\ pt}$ is the same as in direct sampling (of size kn), its constants are different, hence data dependent choices of a will differ as well. The related and interesting problem of estimating the density function $f(x)$ will be dealt with in a different investigation.

Next, let us discuss estimating the moments of X, $E(X^r)$. Note that

$$\mu_r = E(X^r) = \int x^r f(x)dx = \frac{1}{k}\int x^r \left[\overline{G}^{(k-1)/k}(x)\right] dG(x). \tag{21}$$

Based on the sample $Z_1, \ldots, Z_n$ from $\overline{G}(x) = \overline{F}^k(x)$, we propose to estimate μ_r by:

$$\mu_{r,n} = \left(\frac{n-1}{n}\right)^{(k-1)/k} \left\{(nk)^{-1}\sum_{i=1}^{n} Z_i^r / \left(\overline{G}_n(Z_i)\right)^{(k-1)/k}\right\}. \tag{22}$$

Note that $\mu_{r,n}$ can also be written as:

$$\mu_{r,n} = \left(\frac{n-1}{n}\right)^{(k-1)/k} \left\{(nk)^{-1}\sum_{i=1}^{n} Z_{(i)}^r / \left((n-i+1)/n\right)^{(k-1)/k}\right\}, \tag{23}$$

where $Z_{(1)}, \ldots, Z_{(n)}$ denote the order statistics corresponding to $Z_1, \ldots, Z_n$. Thus we easily see that $\mu_{r,n}$ is a generalized L-estimate (cf. Jureckova and Sen (1996) for detailed discussion of this class of statistics) and hence its limiting normality can be attested by known results on L-statistics. It is also possible to obtain a representation of the bias and mean square error of $\mu_{r,n}$ directly. Thus we have,

THEOREM 3. *Up to the second order*

$$Bias(\mu_{r,n}) \simeq \frac{(k-1)(k-2)}{2k^2(n-1)} A_{r,k}(F), \tag{24}$$

where $A_{r,k} = \int x^r \overline{F}^{-k}(x)\left(1 - \overline{F}^k(x)\right) dF(x)$ *assumed to exist and is finite.*

$$\begin{aligned} mse(\mu_{r,n}) \ &\simeq \ \frac{\sigma_r^2}{nk^2} + \frac{(k-1)(2k-1)}{2k^4}\frac{1}{n}[C_{r,k}(F) - \mu_r A_{r,k}(F)] \\ &+ \ \frac{(k-1)(3k-1)}{n(n-1)k^4}\frac{1}{n(n-1)} B_{r,k}(F), \end{aligned} \tag{25}$$

where $\sigma_r^2 = V(X^r)$, $B_{r,k} = \int x^{2r}\overline{F}^{-2k}(x)\left(1-\overline{F}^k(x)\right)^2 dF(x)$ *and* $C_{r,k}(F)$ $= k^2 \int\int x^r y^r \left(\overline{F}(x)\overline{F}(y)\right)^{-2k}\left[\overline{F}^k(\min(x,y)) - \overline{F}^k(x)\overline{F}^k(y)\right] dF(x)dF(y)$.

3. GOODNESS OF FIT TESTING ONE-SAMPLE CASE.

Here we want to test $H_0 : F = F_0$ against $H_1 : F \neq F_0$, F_0 is a known *df*. Based on a sample $Z_1, \ldots, Z_n$ drawn for $G(x) = 1 - \overline{F}^k(x)$. Since testing $F = F_0$ is equivalent to testing $\overline{F}^k(x) = \overline{G}(x)$ is equal to $\overline{F}_0^k = \overline{G}_0(x)$, say, one can base testing on the empirical $\overline{G}_n(x)$, the null $\overline{G}_0(x)$ and a functional (distance) between them but that will not reflect the effort of the parameter k. Thus we proposed the functional (6), which can be estimated by:

$$\begin{aligned}\hat{\Delta}_{n,\gamma}(k) &= \int \overline{F}_n^{2k}(x)\Psi(F_0(x))dF_0(x) \\ &- 2\int \overline{F}_{n,\gamma}^k(x)\overline{F}_0^k(x)\Psi(F_0(x))dF_0(x) + C_0(k), \qquad (26)\end{aligned}$$

where $C_0(k) = \int \overline{F}_0^{2k}(x)\Psi(F_0(x))dF_0(x)$ is a known positive constant,

$$\overline{F}_n^{2k}(x) = \frac{2}{n(n-1)}\sum_{i<j} I(\min(Z_i, Z_j) > x), \qquad (27)$$

and

$$\overline{F}_{n,\gamma}^k = n^{-1}\sum_{i=1}^n C_{i,n}(\gamma)I(Z_i > x), \qquad (28)$$

with $\{C_{i,n}(\gamma)\}_{i=1}^n$ a triangular array of constants such that $\frac{1}{n}\sum_{i=1}^n C_{i,n}(\gamma) \to 1$ and $\frac{1}{n}\sum_{i=1}^n C_{i,n}^2(\gamma) \to C^2(\gamma) > 1$ for all $0 < \gamma \leq 1$. A special choice of these constants, cf. Ahmad (1993) is to choose $C_{i,n}(\gamma) = 1 - \gamma$ for i odd and $C_{i,n}(\gamma) = 1 + \gamma$ for i even. Now, we have the following theorem.

THEOREM 4. *As* $n \to \infty$, $\sqrt{n}\left(\hat{\Delta}_{n,\gamma}(k) - \Delta(k)\right)$ *converges in distribution to a normal distribution with 0 mean and variance* $\sigma_{k,\gamma}^2$ *given in (A.24).*

Under $H_0, \Delta(k) = 0$ and the null variance is

$$\sigma_{k,\gamma}^{(0)2} = (C^2(\gamma)-1)\left\{\iint \overline{F}_0^k(\max(x,y))\overline{F}_0^k(y)\Psi(F_0(x))\Psi(F_0(y))dF_0(x)dF_0(y) - \left[\int \overline{F}_0^{2k}(x)\Psi(F_0(x))dF_0(x)\right]^2\right\}. \quad (29)$$

In the important special case $\Psi(u) = 1$ we get

$$\sigma_{k,\gamma}^{(0)2} = 4k(C^2(\gamma)-1)/(2k+1)^2(3k+2). \quad (30)$$

In the above special case of $C_{i,n}(\gamma)$ we have $C^2(\gamma) = 1+\gamma^2$.

Next, in order to show how to extend the above procedure to the L_p-case where $p \geq 2$ is an integer, note that (7) can be written as follows:

$$\Delta_{k,p} = C_0(k,\gamma,\Psi) - p\delta_{1,p}\left(\overline{F}^k, \overline{F}_0^{k(p-1)}, \Psi\right) + \sum_{r=2}^{p}(-1)^p\binom{p}{r}\delta_{r,p}\left(\overline{F}^{kr}, \overline{F}_0^{k(p-r)}, \Psi\right), \quad (31)$$

where $C_0 = C_0(k,p,\Psi) = \int \overline{F}_0^{kp}(x)\Psi(F_0(x))dF_0(x)$ is a known positive constant and

$$\delta_{k,r,p} = \delta_{k,r,p}\left(\overline{F}^{kr}, \overline{F}_0^{k(p-r)}, \Psi\right) = \int \overline{F}^{kr}(x)\overline{F}_0^{k(p-r)}(x)\Psi(F_0(x))dF_0(x),$$

$r = 1,\ldots,p$. Thus we propose to estimate $\Delta(k,p)$ by:

$$\hat{\Delta}_{n,\nu}(k,p) = C_0 - p\hat{\delta}_{1,p}\left(\overline{F}_{n,\gamma}^k, \overline{F}_0^{k(p-1)}, \Psi\right) + \sum_{r=2}^{p}(-1)^r\binom{p}{r}\delta_{r,p}\left(\overline{F}_n^{kr}, \overline{F}_0^{k(p-r)}, \Psi\right), \quad (32)$$

where $\overline{F}_{n,\gamma}^k(x)$ is as given in (28) and for $r = 2,\ldots,p$,

$$\overline{F}_n^{kr}(x) = \frac{1}{\binom{n}{r}}\sum_{1\leq i_1<\cdots<i_r\leq n} I(\min(Z_{i_1},\ldots,Z_{i_r}) > x). \quad (33)$$

We now state the following theorem.

THEOREM 5. *As $n \to \infty$, $\sqrt{n}\left(\hat{\Delta}_{n,\gamma}(k,p) - \Delta(k,p)\right)$ is asymptotically normal with mean 0 and variance given in (A.24). Under H_0, the variance is equal to*

$$\begin{aligned}\sigma^{(0)2}_{k,p,\gamma} &= p^2(C^2(\gamma)-1)\left\{\iint \overline{F}_0^{k}(\max(x,y))\overline{F}_0^{k(p-1)}(x)\overline{F}_0^{k(p-1)}(y)\right.\\ &\quad \Psi(F_0(x))\Psi(F_0(y))dF_0(x)dF_0(y)\\ &- \left.\left[\int \overline{F}_0^{kp}(x)\Psi(F_0(x))dF_0(x)\right]^2\right\} \qquad (34)\end{aligned}$$

In the special case $\Psi(u) = 1$, the null variance reduces to:

$$\sigma^{(0)2}_{k,p,\gamma} = p^2k(C^2(\gamma)-1)/(kp+1)(2kp-k+2) \qquad (35)$$

Note that with $p = 2$, Theorem 5 yields Theorem 4. Thus it suffices to prove Theorem 5. This we do in the Appendix.

APPENDIX

Proof of Theorem 1:
Recall that if W denotes the standard normal variate, then

$$\begin{aligned}\overline{F}_n(x) &= \overline{G}_n^{\frac{1}{k}}(x) = \left\{\overline{G}(x) + W\left(\frac{\overline{G}(x)G(x)}{n}\right)^{\frac{1}{2}}\right\}^{\frac{1}{k}}\\ &= \overline{G}^{\frac{1}{k}}(x)\left\{1 + W\left(\frac{G(x)}{n\overline{G}(x)}\right)^{\frac{1}{2}}\right\}^{\frac{1}{k}}\\ &\simeq \overline{G}^{\frac{1}{k}}(x)\left\{1 + \frac{W}{k}\left(\frac{G(x)}{n\overline{G}(x)}\right)^{\frac{1}{2}} - \frac{(k-1)}{2k^2}W^2\left(\frac{G(x)}{n\overline{G}(x)}\right)\right.\\ &+ \frac{(k-1)(k-2)}{6k^3}W^3\left(\frac{G(x)}{n\overline{G}(x)}\right)^{\frac{3}{2}}\\ &- \left.\frac{(k-1)(k-2)(k-3)}{24k^4}W^4\left(\frac{G(x)}{n\overline{G}(x)}\right)^2 + \cdots\right\}. \qquad (A.1)\end{aligned}$$

Then

$$
\begin{aligned}
&E\overline{F}_n(x)\\
\simeq\ & \overline{G}^{\frac{1}{k}}(x)\left\{1-\frac{k-1}{2k^2}\left(\frac{G(x)}{n\overline{G}(x)}\right)-\frac{(k-1)(k-2)(k-3)}{8k^4}\left(\frac{G(x)}{n\overline{G}(x)}\right)^2+O(n^{-3})\right\}\\
=\ & \overline{F}(x)\left\{1-\frac{k-1}{2k^2}\left(\frac{1-\overline{F}^k(x)}{n\overline{F}^k(x)}\right)\right.\\
-\ & \left.\frac{(k-1)(k-2)(k-3)}{8k^4}\left(\frac{1-\overline{F}^k(x)}{n\overline{F}^k(x)}\right)^2+O(n^{-3})\right\}. \qquad (A.2)
\end{aligned}
$$

Now, (10) follows from (A.2) above.
Next, by similar argument we see that

$$
\begin{aligned}
E\overline{F}_n^2(x) &= \overline{G}^{2/k}(x)E\left\{1+W\left(\frac{G(x)}{n\overline{G}(x)}\right)^{\frac{1}{2}}\right\}^{2/k}\\
&= \overline{F}^2(x)E\left\{1+W\left(\frac{1-\overline{F}^k(x)}{n\overline{F}^k(x)}\right)^{\frac{1}{2}}\right\}^{2/k}\\
&\simeq \overline{F}^2(x)\left\{1-\frac{k-2}{k^2}\left(\frac{1-\overline{F}^k(x)}{n\overline{F}^k(x)}\right)\right.\\
&- \left.\frac{(k-2)(k-1)(3k-2)}{4k^4}\left(\frac{1-\overline{F}^k(x)}{n\overline{F}^k(x)}\right)^2+O(n^{-3})\right\}\\
&= \overline{F}^2(x)-\frac{k-2}{k^2}\left(\frac{1-\overline{F}^k(x)}{n\overline{F}^{k-2}(x)}\right)\\
&- \frac{(k-2)(k-1)(3k-2)}{4k^4}\left(\frac{(1-\overline{F}^k(x))^2}{n^2\overline{F}^{2k-2}(x)}\right)+O(n^{-3}). \qquad (A.3)
\end{aligned}
$$

Hence, (11) follows from (A.2) and (A.3) after simplification. ▮

Proof of Theorem 2:
Using standard theory, cf. Wand and Jones (1995), we see that

$$
\begin{aligned}
\hat{\overline{G}}_n(x) &\simeq \overline{G}(x)-\frac{a^2}{2}g'(x)\sigma_k^2-\frac{a^4}{24k}g^{(3)}(x)\nu_k(4)+o_p(a^4)\\
&+ W\left[\frac{G(x)\overline{G}(x)}{n}-\frac{a}{n}g(x)S(k)\right]+o_p\left(\left(\frac{a}{n}\right)^{\frac{1}{2}}\right). \qquad (A.4)
\end{aligned}
$$

Using the simple expansion $(1-\theta)^{1/k} \approx 1 - \frac{\theta}{k} - \frac{\theta^2(k-1)}{2k^2}$ we get that

$$\begin{aligned} E\left(\hat{\overline{G}}_n(x)\right)^{\frac{1}{k}} \simeq\ & \left(\overline{G}(x)\right)^{\frac{1}{k}} \left\{1 - \frac{a^2}{2k}\frac{g'(x)}{\overline{G}(x)}\sigma_k^2 - \frac{a^4}{24k}\frac{g^{(3)}(x)}{\overline{G}(x)}\nu_k^2(4) \right. \\ & - \frac{k-1}{4k^2}a^4\left(\frac{g'(x)}{\overline{G}(x)}\right)^2\sigma_k^4 \\ & \left. + \frac{k-1}{2k^2}\left[\frac{G(x)}{n\overline{G}(x)} - \frac{a}{n}\frac{g(x)}{\overline{G}^2(x)}S(k)\right]\right\} \end{aligned} \tag{A.5}$$

Now, (16) follows from (A.5) when we put $\overline{G}(x) = \overline{F}^k(x)$ and simplify.

To evaluate the mse $\left(\hat{F}_n(x)\right)$ we use $(1-\theta)^{2/k} \simeq 1 - \frac{2\theta}{k} + \theta^2\frac{k-2}{k^2}$. Thus,

$$\begin{aligned} & E\left(\hat{\overline{G}}_n(x)\right)^{2/k} \\ \simeq\ & \left(\overline{G}(x)\right)^{2/k}\left\{1 - \frac{a^2}{k}\frac{g'(x)}{\overline{G}(x)}\sigma_k^2 - \frac{a^4}{12k}\left(\frac{g^{(3)}(x)}{\overline{G}(x)}\right)\nu_k(4)\right. \\ - & \left.\frac{k-2}{4k^2}a^4\left(\frac{g'(x)}{\overline{G}(x)}\right)^2\sigma_k^4 + \frac{k-2}{k}\left[\frac{G(x)}{n\overline{G}(x)} - \frac{a}{n}\frac{g(x)}{\overline{G}^2(x)}S(k)\right]\right\}. \end{aligned} \tag{A.6}$$

Hence

$$\begin{aligned} & E\left(\hat{\overline{F}}_n(x) - \overline{F}(x)\right)^2 \\ =\ & E\left(\hat{\overline{G}}_n^{1/k}(x) - \overline{G}^{1/k}(x)\right)^2 \\ =\ & E\hat{\overline{G}}_n^{2/k}(x) - 2\overline{G}^{1/k}(x)E\hat{\overline{G}}_n^{1/k}(x) + \overline{G}^{2/k}(x) \\ \simeq\ & \frac{1}{k^2\overline{G}^{2(k-1)/k}}\left\{\left[\frac{G(x)\overline{G}(x)}{n} - \frac{a}{n}g(x)S(k)\right] + a^4\sigma_k^4(g'(x))^2\right\}. \end{aligned} \tag{A.7}$$

Note that $\left(\overline{G}(x)\right)^{1/k} = \overline{F}(x)$ gives that $g(x) = kf(x)\left(\overline{F}^{k-1}(x)\right)$, $g'(x) = kf'(x)\overline{F}^{k-1}(x) - k(k-1)f^2(x)\overline{F}^{k-2}(x)$. Plugging in these and simplifying yields the result. ▌

Proof of Theorem 3:

For $i = 1, 2, \ldots, n$ denote by

$$\overline{G}_{i,(n-1)}(Z_i) = (n-1)^{-1}\sum_{j\neq i} I(Z_j > Z_i). \tag{A.8}$$

Hence we see that

$$\overline{G}_n(Z_i) = \frac{n-1}{n}\overline{G}_{i,(n-1)}(Z_i), \quad i = 1, 2, \ldots, n. \tag{A.9}$$

Thus, by the central limit theorem of empirical *df*'s and conditioning on Z_i, we see that

$$\overline{G}_n(Z_i) = \left(\frac{n-1}{n}\right)\overline{G}(Z_i)\left\{1 + W\left(\frac{G(Z_i)}{(n-1)\overline{G}(Z_i)}\right)^{\frac{1}{2}}\right\}, \tag{A.10}$$

where W is the standard normal variate. Hence conditioning on Z_i we have,

$$\begin{aligned}&\left(\overline{G}_n(Z_i)\right)^{(k-1)/k}\\ \simeq\ &\left(\frac{n-1}{n}\right)^{(k-1)/k}\left(\overline{G}(Z_i)\right)^{(k-1)/k}\left\{1 - \frac{k-1}{k}W\left(\frac{G(Z_i)}{(n-1)\overline{G}(Z_i)}\right)^{\frac{1}{2}}\right.\\ +\ &\left.\frac{(k-1)(2k-1)}{2k^2}W^2\left(\frac{G(Z_i)}{(n-1)\overline{G}(Z_i)}\right)\right\} + O_p(n^{-3/2}),\end{aligned} \tag{A.11}$$

$i = 1, 2, \ldots, n$. Thus conditioning on Z_i we get that

$$\begin{aligned}&E\{\overline{G}_n^{(k-1)/k}(Z_i)|Z_i\}\\ \simeq\ &\left(\frac{n-1}{n}\right)^{(k-1)/k}\left(\overline{G}(Z_i)\right)^{(k-1)/k}\left\{1 + \frac{(k-1)(2k-1)}{2k^2(n-1)}\frac{G(Z_i)}{\overline{G}(Z_i)}\right\}\\ +\ &O(n^{-3/2}).\end{aligned} \tag{A.12}$$

Hence

$$E\mu_{r,n} \simeq \mu_r + \frac{(k-1)(2k-1)}{2k^2(n-1)}\int x^r\left(\overline{G}(x)\right)^{(2k-1)/k}G(x)dG(x).$$

Substituting for $\overline{G}(x) = \overline{F}^k(x)$ yields the result.

Next, let us evaluate

$$E\mu_{r,n}^2 = E\left\{\left(\frac{n-1}{n}\right)^{(k-1)/k}\frac{1}{nk}\sum_{i=1}^{n} Z_i^n/(\overline{G}_n(Z_i))^{(k-1)/k}\right\}^2.$$

Note that we can write $\hat{\mu}_{r,n}^2 = \left(\frac{n}{n-1}\right)^{(k-1)/k}\mu_{r,n}^2$. Thus,

$$\begin{aligned}E\hat{\mu}_{r,n}^2 &= \left(\frac{1}{nk}\right)^2\left\{\sum_{i=1}^{n} EZ_i^{2r}/(\overline{G}_n(Z_i))^{2(k-1)/k}\right.\\ &\quad+\left.\sum_{i\neq j} EZ_i^n Z_j^n/(G_n(Z_i)G_n(Z_j))^{(k-1)/k}\right\}.\\ &= \left(\frac{1}{nk}\right)^2\{I_{1,t,n} + I_{2,r,n}\},\end{aligned} \tag{A.13}$$

Now, again conditioning on Z_i,

$$\begin{aligned}
&\left(\overline{G}_n(Z_i)\right)^{2(k-1)/k} \\
\simeq\ &\left(\frac{n-1}{n}\right)^{2(k-1)/k}(\overline{G}(Z_i))^{2(k-1)/k}\left\{1-\frac{2(k-1)}{k}W\left(\frac{G(Z_0)}{(n-1)\overline{G}Z_i}\right)^{\frac{1}{2}}\right. \\
+\ &\left.\frac{(k-1)(3k-1)}{k^2}W^2\left(\frac{G(Z_i)}{(n-1)\overline{G}(Z_i)}\right)\right\}. \qquad \text{(A.14)}
\end{aligned}$$

Hence,

$$\begin{aligned}
&E\left\{\left(\overline{G}_n(Z_i)\right)^{2(k-1)/k}|Z_i\right\} \\
\simeq\ &\left(\frac{n-1}{n}\right)^{2(k-1)/k}(\overline{G}(Z_i))^{2(k-1)/k}\left\{1+\frac{(k-1)(3k-1)}{k^2}\frac{G(Z_i)}{(n-1)\overline{G}(Z_i)}\right\},
\end{aligned}$$

which leads to the following:

$$\begin{aligned}
I_{1,r,n} \simeq\ &\left(\frac{n-1}{n}\right)^{2(k-1)/k} n\mu_r^{(2)}+\left(\frac{n-1}{n}\right)^{2(k-1)/k} \\
\times\ &\frac{(k-1)(3k-1)}{k^2}\frac{n}{n-1}\int x^{2r}G(x)\overline{G}^{-(3k-2)/k}(x)dG(x) \qquad \text{(A.15)}
\end{aligned}$$

Next, conditioning on Z_i and Z_j

$$\begin{aligned}
&\overline{G}_n(Z_i)\overline{G}_n(Z_j) \\
=\ &\left(\frac{n-1}{n}\right)^2\overline{G}_{i(n-1)}(Z_i)\overline{G}_{j(n-1)}(Z_j) \\
\simeq\ &\left(\frac{n-1}{n}\right)\overline{G}(Z_i)\overline{G}(Z_j)\left\{1+W\left(\frac{\overline{G}(\min(Z_i,Z_j))-\overline{G}(Z_i)\overline{G}(Z_j)}{(n-1)\overline{G}^2(Z_i)\overline{G}^2(Z_j)}\right)^{\frac{1}{2}}\right\}.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\left(\overline{G}_n(Z_i)\overline{G}_n(Z_j)\right)^{(k-1)/k} \\
\simeq\ &\left(\frac{n-1}{n}\right)^{(k-1)/k}\left(\overline{G}(Z_i)\overline{G}(Z_j)\right)^{(k-1)/k} \\
\times\ &\left\{1-\frac{k-1}{k}W\left(\frac{\overline{G}(\min(Z_i,Z_j))-\overline{G}(Z_i)\overline{G}(Z_j)}{(n-1)\overline{G}^2(Z_i)\overline{G}^2(Z_j)}\right)^{\frac{1}{2}}\right. \\
+\ &\left.\frac{(k-1)(2k-1)}{2k^2}W^2\left(\frac{\overline{G}(\min(Z_i,Z_j))-\overline{G}(Z_i)\overline{G}(Z_j)}{(n-1)\overline{G}^2(Z_i)\overline{G}^2(Z_j)}\right)\right\}.
\end{aligned}$$

Hence

$$
\begin{aligned}
&E\left\{(\overline{G}_n(Z_i)\overline{G}_n(Z_j))^{(k-1)/k}|Z_i,Z_j\right\}\\
&= \left(\frac{n-1}{n}\right)^{2(k-1)/k}\left(\overline{G}_1(Z_i)\overline{G}(Z_j)\right)^{-(k-1)/k}\left\{1+\frac{(k-1)(2k-1)}{2k^2}\right.\\
&\times \left.\left(\frac{\overline{G}(\min(Z_i,Z_j))-\overline{G}(Z_i)\overline{G}(Z_j)}{(n-1)\overline{G}^2(Z_i)\overline{G}^2(Z_j)}\right)\right\}, \qquad \text{(A.16)}
\end{aligned}
$$

which leads to the following:

$$
\begin{aligned}
I_{2,r,n} &\simeq \left(\frac{n-1}{n}\right)^{2(k-1)/k} n(n-1\mu_r^2\\
&+ n(n-1)\frac{(k-1)(2k-1)}{2k^2}\left(\frac{n-1}{n}\right)^{-2(k-1)/k}\\
&\times \frac{1}{(n-1)}\iint x^r y^r(\overline{G}(x)\overline{G}(y))^{-(3k-1)/k}\\
&\times \left[\overline{G}(\min(x,y))-\overline{G}(x)\overline{G}(y)\right]dG(x)dG(y). \qquad \text{(A.17)}
\end{aligned}
$$

Using (A.15) and (A.17) into (A.13) we get that:

$$
\begin{aligned}
E\hat{\mu}_{r,n}^2 &\simeq \left(\frac{n-1}{n}\right)^{-2(k-1)/k}\left\{\frac{1}{nk^2}\mu_r^{(2)}+\frac{(k-1)(3k-1)}{k^4 n(n-1)}B_{r,k}(F)\right.\\
&+ \left.\frac{(n-1)}{nk^2}\mu_r^2+\frac{(k-1)(2k-1)}{2k^4}\left(\frac{1}{n}\right)C_{r,k}(F)\right\}. \qquad \text{(A.18)}
\end{aligned}
$$

Using (24) and (A.18) in the definitions of mse($\mu_{r,n}$) yields the desired result. ▌

Proof of Theorem 5.

Note that

$$
\begin{aligned}
\hat{\delta}_{1,p,\gamma} &= \hat{\delta}_{1,p}\left(\overline{F}^k_{n,\gamma},\overline{F}_0^{k(p-1)},\Psi\right)\\
&= \frac{1}{n}\sum_{i=1}^n C_{0,n}(\gamma)\int_{Z_i}^\infty \overline{F}_0^{k(p-1)}(x)\Psi(F_0(x))dF_0(x)\\
&= \frac{1}{n}\sum_{i=1}^n C_{i,n}(\gamma)\xi_{k,p,F_0}(Z_i), \qquad \text{(A.19)}
\end{aligned}
$$

For $r = 2, \ldots, p$,

$$\begin{aligned}\hat{\delta}_{r,p} &= \binom{n}{r}^{-1} \sum_{1\le i_1<\cdots<i_r\le n} \int_{\min(Z_{i_1},\ldots,Z_{i_r})}^{\infty} \left(\overline{F}_0(x)\right)^{k(p-r)} \Psi(F_0(x))dF_0(x) \\ &= \binom{n}{r}^{-1} \sum_{1\le i_1<\cdots<i_r\le n} \varphi_{k,r,p,F_0}(Z_{i_1},\ldots,Z_{i_r}), \qquad \text{(A.20)}\end{aligned}$$

Set

$$\varphi^{(1)}_{k,r,p,F_0}(Z_1) = E\left[\varphi_{k,r,p,F_0}(Z_1,\ldots,Z_r)|Z_1\right] \qquad \text{(A.21)}$$

Thus by standard theory of U-statistics we easily see that

$$\hat{\delta}_{k,r,p} = \frac{r}{n}\sum_{i=1}^{n} \varphi^{(1)}_{k,r,p}(Z_i) + o_p\left(n^{-\frac{1}{2}}\right),\ r = 2,\ldots,p. \qquad \text{(A.22)}$$

Set for $r = 2, \ldots, p$,

$$\hat{T}_{k,r,p} = \hat{\delta}_{k,r,p} - \delta_{k,r,p} \text{ and } \hat{T}_{k,l,p} = \hat{\delta}_{k,l,p,\gamma} - \delta_{k,l,p}.$$

We get that

$$\begin{aligned}&\left(\hat{\Delta}_{n,\gamma}(k,p) - \Delta(k,p)\right) \\ &= -\hat{T}_{k,l,p} + \sum_{r=2}^{p}(-1)^n \binom{p}{r} \hat{T}_{k,r,p} + o_p\left(n^{-\frac{1}{2}}\right). \qquad \text{(A.23)}\end{aligned}$$

Hence $\sqrt{n}\left(\hat{\Delta}_{n,\gamma}(k,p) - \Delta(k,p)\right)$ is asymptotically normal with mean 0 and variance

$$\begin{aligned}\sigma^2(k,p) &= nVar\{\hat{T}_{k,l,p}\} + n\sum_{r=2}^{p}\binom{p}{r} Var\{\hat{T}_{k,r,p}\} \\ &- 2n\sum_{r=2}^{p}(-1)\binom{p}{r}\operatorname{cov}\left(\hat{T}_{k,l,p},\hat{T}_{k,r,p}\right) \\ &- 2\sum_{r\neq s}\sum(-1)^{r+s}\binom{p}{r}\binom{p}{s}\operatorname{cov}\left(\hat{T}_{k,r,p},\hat{T}_{k,s,p}\right). \qquad \text{(A.24)}\end{aligned}$$

But as $n \to \infty$,

$$\begin{aligned}nVar\left(\hat{T}_{k,l,p}\right) &\to p^2C^2(\gamma)\left\{\iint \overline{F}^k(\min(x,y))\overline{F}_0^{k(p-1)}(x)\overline{F}_0^{k(p-1)}(y)\right. \\ &\times \Psi(F_0(x))\Psi(F_0(y))dF_0(x)dF_0(y) \\ &- \left.\left[\int \overline{F}^k(x)\overline{F}_0^{k(p-1)}(x)\Psi(F_0(x))dF_0(x)\right]^2\right\}. \qquad \text{(A.25)}\end{aligned}$$

Also, for $r = 2, \ldots, p$, as $n \to \infty$,

$$\begin{aligned} nVar\left(\hat{T}_{k,r,p}\right) &\to r^2 \left\{ \iint \overline{F}^k(\min(x,y)) \overline{F}^{k(r-1)}(x) \overline{F}^{k(r-1)}(y) \right. \\ &\times \overline{F}_0^{k(p-r)}(x) \overline{F}_0^{k(p-r)}(y) \Psi(F_0(x)) \Psi(F_0(y)) dF_0(x) dF_0(y) \\ &- \left. \left[\int \overline{F}^{rk}(x) \overline{F}_0^{k(p-r)}(x) \Psi(F_0(x)) dF_0(x) \right]^2 \right\}, \qquad \text{(A.26)} \end{aligned}$$

while we also have as $n \to \infty$,

$$\begin{aligned} & n\mathrm{Cov}\left(\hat{T}_{k,l,p}, \hat{T}_{k,r,p}\right) \\ \to\ & rp \left\{ \iint \overline{F}^k(\min(x,y)) \overline{F}^{k(r-1)}(x) \overline{F}_0^{k(p-r)}(x) \overline{F}_0^{k(p-1)}(y) \right. \\ \times\ & \Psi(F_0(x)) \Psi(F_0(y)) dF_0(x) dF_0(y) \\ -\ & \left[\int \overline{F}^{kr} \overline{F}_0^{k(k-r)}(x) \Psi(F_0(x)) dF_0(x) \right] \\ \times\ & \left. \left[\int \overline{F}^k(x) \overline{F}_0^{k(p-1)}(x) \Psi(F_0(x)) dF_0(x) \right] \right\}, \qquad \text{(A.27)} \end{aligned}$$

and finally for $r \neq s$, as $n \to \infty$,

$$\begin{aligned} & n\mathrm{Cov}\left(\hat{T}_{k,r,p}, \hat{T}_{k,l,p}\right) \\ \to\ & rs \left\{ \iint \overline{F}^k(\min(x,y)) \overline{F}^{k(r-1)}(x) \overline{F}_0^{k(p-r)} \overline{F}^{k(s-1)}(y) \overline{F}^{k(p-s)}(y) \right. \\ \times\ & \Psi(F_0(x)) \Psi(F_0(y)) dF_0(x) dF_0(y) \\ -\ & \left[\int \overline{F}^{kr}(x) \overline{F}_0^{k(p-r)} \Psi(F_0(x)) dF_0(x) \right] \\ \times\ & \left. \left[\int \overline{F}^{ks}(x) \overline{F}_0^{k(p-s)}(x) \Psi(F_0(x)) dF_0(x) \right] \right\}. \qquad \text{(A.28)} \end{aligned}$$

Collecting terms and substituting into (A.24) one gets the asymptotic variable. Under H_0, all terms in $\{\ \ \}$ are equal and the total number of terms is equal to:

$$\begin{aligned} & p^2(C^2(\gamma) - 1) + \left\{ \sum_{r=1}^{p} (-1)^r \binom{p}{r} \right\}^2 \\ =\ & p^2(C^2(\gamma) - 1) + p^2 \left\{ \sum_{r=1}^{p-1} (-1)^r \binom{p-1}{r-1} \right\}^2 \\ =\ & p^2(C^2(\gamma) - 1). \qquad \text{(A.29)} \end{aligned}$$

When $\Psi(u) = 1$, direct integration gives the result. ▌

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