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Equilibria for Pure Exchange Infinite Economies in the Sense of Incomplete Preference *

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In this paper, we introduce a new concept of incomplete preference and cover the known ordering relations such preferences as in economics and semi-order in mathematics. In the sense of the incomplete preference, we obtain a principle of maximal consumption allocations, by which, for a pure exchange economy with infinitely many commodities and infinitely countable agents, we first prove the existence of a quasi-equilibrium, and then conclude that such a quasi-equilibrium can be extended to a general equilibrium of this economy if incomplete preferences are proper in a suitable way.

Key Words: Incomplete preferences; Infinite economies; Maximal consumption allocations; Equilibria.
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1. INTRODUCTION

Many scholars have been doing researches on economies with infinite dimensional spaces and made many important achievements since Bewley's (1972) path breaking work. Most of the results on the existence of equilibria have assumed either a finite number of agents as in Mas-colell (1986) and Monteiro (1996) or a continuum of traders as in Zame (1986) or a measure space of agents as in Jones (1983). In the case of infinitely countable agents, Richard , Srivastara (1988) and Aliprantis et al. (1989) made many studies.

The above arguments critically depend on the preference completeness or an utility function describing some preference. Clearly, the completeness of a preference is a necessary condition for this preference to be represented

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by the utility functions, but certainly not all preferences satisfying the completeness have utility representations, for example, it is observed by Bryan Ellickson (1993) that Lexicographic preference in R_+^2 hasn't. We can go a step further to see that an utility function builds a bridge between mathematics and economics, and is an essential condition for the existing mathematical tools to be applied in some economic problems. Therefore, although it was found early that the preference completeness is often in conflict with some reality markets, let alone utility functions satisfying the completeness, so far there is no choice except the use of the preference completeness because, without the preference completeness, the existing tools are powerless for solving equilibrium theory, utility theory, choice theory and even decision theory, see Fishburn (1991).

In this paper, we attempt to do research in this issue. The main characteristics of this paper lie in two points. First, we delete the assumption of the preference completeness and thus extend many known equilibrium models as in Aliprantis, Brown and Burkinshaw (1989), Mas-colell (1986, 1991) and Richard , Srivastara (1988). Secondly, our method of the existence proof is different from the known those in essence. To overcome the difficulty which we have no way of using any utility function in the absence of the preference completeness, we propose and demonstrate a new principle of maximal consumption allocations, by which we achieve our aim of this paper. In addition, we weaken or delete other limitations, such as the uniformly proper conditions as in Richard and Zame (1986), lower continuity and strictly convexity of preferences as in Aliprantis et al. (1989), so that our economic models can describe the practical markets more precisely.

The present work will utilize the theory of Riesz spaces. For details of the theory of Riesz spaces we refer the reader to Aliprantis and Burkinshaw (1985).

The rest of this paper is organized as follows. Section 2 proposes a concept called the incomplete preference which is more general than either of the preference in economics and the semi-order in mathematics. In the sense of the incomplete preference, Section 3 presents our model and obtains a principle of maximal consumption allocations, by which Section 4 establishes the existence of equilibria for pure exchange infinite economies. Section 5 provides some concluding comments.

2. INCOMPLETE PREFERENCES

In the ordering theory, we often meet the following assumptions related to the ordering relations:

(1) the completeness, that is, either $x \leq y$ or $y \leq x$ for x, $y \in X$;
(2) the reflectivity, that is, $x \leq x$ for any $x \in X$;
(3) the transitivity, that is, $x \leq y$ and $y \leq z$ for $x, y, z \in X$ imply $x \leq z$;

(4) the anti-symmetry, that is, $x \leq y$ and $y \leq x$ for $x, y \in X$ imply $x = y$; where X is a nonempty collection of consumption plans, $\leq$ represents an ordering defined on X. In the equilibrium analysis of economies, it is assumed generally that an ordering relation called preference satisfies completeness, reflectivity and transitivity and may not satisfy anti-symmetry. In the meantime, there exists an ordering relation called semi-order which satisfies reflectivity, transitivity and anti-symmetry and may not satisfy completeness. The semi-order is mainly used in mathematics. To get a more general equilibrium model of economies, we put forward a concept of incomplete preference as follows.

Definition 2.1. Let X be a nonempty collection of consumption plans. An ordering defined on X is called an incomplete preference denoted by $\preceq$ if this ordering $\preceq$ satisfies reflexivity and transitivity, where the relation $x \preceq y$ for $x, y \in \Omega$ means that y is at least as preferable as x; X is called an incomplete preference set termed by $(X, \preceq)$; x is said to be indifferent to y for $x, y \in X$ if both $x \preceq y$ and $y \preceq x$ hold, and this indifference relation is denoted by $x \sim y$; y is said to be preferred to x for $x, y \in \Omega$ if $x \preceq y$ but $x \not\sim y$, and this strong preference relation is denoted by $x \prec y$, where $x \not\sim y$ means that x is not indifferent to y; a nonempty set $M \subset X$ in X is called a complete preference subset if either $x \preceq y$ or $y \preceq x$ holds for any $x, y \in M$.

For clarity, the rest of this paper always terms an incomplete preference by $\preceq$, and terms a semi-order by $\leq$. Obviously, the incomplete preference $\preceq$ indicates that the relations $x \preceq y$ and $y \preceq x$ only imply $x \sim y$ but not necessarily $x = y$ and the relation $x \sim y$ can be deduced from $x = y$. By a direct proof, the indifference relation $\sim$ satisfies reflectivity, symmetry and transitivity required for an equivalence relation on X. Hence we have Lemma 1.

Lemma 1. *The indifference relation $\sim$ on X is an equivalence relation.*

3. MAXIMAL CONSUMPTION ALLOCATIONS

In this section, for convenience we first give some concepts and conclusions.

The letter E will denote a Riesz space, where the Riesz space E means a lattice-ordered space which has a linear operation compatible with the ordering in E. An order bounded linear functional $f : E \to R$ is said to be order continuous whenever $x_\alpha \xrightarrow{o} \theta$ in E implies $f(x_\alpha) \to 0$ in E, where the notation $\xrightarrow{o}$ means that a sequence is order convergent to some element, θ denotes the zero element in E. We term $E^\sim$ the vector space of all order bounded linear functionals and $E_n^\sim$ the vector space of all order continuous linear functionals. Obviously $E_n^\sim \subset E^\sim$. A subset H of E is said to be a solid set whenever $|x| \leq |y|$ and $y \in H$ imply $x \in H$. A solid vector subspace of E is known as an ideal. A Riesz dual system $\langle E, E' \rangle$ is a Riesz space E together with an ideal E' of $E^\sim$ that separates the points of E such that the duality is the natural one, i.e., $\langle x, x' \rangle = x'(x)$ holds for all $x \in E$ and $x' \in E'$. A Riesz dual system $\langle E, E' \rangle$ is called symmetric whenever $\langle E', E \rangle$ is also a Riesz dual system.

In the following, let $\langle E, E_n^\sim \rangle$ be a Riesz dual system which describes the commodity-price duality, where E is the commodity space and $E_n^\sim$ is the price space. Thus the locally convex topology generated by the family of semi-norm $\{|f(x)| : \forall f \in E_n^\sim\}$ (which will be denoted by $\sigma(E, E_n^\sim)$) makes E a locally convex topological space. In the following, we give two known conclusions found in Kantorovitch–Akilov (1955) and in Aliprantis–Burkinshaw (1985) respectively.

Lemma 2. *The topological dual E^* generated by the topology $\sigma(E, E_n^\sim)$ coincides with the order dual $E_n^\sim$ of E, i.e.,*

$$(E, \sigma(E, E_n^\sim))^* = E_n^\sim.$$

Lemma 3. *If $\langle E, E_n^\sim \rangle$ is a symmetric Riesz dual system, then*

(1) every order interval of E is $\sigma(E, E_n^\sim)$-compact;

(2) E is an Archimedean Riesz space, that is, $\frac{1}{n}x$ is order convergent to θ for $x \in E$ and $x > \theta$ when $n \to \infty$.

Let the positive cone E^+ of E denote a consumption set, where $E^+ = \{x \in E | x \geq \theta\}$. An incomplete preference $\preceq$ on the positive cone E^+ is said to be:

(1) *monotone* (or *strictly monotone*) , whenever $\theta \leq x < y$ implies $x \preceq y$ or $(x \prec y)$ for $x, y \in E^+$;

(2) *upper convex* (or *lower convex*), whenever the set $\{y \in E^+ : x \preceq y\}$ (or $\{y \in E^+ : y \preceq x\}$) is convex for each $x \in E^+$; and

(3) *upper continuous* (or *lower continuous*), if the set$\{y \in E^+ : x \preceq y\}$ (or $\{x \in E^+ : y \preceq x\}$) is $\sigma(E, E_n^\sim)-$closed in E^+ for each $x \in E^+$.

Definition 3.1. A pure exchange infinite economy $\mathcal{E}$ is a triplet

$$\varepsilon = (\langle E, E_n^\sim \rangle, \{w_i : i = 1, 2, \cdots\}, \{\preceq_i : i = 1, 2, \cdots\}),$$

where the components of ε satisfy the following properties:

(1) the commodity-price duality $\langle E, E_n^\sim \rangle$ is a symmetric Riesz dual system;

(2) there is an infinite countable number of consumers;

(3) each consumer i has an initial endowment $w_i > 0$, and the total endowment w is defined by $w = \sum_{i=1}^{\infty} w_i$, where the supremum w is assumed to exist in E^+;

(4) each consumer i has an upper convex and monotone incomplete preference $\preceq_i$ on his consumption set E^+.

Remark 3.1. Since, under the same conditions, the equilibrium existing in an economy with infinitely countable consumers must also exist in an economy with finite consumers, without loss of generality, we assume an economy with infinitely countable consumers in Definition 2 directly, and only provide the proof of this case in this paper. It is obvious that the conclusions of this paper are true for economies with finite consumers. In addition, the supremum w in general exists because of the finiteness of the endowment in the reality world, i.e., the assumption (3) in Definition 2 is reasonable.

Here are many examples of symmetric Riesz dual systems (see Aliprantis, Brown and Burkinshaw (1989)):

$$\langle L_p(\mu), L_q(\mu) \rangle, \ \langle l_p, l_q \rangle \ (1 < p, q < +\infty, \ \frac{1}{p} + \frac{1}{q} = 1), \ \langle R^n, R^n \rangle, \ \langle l_\infty, l_1 \rangle.$$

If the measure μ is $\sigma-$finite, then

$$\langle L_1(\mu), L_\infty(\mu) \rangle, \langle L_\infty(\mu), L_1(\mu) \rangle$$

are also symmetric Riesz dual systems. In particular,

$$(L_p(\mu))_n^\sim = L_q(\mu), \ (l_p)_n^\sim = l_q, \ (R^n)_n^\sim = R^n,$$

$$(l_\infty)_n^\sim = l_1, \ (L_1(\mu))_n^\sim = L_\infty(\mu), \ (L_\infty(\mu))_n^\sim = L_1(\mu).$$

The above examples show that many widely used Riesz dual systems are all symmetric.

Definition 3.2. A vector $(x_1, \cdots, x_i, \cdots)$ is called a feasible allocation of the infinite economy $\mathcal{E}$ for $x_i \in E^+$ and $i = 1, 2, \cdots$, if $\sum_{i=1}^{\infty} x_i \leq w$; the set of all feasible allocations in the infinite economy $\mathcal{E}$ is denoted by A. A vector $(x_1, \cdots, x_i, \cdots)$ is called an effective allocation of the infinite economy $\mathcal{E}$ for $x_i \in E^+$ if $\sum_{i=1}^{\infty} x_i = w$; the set of all effective allocations in the infinite economy $\mathcal{E}$ is denoted by B.

Clearly $A \subset (E^+)^\infty$, where $(E^+)^\infty$ represents a product space formed by infinite countable Riesz spaces E^+, the product topology on $(E^+)^\infty$ is induced by the topology $\sigma(E, E_n^\sim)$ on E^+ and denoted by τ. For any $x = (x_1, \cdots, x_i, \cdots) \in (E^+)^\infty$, we define a mapping ρ_i from the product space $(E^+)^\infty$ into the $i-th$ space E^+ by

$$\rho_i x = x_i, \; i = 1, 2, \cdots . \tag{1}$$

We know that the mapping $\rho_i : (E^+)^\infty \longrightarrow E^+$ is continuous.

Definition 3.3. For $x = (x_1, \cdots, x_i, \cdots)$, $y = (y_1, \cdots, y_i, \cdots) \in (E^+)^\infty$, y is said to be at least as preferable as x, which is denoted by $x \preceq y$, if $x_i \preceq_i y_i$ holds for each i; y is said to be preferred to x, which is denoted by $x \prec y$, if $x_i \preceq_i y_i$ holds for each i, and there is at least one i_0 with $x_{i_0} \prec_{i_0} y_{i_0}$.

Clearly, if each $\preceq_i$ is an incomplete preference for $i = 1, 2, \cdots$, the relation $\preceq$ given by Definition 4 is also an incomplete preference, and thus the product space $(E^+)^\infty$ is an incomplete preference set under $\preceq$. Moreover, we can get the following property further.

Lemma 4. *If each incomplete preference $\preceq_i$ on E^+ is upper continuous for $i = 1, 2, \cdots$, then $\preceq$ on $(E^+)^\infty$ is also upper continuous.*

Definition 3.4. Suppose that $(\Omega, \preceq)$ is a nonempty incomplete preference set. An element $x^* \in \Omega$ is called a maximal element of Ω if there exists no $x \in \Omega$ such that $x^* \preceq x$ and $x^* \not\sim x$. A element $x^* \in E^+$ is called an maximal consumption allocation of the infinite economy $\mathcal{E}$ if x^* is a maximal element of A and an effective allocation of the economy $\mathcal{E}$, where A is a set of all feasible allocations in the economy ε.

Theorem 1. *Assume that*

$$\varepsilon = (\langle E, E_n^\sim \rangle, \{w_i : i = 1, 2, \cdots\}, \{\preceq_i : i = 1, 2, \cdots\})$$

is a pure exchange infinite economy and each incomplete preference $\preceq_i$ is upper continuous for $i = 1, 2, \cdots$, then the economy $\mathcal{E}$ must have a maximal consumption allocation.

4. ECONOMIC EQUILIBRIA

In this section, we will discuss the existence of the quasi-equilibrium and the general equilibrium of economies with infinitely many commodities and countably many agents in the sense of incomplete preference.

DEFINITION 4.1. An effective allocation $(x_1^*, \cdots, x_i^*, \cdots) \in A$ is a quasi-equilibrium for the economy $\mathcal{E}$ if there exists a non-zero price $p^* \in E_n^\sim$ such that for each i,
(a) $p^* x_i^* = p^* w_i$;
(b) $x_i^* \preceq_i x$ in E^+ implies $p^* w_i \leq p^* x$; and
(c) $\sum_{i=1}^{\infty} x_i^* = w$.

DEFINITION 4.2. An effective allocation $(x_1^*, \cdots, x_i^*, \cdots) \in A$ is a general equilibrium for the economy $\mathcal{E}$ if there exists a non-zero price $p^* \in E_n^\sim$ such that
(1) for each i, x_i^* is a maximal element in the ith consumer's budget set $B_i(p^*) = \{x \in E^+ : p^* x \leq p^* w_i\}$; and
(2) $\sum_{i=1}^{\infty} x_i^* = w$.

Any non-zero price p^* for which $x_i^* \preceq_i x$ in E^+ implies $p^* w_i \leq p^* x$ is called a supporting price of the effective allocation $(x_1^*, \cdots, x_i^*, \cdots)$. Any supporting price p^* is necessarily a positive price. To see this, let $y \in E^+$, then $x_i^* \preceq_i x_i^* + ny$ for any positive integer n, and so $p^* w_i \leq p^*(x_i^* + ny)$. That is , $p^* y \geq \frac{1}{n} p^*(w_i - x_i^*)$ holds for any n, by which we have that $p^* y \geq 0$ for $n \to \infty$, i.e., p^* is a positive price. In addition, it is easy to see that a general equilibrium must be a qusi-equilibrium.

THEOREM 2. *If the infinite economy $\mathcal{E}$ satisfies that*
(i) for each $i = 1, 2, \cdots$, $\preceq_i$ is strictly monotone and upper continuous;
(ii) for any effective allocation $(x_1, \cdots, x_i, \cdots)$, there exists some i_0 such that the set $\{y \in E^+ | x_{i_0} \prec_{i_0} y\}$ has at least one interior point;
(iii) the set B of all effective allocations is a τ_{-}closed set in $(E^+)^\infty$.
Then the economy $\mathcal{E}$ has a quasi-equilibrium.

Remark 4.1. In some widely used spaces such as all finite dimensional spaces, l_∞ and $L_\infty(\mu)$, the assumptions (ii) and (iii) of Theorem 2 are automatically satisfied.

THEOREM 3. *Assume that* $(x_1^*, \cdots, x_i^*, \cdots) \in (E^+)^\infty$ *is a quasi-equilibrium supported by a price* $p^* \in E_n^\sim$*, and for each* i,

(i) the quasi-equilibrium income distribution is strictly positive, i.e., $p^* w_i > 0$ *for each* i*;*

(ii) there exists some $\delta \in (0,1)$ *such that* $x_i^* \prec_i \delta x_i$ *holds if* $x_i^* \prec_i x_i$ *for some* $x_i \in E^+$.

Then $(x_1^*, \cdots, x_i^*, \cdots)$ *is a general equilibrium allocation supported by the price* p^*.

Remark 4.2. The straight proof indicates that when each preference $\preceq_i$ is complete and upper continuous, $i = 1, 2, \cdots$, the assumption (ii) of Theorem 3 is automatically fulfilled.

The next theorem implies that the quasi-equilibria possess the following welfare property which was also established in Aliprantis et al. (1989). The proof of Theorem 4 is also analogue of Theorem 6.3 in Aliprantis et al. (1989).

THEOREM 4. *If there exists a pure exchange infinite economy in which each incomplete preference is strictly monotone and upper continuous, then every quasi-equilibrium which is supported by a price with strictly positive income distribution is a Pareto maximal allocation.*

5. CONCLUSIONS

Without the preference completeness required in all the known documents for the equilibrium existence problems, this paper still show the existence of maximal consumption allocations for infinite economies, which is a main contribution in this paper. The proving idea and the principle of maximal allocations make us avoid the standard method of using utility representations and succeed in finding a quasi-equilibrium of an infinite economy in the absence of the preference completeness. Then, under some suitable conditions for preferences, we extend such a quasi-equilibrium to a general equilibrium of the infinite economy further. The basic idea of our proof is due to Aliprantis et al. (1989) who first looked for a quasi-equilibrium and then extended this quasi-equilibrium to the original economy, which has been also applied by Konrad (1996). However, as mentioned in the introduction of our paper, the lack of the preference completeness adds the difficulty greatly and makes the existing method powerless for proving the equilibrium existence, which directly results in such a fact that our arguments in this paper differ substantially from those in Aliprantis et al. (1989) and Konrad (1996). In particular, it is worth noting that we can no longer make use of any convenience of utility representations which is crucial for arguments in almost all known documents with respect to the equilibrium analysis. Instead, and this is the technical innovation of our paper, we employ arguments of ordering theory.

With regard to the assumption (ii) of Theorem 2 , which requires that the positive cones of Riesz spaces have nonempty interior points and has been used by many researchers such as Richard, Zame (1986), although all finite dimensional spaces, infinite spaces l_∞ and $L_\infty(\mu)$ having nonempty interior points are the most widely used spaces in the arguments for the equilibrium problems, such as in Bewly (1972), Richard and Srivastara (1988), we can, of course, see that since the positive cones of most infinite dimensional Riesz spaces haven't interior points, this assumption is not fully satisfactory. To tackle this problem, some researchers formulate several improved ways for preferences or commodity spaces, but the results are not too ideal yet. Hence, we have to leave this open problem to be solved in the future.

APPENDIX A

In this appendix, we shall prove Theorem 1–Theorem 3 by a series of lemmas.

Proof of Lemma 4 We only need to show that for $\forall x = (x_1, \cdots, x_i, \cdots) \in (E^+)^\infty$, the set $\{y \in (E^+)^\infty | x \preceq y\}$ is closed in $(E^+)^\infty$. Let

$$U_i = \{y_i \in E^+ | x_i \preceq_i y_i\},$$

where $i = 1, 2, \cdots, x_i = \rho_i x$, and the mapping ρ_i is defined by (1) in Section 3. In light of the relation between $\preceq_i$ and $\preceq$ as well as by direct proof, we can have

$$\{y \in (E^+)^\infty | x \preceq y\} = \bigcap_{i=1}^{\infty} \rho_i^{-1}(U_i).$$

Since each incomplete preference $\preceq_i$ is upper continuous, the set U_i is closed in E^+. By the continuity of ρ_i, the set $\rho_i^{-1}(U_i)$ is also closed in $(E^+)^\infty$, and thus so is the set $\bigcap_{i=1}^{\infty} \rho_i^{-1}(U_i)$. Therefore, for $\forall x = (x_1, \cdots, x_i, \cdots) \in (E^+)^\infty$, the set $\{y \in (E^+)^\infty | x \preceq y\}$ is closed in $(E^+)^\infty$, i.e., the incomplete preference $\preceq$ is upper continuous. ▌

LEMMA 5. *With the assumptions of Theorem 1, the set A of all feasible allocations is compact closed under the product topology τ of $(E^+)^\infty$.*

Proof. Since the commodity-price duality $\langle E,\, E_n^\sim\rangle$ is a symmetric Riesz dual system, $E_n^\sim$ is complete, i.e., if $x \neq \theta$, there exists some $f \in E_n^\sim$ with $f(x) \neq 0$. Thus E is a Hausdorff space under the topology $\sigma(E, E_n^\sim)$. In view of Lemma 3, the ordering interval $[\theta,\, w] = \{x \in E | \theta \leq x \leq w\}$ is σ-compact. Since all compact subset in a Hausdorff space are closed, the ordering interval $[\theta,\, w]$ is a closed set in E.

In the following, we show that A is $\tau-$closed. Clearly, $A \subset [\theta,\, w]^\infty$, where $[\theta,\, w]^\infty$ represents a product space formed by infinite countable ordering intervals $[\theta,\, w]$. Suppose that a directed set $\{x_\alpha\}_{\alpha\in\Gamma}$ in A has a limit $x = (x_1, x_2, \cdots, x_i, \cdots)$ of net convergence under the topology τ, which is denoted by $x_\alpha \overset{net}{\longrightarrow} x$, where $x_\alpha = (x_{1,\alpha}, \cdots, x_{i,\alpha}, \cdots)$, we prove $x \in A$. It is clear that for $\alpha \in \Gamma$ and any positive integer n, $\sum_{i=1}^n x_{i,\alpha} \leq w$, and $\sum_{i=1}^n x_{i,\alpha} \overset{net}{\longrightarrow} \sum_{i=1}^n x_i$ under the topology $\sigma(E, E_n^\sim)$. Since the ordering interval $[\theta,\, w]$ is a closed set, $\sum_{i=1}^n x_i \in [\theta,\, w]$, i.e., $\sum_{i=1}^n x_i \leq w$. Noting that $\sum_{i=1}^n x_{i,\alpha}$ is order convergent to $\sum_{i=1}^\infty x_i$ for $n \to \infty$, we have that $\sum_{i=1}^\infty x_i \leq w$, that is, $x \in A$, and hence A is a $\tau-$closed set in $[\theta,\, w]^\infty$. By virtue of Tychonoff theorem and on account of the compactness of the ordering interval $[\theta,\, w]$, the product space $[\theta,\, w]^\infty$ is a compact set, which, along with the closeness of A in $[\theta,\, w]^\infty$, implies that the set A is compact closed under the product topology τ. ▮

Proof of Theorem 1 Regarding the subset A as a topological subspace of $[\theta,\, w]^\infty$, we get a quotient space denoted by $A/\!\sim$ relative to the equivalence relation $\sim$ which is shown by Lemma 1, that is, $A/\!\sim\, = \{\tilde{x} | \forall\, x{\in}A\}$, where $\tilde{x} = \{y{\in}A | y{\sim}x\}$ is an equivalence set for $x{\in}A$, the topology of $A/\!\sim$ is a quotient topology with respect to the topology of A and the natural mapping r, here the nature mapping r from A into $A/\!\sim$ is defined by $rx = \tilde{x}{\in}A/\!\sim$ for any $x{\in}A$. Then, an ordering on $A/\!\sim$ is defined by $\tilde{x}{\leq}\tilde{y}$ for $\tilde{x}, \tilde{y}{\in}X/\!\sim$ if and only if there exist $u{\in}\tilde{x}$ and $v{\in}\tilde{y}$ such that $u{\preceq}v$, here $\tilde{x}$ and $\tilde{y}$ are viewed as equivalence sets in A. It is easy to follow that the ordering $\leq$ in $A/\sim$ satisfies reflexivity, transitivity and anti-symmetry, i.e., the ordering $\leq$ is a semi-order, and so $A/\!\sim$ is a semi-order set. For $a{\in}A$, set

$$A(a) = \{x{\in}A | a{\preceq}x\}, A(\tilde{a}) = \{\tilde{x}{\in}A/\!\sim | \tilde{a}{\leq}\tilde{x}\},$$

where $\tilde{a} = ra$. Obviously, $A(a) = A \bigcap \{x{\in}(E^+)^\infty | a{\preceq}x\}$. Noting the upper continuity of $\preceq_i$ for each i as well as Lemma 4 and Lemma 5, we have that $A(a)$ is compact closed in A. By direct proof, it is easy to follow the following conclusions:

Conclusion (a) The natural mapping

$$r : A{\longrightarrow}A/\!\sim$$

is continuous and surjective, and $r(r^{-1}(S)) = S$ for $S{\subset}A/\!\sim$;

Conclusion (b) $r^{-1}(A(\tilde{a})) = A(a)$.

Take a complete semi-order set N from $A(\tilde{a})$, and let

$$N^* = r[c(r^{-1}(N))].$$

where $c(.)$ denotes a closure of the set in brackets (), which is the same sense in the following. Frist, we shall show that N^* is a compact set in $A/\sim$ and $N \subset N^* \subset A(\tilde{a})$. Since $A(a)$ is compact closed in A, by Conclusion (b)

$$c(r^{-1}(N)) \subset r^{-1}(A(\tilde{a})) = A(a), \tag{A.1}$$

which means that the closure $c(r^{-1}(N))$ is a compact set in $A(a)$. Observing the continuity of r and the definition of N^*, we have that N^* is a compact subset in $A/\sim$. Since $r^{-1}(N) \subset r^{-1}[c(N)] \subset r^{-1}(A(\tilde{a})) = A(a)$ holds and the continuity of r implies that $r^{-1}(c(N))$ is a closed set in $A(a)$, by (A.1)

$$r^{-1}(N) \subset c(r^{-1}(N)) \subset r^{-1}(c(N)) \subset r^{-1}(A(\tilde{a})),$$

and by Conclusion (a)

$$N = r(r^{-1}(N)) \subset r[c(r^{-1}(N))] = N^* \subset r[r^{-1}(c(N))] \subset A(\tilde{a}). \tag{A.2}$$

Hence, by the above discussions, N^* is a compact set in $A(\tilde{a})$ and $N \subset N^* \subset A(\tilde{a})$ holds.

Secondly, we shall show that N has an upper bound in $A(\tilde{a})$. For each $\tilde{x} \in N$. , we let

$$A(\tilde{x}) = \{\tilde{y} \in N^* | \tilde{x} \leq \tilde{y}\}.$$

Obviously $A(\tilde{x}) = A(\tilde{x}) \cap N^*$, which, by the closeness of $A(\tilde{x})$ in $A/\sim$, means that $A(\tilde{x})$ is a closed set in the topological subspace N^*. Take any finite members $\{A(\tilde{x}_i) | i = 1, 2, \ldots, n\}$ in $\{A(\tilde{x}) | \tilde{x} \in N\}$ and let

$$\tilde{x}_0 = max\{\tilde{x}_i | i = 1, 2, \ldots, n\}.$$

Since N is a complete ordered set and $\{\tilde{x}_i | i = 1, 2, \ldots, n\} \subset N$, $\tilde{x}_0 \in N$ makes sense. Clearly $\tilde{x}_0 \in \bigcap_{i=1}^n A(\tilde{x}_i)$, thus $\bigcap_{i=1}^n A(\tilde{x}_i) \neq \emptyset$. By the compactness of N^* and according to the finite intersection property of compact sets, we get $\bigcap_{\tilde{x} \in N} A(\tilde{x}) \neq \emptyset$. Taking $\tilde{y} \in \bigcap_{\tilde{x} \in N} A(\tilde{x})$ and by the definition of $A(\tilde{x})$, we have $\tilde{y} \in N^* \subset A(\tilde{a})$ and $\tilde{x} \leq \tilde{y}$ for any $\tilde{x} \in N$. Hence $\tilde{y}$ is an upper bound of N in $A(\tilde{a})$. Thus it follows from Zorn's lemma that $A(\tilde{a})$ has a maximal element $\tilde{x}^*$. Then, by way of contradiction and in view of the definitions of the sets $A(a)$ and $A(\tilde{a})$, it is easy to prove that any x^* of the equivalent set $\tilde{x}^* = \{y \in A | y \sim x^*\}$ is a maximal element of A.

Finally, letting $x^* = (x_1^*, \cdots, x_i^*, \cdots) \in \tilde{x}^*$, we show that $\sum_{i=1}^{\infty} x_i^* = w$, i.e., x^* is an effective allocation of the infinite economy $\mathcal{E}$. If otherwise, $\sum_{i=1}^{\infty} x_i^* < w$ must hold because of $x^* \in A$. Let

$$\eta = w - \sum_{i=1}^{\infty} x_i^*,$$

thus $\eta > \theta$. We construct an element $x = (x_1, \cdots, x_i, \cdots)$ as follows: $x_1 = x_1^* + \eta$ and $x_i = x_i^*$ for $i \geq 2$. Clearly $\sum_{i=1}^{\infty} x_i = w \in A$ and $x^* \prec x$ by Definition 4, which contradicts that x^* is a maximal element of A. To sum up, we show that x^* is a maximal consumption allocation of the infinite economy $\mathcal{E}$. ▌

Proof of Theorem 2 Let

$$C = \bigcap_{i=1}^{\infty} \rho_i^{-1}\Big(\bigcap_{p \in (E_n^{\sim})^+} (\{x_i \in E^+ | px_i \leq pw_i\})\Big),\ D = B \bigcap C,$$

where the mapping $\rho_i : (E^+)^\infty \longrightarrow E^+$ is defined by (1) in Section 3. Clearly $D \neq \phi$ due to $(w_1, \cdots, w_i, \cdots) \in D$, and by condition (iii) and the continuity of ρ_i, the set D is nonempty τ-closed in $(E^+)^\infty$. The relation $D \subset B \subset A$ and Lemma 5 imply that D is τ-compact closed in $[0,\, w]^\infty$. On account of Theorem 1 and noting the upper continuity of $\preceq_i$, we have that D has a maximal allocation $x^* = (x_1^*, \cdots, x_i^*, \cdots)$.

Next, we prove that if $x_i^* \preceq_i x_i$ for each i, there exists a non-zero price $p^* \in E_n^{\sim}$ with

$$p^* w \leq p^* \sum_{i=1}^{\infty} x_i. \tag{A.3}$$

Since $\sum_{i=1}^{\infty} x_i^* = w$ and $\theta < w$, we can find an element $x_{i_1}^* \in \{x_i^* | i = 1, 2, \cdots\}$ with $x_{i_1}^* > \theta$. Obviously there exists $y \in E^+$ such that $x_{i_1}^* \prec_{i_1} y$. In fact, on account of the strict monotoneity of $\preceq_{i_1}$, $x_{i_1}^* \prec_{i_1} y$ must hold for $y = \alpha x_{i_1}^*$ and $\alpha > 1$. Let

$$U(x_{i_1}^*) = \{y \in E^+ | x_{i_1}^* \prec_{i_1} y\},\ V(x_i^*) = \{y \in E^+ | x_i^* \preceq_i y\},$$

$$G = U(x_{i_1}^*) + \sum_{i \neq i_1} V(x_i^*),\ \ H = \sum_{i=1}^{\infty} V(x_i^*).$$

By the fact just discussed above, $U(x_{i_1}^*) \neq \emptyset$, and it is clear that $G \subset H$, and $V(x_i^*)$ is also nonempty due to $x_i^* \in V(x_i^*)$ for each i. Since each consumer i has an upper convex incomplete preference $\preceq_i$, $U(x_{i_1}^*)$, $V(x_i^*)$

are all nonempty convex sets for each i, and so G is also a nonempty convex set. Uniting $\sum_{i=1}^{\infty} x_i^* = w$ with the definition of G, we know $w \notin G$. In addition, it is easy to follow from condition (ii) that G has at least one interior point. On account of separation theorem of convex sets, there exists $p^* \in E^*$ for any $x \in G$ with

$$p^* w \leq p^* x, \tag{A.4}$$

where E^* is a conjugate space of the topological space E. By Lemma 2,

$$(E, \sigma(E, E_n^{\sim}))^* = E_n^{\sim},$$

hence $p^* \in E_n^{\sim}$ also holds. By means of the strict monotoneity of $\preceq_i$, the order continuity of p^* and (A.5), it is easy to prove that for each $x \in H$, $p^* w \leq p^* x$, which, along with the definition of the set H, implies that (A.4) holds.

Now, we prove, for any k and any element $y \in E^+$ with $x_k^* \preceq_k y$,

$$p^* x_k^* \leq p^* y. \tag{A.5}$$

Define an element $(y_1, \cdots, y_i, \cdots) \in (E^+)^{\infty}$ by $y_i = x_i^*$ for $i \neq k$ and $y_i = y$ for $i = k$. Clearly $x_i^* \preceq_i y_i$ for each i, and so by (A.4) $p^* w \leq p^*(\sum_{i=1}^{\infty} y_i)$. Taking $w = \sum_{i=1}^{\infty} x_i^*$ into account, we get

$$p^*(\sum_{i=1}^{\infty} x_i^*) \leq p^*(\sum_{i=1}^{\infty} y_i). \tag{A.6}$$

Clearly $\sum_{i=1}^{n} x_i^*$ and $\sum_{i=1}^{n} y_i$ converge in order to $\sum_{i=1}^{\infty} x_i^*$ and $\sum_{i=1}^{\infty} y_i$ for $n \to \infty$ respectively. Hence, by the order continuity of p^* and (A.7), we have $\sum_{i=1}^{\infty} p^* x_i^* \leq \sum_{i=1}^{\infty} p^* y_i$, which, by $y_i = x_i^*$ for $i \neq k$, implies that (A.6) holds.

Finally, we show that the conclusions (a) and (b) in Definition 6 hold. Since $w = \sum_{i=1}^{\infty} w_i = \sum_{i=1}^{\infty} x_i^*$ holds, and $\sum_{i=1}^{n} x_i^*$ and $\sum_{i=1}^{n} w_i$ converge in order to $\sum_{i=1}^{\infty} x_i^*$ and $\sum_{i=1}^{\infty} w_i$ for $n \to \infty$ respectively,

$$\sum_{i=1}^{\infty} p^* w_i = \sum_{i=1}^{\infty} p^* x_i^*,$$

that is,

$$\sum_{i=1}^{\infty} (p^* w_i - p^* x_i^*) = 0. \tag{A.7}$$

By the discussions in Section 4 and according to (6), we get $\theta \leq p^*$, i.e., $p^* \in (E_n^\sim)^+$, which, together with $(x_1^*, \cdots, x_i^*, \cdots) \in C \subset (E^+)^\infty$, indicates $p^* x_i^* \leq p^* w_i$ for each i. Taking (A.8) into account, we get $p^* x_i^* = p^* w_i$, i.e., the conclusion (a) in Definition 6 holds. From the conclusion (a) and (A.6), it follows that the conclusion (b) in Definition 6 holds. Therefore, the economy $\mathcal{E}$ has a quasi-equilibrium. ▮

Proof of Theorem 3 If the conclusion is not true, there exists some i_0 and $x_{i_0} \in B_i(p^*)$ defined by Definition 7 such that $x_{i_0}^* \prec_{i_0} x_{i_0}$, and so $p^* x_{i_0} \geq p^* w_{i_0}$, which, together with $x_{i_0} \in B_i(p^*)$, means $p^* x_{i_0} = p^* w_{i_0}$. Since condition (ii) indicates that there exists $\delta \in (0, 1)$ with $x_{i_0}^* \prec_{i_0} \delta x_{i_0}$, $\delta p^* x_{i_0} \geq p^* w_{i_0}$. Noting condition (i), i.e., $p^* w_{i_0} > 0$, we have

$$p^* w_{i_0} \leq \delta p^* x_{i_0} = \delta p^* w_{i_0} < p^* w_{i_0},$$

which is impossible. Therefore, each x_i^* is a maximal element $B_i(p^*)$, and thus $(x_1^*, \cdots, x_i^*, \cdots)$ is a general equilibrium allocation supported by the price p^*. ▮

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