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Uniqueness and Stability of Equilibria in a Model with Endogenous Markups and Labor Supply

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The presence of public policy in models with multiple steady states is known to be capable of reducing the set of equilibria. This paper shows that in a simple growth model with endogenous markups, introducing an endogenous labor-leisure choice also helps eliminate multiple steady state equilibria. Moreover, it alters the stability condition of the unique steady state as well; namely, the steady state may display damped oscillations and admit periodic orbits.

Key Words: Multiple equilibria; Endogenous markups; Endogenous labor supply.
JEL Classification Numbers: O40, C62.

1. INTRODUCTION

Many dynamic models have been shown to be capable of generating multiple steady state equilibria. The sources of such multiplicity vary from model to model, ranging from certain types of externalities to self-fulfilling expectations. This poses a serious problem concerning the asymptotic nature of which steady state the economy will reach and the validity of comparative statics analysis. To overcome this difficulty, economists have explored some ways which enable them to reduce the set of equilibria. One

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widely recognized way is to use properly designed public policy. For example, Grandmont (1985) in a monetary economy with endogenous business cycles argues that if the government precommit to proportional monetary transfers, it may help select the unique steady state; Woodford (1986) in a model with finance constraints recommends the use of government deficits to eliminate sunspot equilibria; Matsuyama (1991) and Boldrin (1992) in endogenous growth models in which externalities generate multiple growth paths suggest that public policy may guide the economy to bypass the poverty trap and move it to the high growth equilibrium; and Evans and Honkapohja (1993) in a model with external increasing returns and learning advocate the use of some fiscal policy to eliminate a low employment equilibrium.

This paper attempts to explore another avenue and suggests that endogenizing labor supply may help reduce the set of equilibria in some dynamic models. Fairly recently, it is known that allowing for an endogenous labor choice in models with aggregate increasing returns alters the stability condition of a unique steady state (e.g., see Benhabib and Farmer 1994 and Benhabib and Perli 1994). Specifically, these models display an indeterminate steady state in a sense that there exists a continuum of convergent equilibrium paths for any initial condition. However, the possibility of endogenous labor to reduce the set of equilibria has not yet been explored in the literature.

We demonstrate this possibility by introducing an endogenous labor-leisure choice into a simple growth model with endogenous markups developed by Gali (1995). In the original Gali model, there exists a continuum of monopolistically competitive intermediate goods producers. Unlike that in the conventional constant-markups model by Dixit and Stiglitz (1977), the competition here intensifies with the level of economic development and the associated range of products available. The presence of market power drives a wedge between the marginal product of capital and the return of investment. Consequently, this introduces nonmonotonicity into the interest rate schedule, which is shown to be a potential source of multiple steady states; with specific functional forms, Gali shows explicitly that the model may possess three distinct steady states. Moreover, once an equilibrium is embedded in a neighborhood of diminishing marginal return of capital, it is a saddle point.

We show, instead, that once elastic labor supply is allowed for, which also derives a wedge between the marginal product of labor and its return, the kind of multiple steady states disappears and they merge into a unique steady state, as long as the optimal markup is a weakly convex function of the number of firms. The intuition is that the presence of endogenous labor enlarges the usual negative effect of investment on the marginal product of capital more than the positive effect on the optimal markups (see the discussion in Section 3 for details). As a result, the range of the nonmonotonic interest rate schedule is removed away from the feasible domain of capital. Monotonicity then yields uniqueness. In addition, our results suggest that even when the steady state displays monotonic return of capital, the dynamics can be rather complex, including the possibility of indeterminacy (or damped oscillations) and periodic orbits. In a word, introducing endogenous labor in the Gali (1995) model not only reduces the set of equilibria but also alters the stability condition of the equilibrium.

2. A MODEL WITH ENDOGENOUS LABOR

Consider a closed economy with many identical, rational, and perfect foresight agents. For simplicity, population is normalized to unity. A representative consumer seeks to maximize

$$\int_0^\infty \left(\frac{1}{1-\sigma} C_t^{1-\sigma} - \frac{B}{1+\gamma} H_t^{1+\gamma} \right) e^{-\rho t} dt \tag{1}$$

where C_t and H_t represent consumption and labor effort at time t respectively, and the discount rate, ρ, and the preference parameters, B, σ and γ are positive constants. If σ equals one, then the corresponding component of the utility function is logarithmic; furthermore, if γ equals zero, the underlying utility function is a standard one that represents indivisible labor (e.g., see Hansen 1985).

The budget constraint that the representative agent faces is given by[1]

$$C_t + \dot{K}_t = w_t H_t + r_t K_t - \delta K_t, \tag{2}$$

where K_t is capital, w_t is the wage rate, r_t is the interest rate, and δ is the rate of capital depreciation.

Given w_t and r_t, the representative agent's problem is to choose C_t, K_t, and H_t so as to maximize (1) subject to (2), the nonnegativity constraints $C_t \geq 0$ and $K_t \geq 0$, and an initial condition for capital K_0.[2] It can then be easily shown that the first-order conditions consist of (the time index is dropped for ease of exposition):

$$\frac{\dot{C}}{C} = \frac{1}{\sigma}(r - \rho - \delta), \tag{3}$$

$$BC^\sigma H^\gamma = w, \tag{4}$$

[1] Although monopolistic competition is a key feature of the model, we nonetheless consider an equilibrium with free entry and zero profit. Hence, factor payments are the only source of individual's income.

[2] More precisely, for a given value of C, the value of K must be bounded. See footnote 5 for details.

$$\lim_{T\to\infty} e^{-\rho T} C_T^{-\sigma} K_T = 0 \tag{5}$$

Evidently, Eqs. (3) and (4) bear standard marginal benefit vs. marginal cost interpretations, while Eq. (5) is the associated transversality condition.

On the production side, there is a continuum of monopolistically competitive intermediate goods producers, indexed by $j \in [0, M]$, where M can be viewed as the range of intermediate goods. Final output, produced in a competitive sector, is given by

$$Y = \left[M^{-(1-1/\mu(M))} \int_0^M Y(j)^{1/\mu(M)} dj \right]^{\mu(M)}, \tag{6}$$

where Y is output, $Y(j)$ is the quantity of intermediate goods used, and $\mu(M)$ measures the degree of monopoly power in the markets for intermediate products; moreover, $\mu'(M) < 0$, $\lim_{M\to 0} \mu(M) = \bar{\mu} \in (1, \infty)$, and $\lim_{M\to\infty} \mu(M) = 1$.[3] The smaller the number of firms in the industry, the less intense the competition, and then the higher the $\mu(M)$ and the monopoly profits. In this paper, $\mu(M)$ is interpreted as the optimal markup. Notice that (6), adapted from Gali (1995), extends the conventional Dixit-Stiglitz (1977) technology by endogenizing entry or markups. As pointed out by Gali, it is this factor that potentially generates multiplicity of steady state equilibria in such type of models with inelastic labor supply.

Let $P(j)$ be the relative price of the jth intermediate good in terms of the final good, the profits of a final goods producer are

$$\Pi = Y - \int_0^M P(j) Y(j) dj. \tag{7}$$

Profit maximization yields the following demand functions for intermediate goods:

$$Y(j) = P(j)^{-\xi(M)} (Y/M). \tag{8}$$

where $\xi(M) = \mu(M)/[\mu(M) - 1]$ is the elasticity of substitution among intermediate goods.

The technology for producing every intermediate commodity is assumed to exhibit the same functional form:

$$Y(j) = F(K(j) - v, H(j)) = A(K(j) - v)^{\alpha} H(j)^{1-\alpha}, \tag{9}$$

[3] Alternatively, (6) implies that the elasticity of substitution among intermediate goods, which is defined as $\mu(M)/[\mu(M) - 1]$, is increasing in M; i.e., it becomes more elastic when the range of inputs available expands.

where α is the (constant) capital share and ν is the overhead requirement. Solving (8) for $P(j)$ and making use of (9), we have the profit function of the jth intermediate good producer:

$$\begin{aligned}\Pi(j) &= (Y/M)^{1/\xi(M)} A^{1/\mu(M)} [K(j) - v]^{\alpha/\mu(M)} H(j)^{(1-\alpha)/\mu(M)} \\ &- wH(j) - rK(j). \end{aligned} \tag{10}$$

The first-order conditions for the problem (9) are:

$$w = \frac{1-\alpha}{\mu(M)} \frac{P(j)Y(j)}{H(j)}, \tag{11}$$

$$r = \frac{\alpha}{\mu(M)} \frac{P(j)Y(j)}{K(j) - v}. \tag{12}$$

Eqs. (11) and (12) imply that the wage and interest rates equal the marginal revenue products of labor and capital, respectively. When $\mu(M) = 1$ and $\nu = 0$, the intermediate goods are perfect substitutes in the production of the final good, and hence this model of monopolistic competition becomes a standard one of perfect competition. When $\mu(M)$ is independent of M, which is larger than one by assumption, it corresponds to a standard one of monopolistic competition with constant markups.

Given the symmetry of the model, we obtain that in equilibrium $K(j) = K/M, H(j) = H/M, Y(j) = Y/M$, and $P(j) = 1$. Finally, substituting (11), (12) and these symmetry conditions into (9), we can derive the free-entry, zero-profit equilibrium of the model:

$$K = \left(1 + \frac{\alpha}{\mu(M) - 1}\right) vM, \tag{13}$$

which defines the range of intermediate firms, M, as a function of the capital stock; in addition, $0 \leq M \leq K/\nu$. It is straightforward to verify that the number of firms, M, is increasing in the capital stock.

3. STEADY STATE ANALYSIS: UNIQUENESS

An interior perfect foresight equilibrium of our model economy is a path $\{C_t, K_t, H_t\}_{t=0}^{\infty}$ of the dynamical system (2)-(4), satisfying (11)-(12), the initial condition K_0 and the transversality condition (5). Once the equilibrium paths of C_t, K_t and H_t are determined, we can easily obtain the paths of all other variables by substituting in the appropriate equations. To characterize the equilibrium, we start by focusing on the stationary solutions of (2)-(4) in this section, and then we turn to the transitional dynamics in the subsequent section.

At the steady state, (C^*, K^*, H^*), the system is described by

$$C^* = w(K^*, H^*)H^* + r(K^*, H^*)K^* - \delta K^*, \tag{14}$$

$$r(K^*, H^*) = \rho + \delta, \tag{15}$$

$$BC^{*\sigma}H^{*\gamma} = w(K^*, H^*), \tag{16}$$

where $w(K^*, H^*)$ and $r(K^*, H^*)$ are given by (11) and (12) respectively. Next, to reduce the dimensionality, we shall write the model variables, (C^*, K^*, H^*), as functions of M. Since K^* is given by (13), to C^* and H^* we now turn. To obtain an expression for H^*, we first substitute r from (9) and (12) into (15) and then make use of (13):

$$H^* = \frac{\alpha v M}{\mu(M) - 1}\left[\frac{\mu(M)(\rho+\delta)}{\alpha A}\right]^{1/(1-\alpha)}. \tag{17}$$

Similarly, after a little algebra, we find C^* to be

$$C^* = vM\left[\frac{\rho + (1-\alpha)\delta}{\mu(M) - 1} + \rho\right]. \tag{18}$$

Then, substituting (11), (17) and (18) into (16) and rearranging yield a non-linear equation for the steady state value of M:

$$D_1 M^{\sigma+\gamma}(\mu(M) - 1)^{-\gamma}\left[\frac{\rho + (1-\alpha)\delta}{\mu(M) - 1} + \rho\right]^{\sigma} = D_2\mu(M)^{-(1+\gamma)/(1-\alpha)}, \tag{19}$$

where $D_1 = B\alpha^{\gamma}v^{\sigma+\gamma}[(\rho+\delta)/\alpha A]^{\gamma/(1-\alpha)}$ and $D_2 = (1-\alpha)A[\alpha A/(\rho + \delta)]^{\alpha/(1-\alpha)}$ are positive constant parameters. Clearly, Eq. (19) cannot be solved analytically; we have to resort to diagrammatic illustrations or numerical methods. We first establish the existence of a solution and then demonstrate that the solution is unique under economically plausible conditions.

To visualize a solution of (19), let Γ and Θ be the expressions in the left- and right-hand sides, respectively. It is easy to show that function Θ is a continuous, monotonically downward-sloping curve of $\mu(M)$, while function Γ is also a continuous, monotonically downward-sloping curve as M is inversely related with $\mu(M)$ (i.e., $\mu\prime(M) < 0$). Since by assumption that $\lim_{M\to 0}\mu(M) = \bar{\mu} \in (1, \infty)$, and $\lim_{M\to\infty}\mu(M) = 1$, the domain for $\mu(M)$ is $[1, \bar{\mu}]$. When $\mu(M)$ approaches 1, Γ and Θ curves approach $+\infty$

and $D_2 > 0$, respectively; on the other hand, when $\mu(M)$ approaches $\bar{\mu}, \Gamma$ and Θ curves approach 0 and $D_2\bar{\mu}^{-(1+\gamma)/(1-\alpha)} > 0$, respectively. The above analysis establishes the existence of a positive solution of (19). Figure 1 depicts the Γ and Θ curves and a solution.

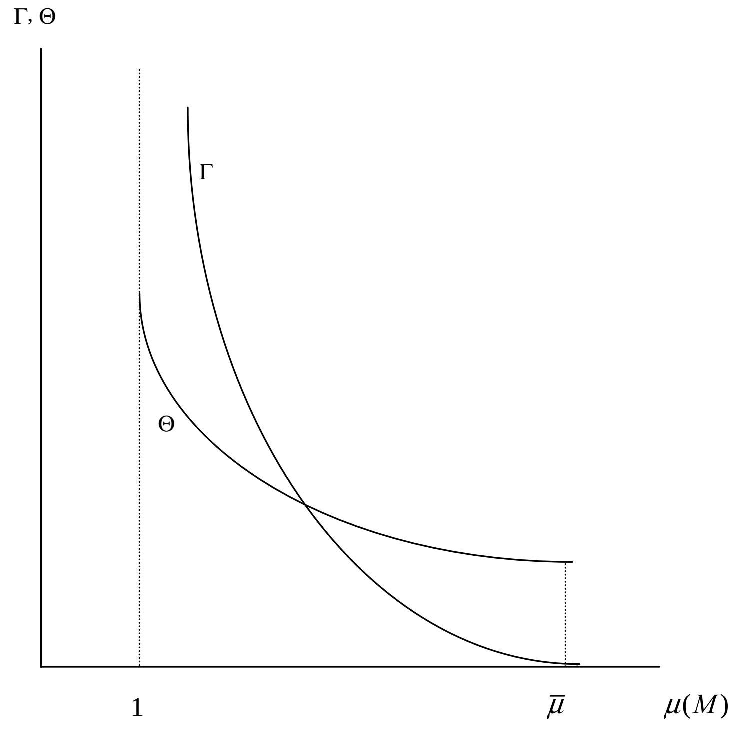

FIG. 1. Determination of the optimal markup $\mu(M)$

A set of sufficient conditions for the solution of (19) to be a unique one is that both Γ and Θ are convex functions of μ. Plainly, Θ is convex in μ. For Γ to be convex, we make two additional assumptions:

(i) $\sigma + \gamma \geq 1$; and (ii) $\mu''(M) \geq 0$,

which can be shown to be economically plausible with the following brief justifications.

The first condition stems from the fact that the coefficient of relative risk aversion (or equivalently, the inverse of the intertemporal elasticity of substitution), σ, is often estimated to be larger than 1. In an influential paper, Mehra and Prescott (1985) find that the bulk of micro evidence on σ points to a value between 1 and 2. Hall (1988) obtains primarily the same result. On the other hand, it is a common practice to incorporate this empirical finding in theoretical models that require calibrations, see, e.g., Mulligan and Sala-i-Martin (1993) and Gomme (1993), among others. The standard logarithmic utility function corresponds to $\sigma = 1$.

The second condition requires the μ function to be weakly convex in M, which accords reasonably well with intuition. It is known that in this model, as more firms enter, the variety of intermediate goods available increases and the competition intensifies, which in turn drives down each incumbent firm's markup. Weak convexity implies that as more and more firms enter, each successive decline in markup falls. This may be termed as diminishing marginal influence. An extreme case would be as follows: when the firm number is very large so as to be close to infinity, the marginal influence will be zero, since each firm now behaves competitively.

To see that Γ is convex, without loss of generality, we take $\delta = 0$ and rewrite it as:

$$\Gamma(\mu(M)) = D_1 \rho^{\sigma} \mu(M)^{-\gamma} \left[\frac{M\mu(M)}{\mu(M) - 1} \right]^{\sigma+\gamma}. \tag{20}$$

Directly taking the second derivative with respect to μ yields (with the superfluous positive constant $D_1\rho^{\sigma}$ dropped for convenience):

$$\begin{aligned}
\frac{d^2\Gamma}{d\mu^2} &= \gamma(\gamma+1)\mu^{-\gamma-2}\left(\frac{M\mu}{\mu-1}\right)^{\sigma+\gamma} \\
&- 2\gamma(\sigma+\gamma)\mu^{-\gamma-1}(\left(\frac{M\mu}{\mu-1}\right))^{\sigma+\gamma-1}\left[\frac{\mu dM/d\mu}{\mu-1} - \frac{M}{(\mu-1)^2}\right] \\
&+ (\sigma+\gamma)(\sigma+\gamma-1)\mu^{-\gamma}\left(\frac{M\mu}{\mu-1}\right)^{\sigma+\gamma-2}\left[\frac{\mu dM/d\mu}{\mu-1} - \frac{M}{(\mu-1)^2}\right]^2 \\
&+ (\sigma+\gamma)\mu^{-\gamma}\left(\frac{M\mu}{\mu-1}\right)^{\sigma+\gamma-1}\left[\frac{\mu d^2M/d\mu^2}{\mu-1} - \frac{2dM/d\mu}{(\mu-1)^2} + \frac{2M}{(\mu-1)^3}\right].
\end{aligned} \tag{21}$$

It is clear that the first two terms are always positive. With condition (i), the third term is non-negative. Notice that $d^2M/d\mu^2 = -[\mu''/(\mu')^2](dM/d\mu)$ $= -\mu''/(\mu')^3 \geq 0$ if condition (ii) is satisfied. Therefore the fourth term on the RHS of (20) is also positive. Combining these, we obtain $d^2\Gamma/d\mu^2 > 0$, or Γ is convex as required. Moreover, in order to have a unique solution, it requires the tangent line evaluated at the equilibrium for the Γ curve to be steeper than that for the Θ curve. It can be shown that this is equivalent to the following condition $\bar{\mu} \leq 1/\alpha$ (also see Wu and Zhang, 2000, 2001). Estimates of the capital elasticity of production, α, ranges from 0.25 (e.g., Lucas, 1988) to 0.42 (e.g., Rotemberg and Woodford, 1994), while those of markups, based on either the gross output measure or the value added measure, lie within the range from 1.05 to 2.3 (e.g., Morrison, 1990; Norrbin, 1993; Roeger, 1995). Based on these conditions, the uniqueness of the solution is thus verified.

To compare our results with Gali's (1995), we employ his functional form of $\mu(M)$ to illustrate the impossibility of multiple steady states in our model. The elasticity of substitution among inputs is linear and given by $\xi(M) = \mu(M)/(\mu(M) - 1) = \phi + \varepsilon M$, where $\phi = \bar{\mu}/(\bar{\mu} - 1) > 1$. One can easily verify that this formulation satisfies condition (ii). Rewriting it as $M = [1 - \phi + 1/(\mu(M) - 1)]/\varepsilon$ and substituting into (19), we obtain the following closed-form equation for $\mu(M)$:

$$D_1 \varepsilon^{-(\sigma+\gamma)} (\mu(M) - 1)^{-\gamma} \left[1 - \phi + \tfrac{1}{\mu(M)-1}\right]^{\sigma+\gamma} \left[\tfrac{\rho+(1-\alpha)\delta}{\mu(M)-1} + \rho\right]^{\sigma} = D_2 \mu(M)^{-(1+\gamma)/(1-\alpha)}. \tag{22}$$

It is easy to show that when $\mu(M) = \bar{\mu}$ the term in the first square bracket becomes zero and so is the Γ curve. Once again, uniqueness is warranted, or the possibility of multiple steady states is ruled out in our model. In addition, given the specific functional form for the elasticity of substitution, we obtain $\mu(M) = (\phi + \varepsilon M)/(\phi + \varepsilon M - 1)$. Substituting this result into the zero-profit equilibrium condition (13) yields the following expression describing the relationship between the number of firms, M, and the capital stock, K:

$$M = (\psi/2)(\sqrt{1 + 4\alpha\varepsilon K/[v(1 + \alpha(\phi - 1))^2]} - 1), \tag{23}$$

where $\psi = [1 + \alpha(\phi - 1)]/(\alpha\varepsilon)$. Notice that (23) coincides with the one obtained in Gali (1995), implying that allowing for elastic labor supply does not alter the generic relationship between the number of firms and the capital stock.

Numerically, we impose the same parameter values; i.e., $\alpha = 0.8, A = 0.397, \nu = 0.15$, $\phi = 1.27, \varepsilon = 0.05$, $\delta = 0.1$, and $\rho = 0.04$. Three preferences parameters remain to be specified. For B, we set it to 2.86, the valued used in the literature of indivisible labor (e.g., see Cooley and Hansen 1989); for σ and γ, they are allowed to vary in the ranges of $[1, 4]$ and $[0, 4]$, respectively. Hence, condition (i) is warranted. Table 1 presents the equilibrium values of the markups. Evidently, not only do multiple steady state equilibria a disappear, but also the equilibrium markups change with σ and γ which are not the case in the Gali model. In addition, we document the respective markups by choosing a conventionally used (see, e.g. Cooley and Hansen 1989) value of the capital share, $\alpha = 0.36$, since we view that $\alpha = 0.8$ used by Gali is exceedingly high.

Finally, we close this section by looking at the implications of our uniqueness result on the behavior of the interest rate, r. It is known that with inelastic labor supply, r can be written as a function of capital, K, only; with elastic labor supply, however, both endogenous model variables, K

TABLE 1.

The Equilibrium Markups

		The Markups μ	
σ	γ	$\alpha = 0.8$	$\alpha = 0.36$
1.00	2.00	4.48	3.19
2.00	2.00	2.95	2.51
3.00	2.00	1.77	2.21
4.00	2.00	1.65	2.05
2.00	0.00	2.71	2.41
2.00	1.00	2.81	2.47
2.00	3.00	3.09	2.53
2.00	4.00	3.21	2.55

Notes: The parameter values are set at: $\alpha = 0.8, A = 0.397$, $\nu = 0.15, \phi = 1.27$, $\varepsilon = 0.05, \delta = 0.1$, $\rho = 0.04$, and $B = 2.86$; the benchmark case is when $\sigma = 2.00$ and $\gamma = 2.00$.

and C, affect r. To obtain an expression for r in our model, we first solve for H from (4) and (11):

$$H = B^{-1/(\alpha+\gamma)} C^{-\sigma/(\alpha+\gamma)} \left[\frac{1-\alpha}{\mu(M)} A(K - vM)^{\alpha} \right]^{1/(\alpha+\gamma)}. \tag{24}$$

Then we substitute (24) into (12) to produce

$$r(K, C) = r_0 \mu(M)^{-(1+\gamma)/(\alpha+\gamma)} (K - vM)^{-\gamma(1-\alpha)/(\alpha+\gamma)} C^{-\sigma(1-\alpha)/(\alpha+\gamma)}, \tag{25}$$

where r_0 is a combination of constant model parameters. From (25), variations in K affect r for a given amount of C through two offsetting effects: the usual negative effect on the marginal product of capital $((K - vM)^{-\gamma(1-\alpha)/(\alpha+\gamma)})$ and the positive effect on the optimal markups $(\mu(M)^{-(1+\gamma)/(\alpha+\gamma)})$. In the special case of a high elasticity of marginal disutility of work ($\gamma \to \infty$), (25) collapses to the one in the Gali model. For any other values of $\gamma \quad \geq 0$, the relationship between $r(K, C)$ and K given C crucially depends on wether or not γ equals zero. Let us start by analyzing the case where $\gamma \neq 0$. As is known in the exogenous labor model, multiplicity stems from the fact that the net outcome of the above two competing effects alternates its directions over the feasible domain of K between zero and an upper bound (Gali 1995). But in our model, albeit the presence of endogenous labor enlarges both effects, it disproportionately affects the negative one via the marginal product of capital more than the positive one via the markups (see Eq. (25)). As a result, the uniqueness solution suggests that the kind of nonmonotonicity of $r(K, C)$ has been pushed away from the domain of $K \in [0, \bar{K}_c]$, where $\bar{K}_c$ is the upper bound for a given C (see Figure 2). Therefore, we conclude that $\partial r(K, C)/\partial K < 0$.

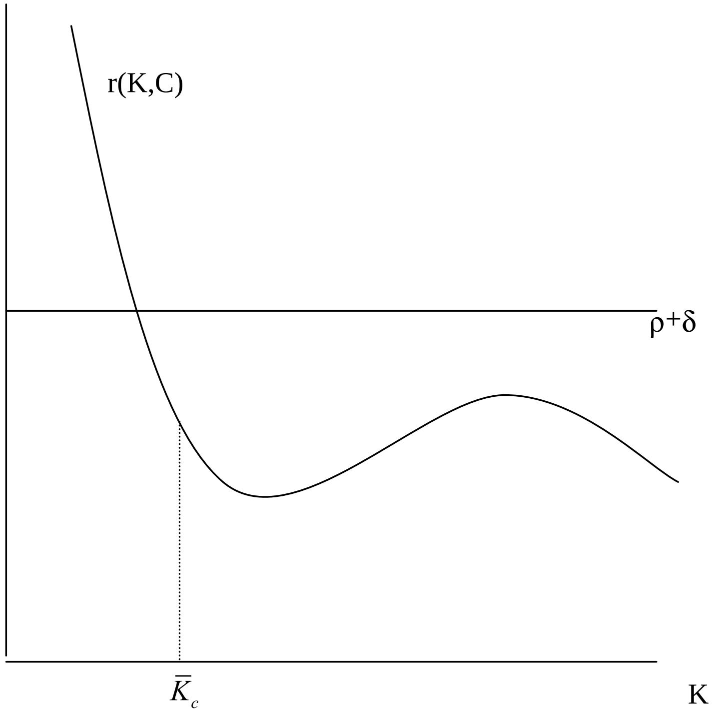

FIG. 2. The interest rate schedule (for a given C) when $\gamma > 0$

Interestingly, when $\gamma = 0$, or the utility function reflects indivisible labor, Eq. (25) implies that for a given C, r increases with K (recall that $\mu'(M) < 0$ and $\partial M/\partial K > 0$). Figure 3 indicates that the r curve slopes upward, resulting from only the positive effect on the markups. Hence, in this case, $\partial r(K, C)/\partial K > 0$.

4. TRANSITIONAL DYNAMICS ANALYSIS: STABILITY

In the Gali (1995) model with exogenous labor, it has been shown that a steady state with diminishing marginal revenue product of capital in its neighborhood (i.e., $r' \equiv dr(K)/dK < 0$) is always a saddle. In our model with endogenous labor, on the other hand, one would conceive that such a result may not hold in that both endogenous model variables, C and K,

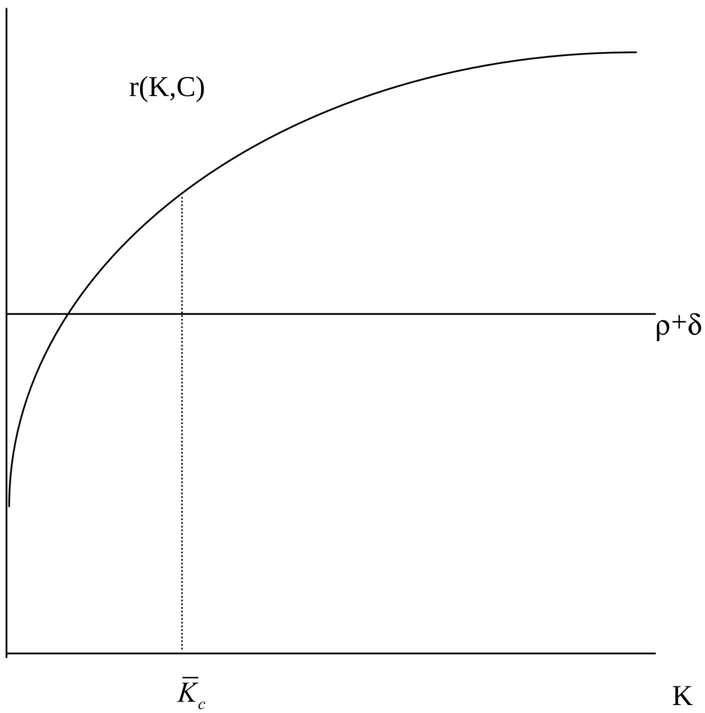

FIG. 3. The interest rate schedule (for a given C) when $\gamma = 0$

affect the dynamic paths in a more complicated way. This section briefly explores the complex dynamics of the model.

Formally, let

$$g(K, C) \equiv wH + rK = w(K, H(K, C))H(K, C) + r(K, H(K, C))K.$$

From Eqs. (11), (12) and (24), it follows that

$$\begin{aligned} g(K, C) &= g_0\left(1 + \frac{\alpha v M}{K - vM}\right)\mu(M)^{-(1+\gamma)/(\alpha+\gamma)} \\ &\times (K - vM)^{\alpha(1+\gamma)/(\alpha+\gamma)} C^{-\sigma(1-\alpha)/(\alpha+\gamma)}, \end{aligned} \tag{26}$$

where g_0 is a combination of constant model parameters. Then the dynamical system consists of two differential equations in K and C:

$$\dot{K} = g(K, C) - C - \delta K, \tag{27}$$

$$\dot{C} = \sigma^{-1} C[r(K, C) - \rho - \delta], \tag{28}$$

where $g(K, C)$ and $r(K, C)$ are given by (25) and (25) respectively.[4] Straightforward differentiation yields $g_C \equiv \partial g(K, C)/\partial C < 0$ and $r_C \equiv \partial r(K, C)/\partial C < 0$. From the preceding section, we also obtain that $r_K \equiv \partial r(K, C)/\partial K < 0$ if $\gamma > 0$ and $K < \bar{K}_c$, and $r_K > 0$ if $\gamma = 0$. As for the sign of $g_K \equiv \partial g(K, C)/\partial K$, unfortunately, it depends in a complicated way on μ and the values taken by all the exogenous model parameters. Thus, g_K can be of either sign and the magnitude is also indeterminate.

We now turn to an analysis of the behavior of the pair of differential equations in (27) and (28). Linearizing them around the unique interior steady state (K^*, C^*) gives:

$$\begin{bmatrix} \dot{K} \\ \dot{C} \end{bmatrix} = \begin{bmatrix} g_K - \delta & -1 + g_C \\ \sigma^{-1} C^* r_K & \sigma^{-1} C^* r_C \end{bmatrix} \begin{bmatrix} K - K^* \\ C - C^* \end{bmatrix}. \tag{29}$$

Using the steady state values of K^*, C^*, and Eqs. (25)-(25), we can compute the Jacobian of (29). The trace and the determinant of this Jacobian are given by the following expressions

$$tr = \sigma^{-1} C^* r_C + (g_K - \delta), \tag{30}$$

$$\det = \sigma^{-1} C^* [r_C (g_K - \delta) + r_K (1 - g_C)], \tag{31}$$

where it is assumed that $\det \neq 0$. It should be noted that the first term in the tr equation and the second term in the det equation are both negative, so the signs of tr and det hinge on the sign and size of g_K. Let us define two critical values of g_K. First, define $g_K^* \equiv \delta - r_K(1 - g_C)/r_C$ (recall that $r_C < 0$) such that $\det \gtrless 0$ if $g_K \gtrless g_K^*$. Second, define $g_K^{**} \equiv \delta - \sigma^{-1} C^* r_C$ such that $tr \gtreqqless 0$ if $g_K \gtreqqless g_K^{**}$ Notice that by construction when $\gamma > 0, r_K < 0$ and hence $g_K^* < g_K^{**}$. We then summarize the stability of the steady state for this case by the following:

Case 1. $g_K > g_K^*$ or $\det < 0$. Since the eigenvalues of the system have opposite signs, the steady state (K^*, C^*) is a saddle point.

Case 2. $g_K < g_K^*$ or $\det > 0$. Since $g_K^* < g_K^{**}$, we must have $g_K < g_K^* < g_K^{**}$, or $\det > 0$ and $tr < 0$, implying that the eigenvalues are both negative or else the Jacobian has a conjugate pair of complex eigenvalues each with a negative real part. Therefore, in the former scenario, the steady state (K^*, C^*) is a stable node, while in the latter, it is a stable spiral.

Interestingly, it can be readily shown that when $\gamma = 0$ (recall that $r_K > 0$), the comparison between g_K^* and g_K^{**} cannot be explicitly made. As a result, two additional possible cases would emerge.

[4]It can be easily shown that the Inada conditions for the production function also carry over to function g. These guarantee the existence of a maximum sustainable capital stock, $\bar{K}_c$, for a given C, defined as the smallest possible solution to the equation $g(K, C) - \delta K = 0$. A similar condition is also derived in Gali (1995).

Case 3. $g_K^{**} < g_K < g_K^*$, or det > 0 and tr > 0. The steady state (K^*, C^*) is either an unstable node if the two roots are positive in real values, or an unstable spiral if the two roots are complex conjugates with positive real parts.

Case 4. $g_K < g_K^*$ or det > 0, and discriminant $\Delta \equiv \mathrm{tr}^2 - 4 \cdot \det < 0$ or the eigenvalues are complex conjugates. Suppose there exists some value g_K, say $\hat{g}_K$, such that the steady state is with purely imaginary eigenvalues, i.e., $\hat{g}_K = g_K^{**}$ or tr $= 0$; moreover, as g_K crosses $\hat{g}_K$ in some direction, the real parts of the complex roots change from negative to positive. Then, for all values of g_K on the side of that specific value $\hat{g}_K$ and close enough to it, there is a periodic orbit surrounding the steady state, which is known as the *Hopf* bifurcation.

Four comments should be made here. First, in conventional models with constant markups, net investment must increase with capital, or $g_K - \delta > 0$, because net investment is inversely related with the interest rate which declines as capital rises. Our results show that in a general utility function with endogenous labor ($\gamma > 0$), the steady state then corresponds to a saddle in this circumstance. Second, as noted earlier that in models with exogenous labor, a steady state with a diminishing marginal return of capital surrounding it corresponds simply to a saddle. However, in our case, it generates much richer dynamics, implying that not only do multiple steady states disappear under elastic labor but also the stability conditions merge into more complicated ones. Third, case 2 suggests that an indeterminate steady state may arise here, consistent with the findings in related models without endogenous markups (e.g., see Benhabib and Farmer 1994). Finally, once the system gives birth to limit cycles as in case 4, small perturbations of g_K can change the stability of the equilibrium entirely, although by the implicit function theorem the equilibrium changes smoothly with g_K and remains locally unique.

5. CONCLUSION

In a simple growth model with endogenous markups developed by Gali (1995), which has become a popular model in recent studies, we have shown that introducing an endogenous labor-leisure choice can reduce the set of equilibria. Moreover, it can alter the stability condition of the unique steady state as well; in particular, the steady state may display damped oscillations and admit periodic orbits. Finally, we caution readers who tend to generalize our findings to other dynamic models with multiplicity of steady state equilibria, and emphasize the need for more further research.

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