ANNALS OF ECONOMICS AND FINANCE **5**, 93–126 (2004)

Optimal Stopping of Active Portfolio Management*

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We study an investor's decision to switch from active portfolio management to passive management. This problem is mathematically modelled by a mixture of a consumption-portfolio selection problem and an optimal stopping problem. We assume that the investor has stochastic differential utility with ambiguity aversion and incurs utility loss from active portfolio management that can be avoided by switching to passive management, and show that she manages actively as long as her level of wealth is above a certain threshold. The threshold wealth level is shown to be an increasing function of both the coefficient of ambiguity aversion and the utility cost of active management.

Key Words: Consumption-portfolio selection; Active management; Passive management; Discretionary stopping time; Recursive utility; Stochastic differential utility; Optimal switching; Ambiguity.
JEL Classification Numbers: G11, D81, C61.

* We thank Guofu Zhou, co-editor, for many helpful comments

1529-7373/2002

1. INTRODUCTION

In this paper we study the decision problem of switching from active portfolio management to passive management. We consider a model of optimal consumption-investment selection in which an investor has an option to stop her active role of consumption and portfolio selection. For example, an individual investor who actively manages her investment may often consider the option to put her wealth to a third party manager, say a mutual fund or an investment trust, or simply to a bank account. An institutional investor may want to save management costs by switching to a passive management style that simply follows a target index. [1]

For example, let us imagine a potential new investor who needs to make an investment decision. If the characteristics of the financial assets, including the current expected returns and risks and the laws governing their future evolution, are well known, then she is able to choose her portfolio according to a classical recipe (e.g., see Merton 1971, 1973). In particular, if there are n-sources of uncertainty that affect the means and variances of the returns on risky assets, then the n-fund separation theorem is valid and she can simply buy shares of n-mutual funds whose returns have highest correlation with the sources of uncertainty (Merton 1973). But this is only an ideal situation, and in reality, the investor does not fully understand the financial market and the characteristics of asset returns. Therefore, if she decides to manage actively, she will need to make efforts to study and understand the characteristics and factors that affect them. If the cost of efforts seems to outweigh investment returns, then she might choose to select an investment manager, or a trust, or a mutual fund, or a hedge fund, and in addition she can just follow the manager's recommendation, or she can decide to put her money simply in a money market or a bank account.

In this paper we model the situation of such an investor as explained in the previous paragraph by introducing the utility cost of active management and ambiguity about the market parameters. Then, we show that the investor continues her role of active management as long as her wealth is above a threshold level and stops when her wealth falls to or below this threshold wealth level.[2] Ambiguity about market parameters plays an im-

[1] The problem can be posed in both ways: When does an investor change to passive management if she currently adopts active portfolio management? Or, when does she switch to active management from the current passive position? We can deal with both problems. But, it is sufficient to consider the first problem in order to derive economic intuition, since the answer to the second question will be easily conjectured once we obtain the answer to the first one. Thus, in this paper we only discuss when the investor stops active management and switch to passive management.

[2] This result depends on our choice of cost structure, which we here assume to be the investor's utility loss due to loss of leisure time and hardship involved with information acquisition and independent of wealth level. Therefore, due to economies of scale the investor manages her portfolio actively only if her wealth is large enough. Other costs

portant role in this decision; the more ambiguous the financial market is, the higher the threshold wealth level, therefore, the earlier the stopping time. The cost of active management is also an important factor in the decision; the larger the cost of active management, the higher the threshold wealth level is and the earlier the stopping time will be. Consequently, an investor with small wealth immediately chooses passive management. These results can give a partial explanation to the common observation that people with small wealth do not participate in the stock market (Bartaut and Haliassos 1995).

We also compare the optimal consumption and portfolio amount in our model with those of the classical problem in which there is no option to stop active management. A typical aspect about consumption as a function of wealth is that there is a big jump at the threshold wealth level. The consumption function is almost linear when the wealth is higher than the threshold level, and asymptotically approaches that of the classical problem for a very large wealth level. Optimal portfolio selection exhibits the same feature when the investor is rich enough, i.e., it approaches that of the classical problem. However, a very interesting portfolio behavior occurs when the wealth level is slightly greater than the critical wealth level. As the wealth level approaches down to the critical level, the investor takes much more risk than other investors who have no option to stop active management, so the optimal amount invested in risk assets actually increase as wealth declines to the critical wealth level. The investor is confronted with a situation that encourages her risk taking at a level of wealth that is slightly greater than the critical level. If the result of risk taking is successful then wealth will increase and the cost of active management will become small relative to the benefit of active management. If the result is a failure then she can simply switch to passive management and save the cost of active management.

Mathematically, our problem can be modelled by a mixture of an optimization problem with two control variables, consumption and a portfolio vector (c, π), and an optimal stopping problem with stopping time τ. If the investor has von Neumann-Morgenstern utility, the problem is simply expressed by

$$\sup_{(c,\pi,\tau)} \mathbf{E}\left[\int_0^\tau e^{-\beta t} u(c_t)\,dt + e^{-\beta\tau} U(X_\tau)\right]$$

of active management such as transaction costs are left for future study. For instance, if there are transactions costs that increase non-proportionally with the total value of assets under management, then there might be two threshold wealth levels between which the manager manages actively.

with the usual notation. Karatzas and Wang (2000) first studied this type of problem. They showed the existence of a value function and analyzed the stopping time problem by using a duality method of the martingale approach. Choi, Koo, and Kwak (CKK 2003) have extended the results when the agent has stochastic differential utility (SDU).

In this paper we apply CKK's model to study the problem of optimal switching from active management to passive management. In order to model ambiguity aversion we follow Chen and Epstein (2002) who extended the multiple-prior preferences of Gilboa and Schmeidler (1989). Chen and Epstein's utility with ambiguity aversion can be represented by stochastic differential utility that fits nicely into CKK's framework.

The portfolio selection problem goes back to Markowitz (1959). Samuelson and Merton (1969) and Merton (1969, 1971) first solved the multi-period model in continuous time by using dynamic programming. Pliska (1986), Cox and Huang (1989) and Karatzas (1989) have developed a general martingale approach to consumption and portfolio selection.

SDU was first introduced by Duffie and Epstein (1992) as a continuous time limit of recursive utility of Kreps and Porteus (1978) and Epstein and Zin (1989). The SDU is not temporally additive, and can differentiate between the coefficient of relative risk aversion and the elasticity of intertemporal substitution [3]. Duffie and Skiadas (1994), Schroder and Skiadas (1999), and Skiadas (1998) have studied a consumption-investment problem of an economic agent who has SDU. El Karoui, Peng and Quenez (1997, 2001) have developed the theory of the backward stochastic differential equations(BSDE) and solved the optimization problem with SDU using BSDE. Epstein and Wang (1994, 1995) have proposed the notion of ambiguity aversion and Chen and Epstein (2002) have shown that a preference with ambiguity aversion can be represented as SDU in continuous time framework.

The paper proceeds as follows. In Section 2, the model is given, where the general assumptions about the market, portfolio and consumption processes are specified and the general optimal stopping problem is defined. Section 3 defines the wealth and utility processes and briefly sketches the methods in Choi, Koo, and Kwak (2003). Section 4 considers the problem of optimal stopping of active portfolio management and derives the result discussed above. Section 5 concludes.

[3] Two coefficients are reciprocals of each other with time-separable von-Neumann and Morgenstein utility.

2. THE ECONOMIC MODEL

2.1. The Economy and The Investment Opportunity

Suppose there is an economic agent who are infinitely-lived. There is one consumption good in the economy. The investment opportunity set consists of an instantaneously risk-free asset, whose price P_t^0 evolves according to

$$dP_t^0 = P_t^0 r_t dt, \quad P_0^0 = p_0,$$

and m risky assets whose prices-per-share P_{it} evolve according to

$$dP_t^i = P_t^i \left[\mu_t^i dt + \sum_{j=1}^{m} \sigma_t^{ij} dW_t^j \right], \quad P_0^i = p_i, \quad i = 1, \ldots, m.$$

Here $W_t = (W_t^1, W_t^2, \ldots, W_t^m)$ is an m-dimensional standard Brownian motion on a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$ where $\mathcal{F} = \{\mathcal{F}_t\}_{\{0 \le t\}}$ is defined by $\mathcal{P}$-augmentation of the filtration generated by $\{W_t\}$.[4] The coefficients of the model, the interest rate r_t, the mean rate of return of each risky asset μ_t^i, the volatility matrix $\sigma_t = (\sigma_t^{ij})$ are all assumed to be bounded and progressively measurable with respect to $\mathcal{F}$. Also, σ_t is assumed to have a full rank almost surely. We now define the market-price-of-risk process,

$$\theta(t) \triangleq \sigma^{-1}(t)[\mu_t - r_t \mathbf{1}_m],$$

the discount process,

$$\zeta(t) \triangleq \frac{1}{P_{0t}} = \exp\left\{ -\int_0^t r_s \, ds \right\},$$

the exponential martingale process,

$$Z_0(t) \triangleq \exp\left\{ -\int_0^t \theta^*(s) \, dW_s - \frac{1}{2} \int_0^t \|\theta(s)\|^2 ds \right\},$$

where the superscript $*$ denotes the transpose of a matrix(or a vector) and the pricing kernel process,

$$H(t) \triangleq \zeta(t) Z_0(t).$$

We define a new probability measure

$$\tilde{P}(A) \triangleq \mathbf{E}[Z_0[T] 1_A].$$

[4]See Section 1.7 of Karatzas and Shreve (1998) for the measure theoretical detail on the infinite horizon setup of probability space.

By Girsanov Theorem the process

$$\tilde{W}_t = W_t + \int_0^t \theta(s)\, ds, \quad 0 \le t \le T$$

is a standard Brownian motion under the new probability measure $\tilde{\mathcal{P}}$. (See Karatzas and Shreve(1992))

2.2. Stochastic Differential Utility

Stochastic differential utility(SDU) was introduced by Duffie and Epstein (1992). In this section we extend their definition to the case where the final time is a random stopping time.

Let τ be a stopping time in S and Y be an $\mathcal{F}_\tau$-measurable random variable (Y is called a final reward). By the same argument as in El-karoui, Peng, and Quenez (1997) there exists a unique solution $\{Y_t, Z_t\}_{t=0}^{\tau}$ to the backward stochastic differential equation

$$dY_t = -f(t, c_t, Y_t, Z_t)dt + Z_t^* dW_T, \qquad Y_\tau = Y, \tag{1}$$

where $f : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^m \longrightarrow \mathbb{R}$ is a continuously differentiable function. f is called an intertemporal aggregator or a standard driver.[5] The investor's utility process is defined to be the first argument $\{Y_t\}_{t=0}^{\tau}$ of the above pair $\{(Y_t, Z_t)\}_{t=0}^{\tau}$. Equivalently, the pair is a unique solution satisfying the following equation

$$Y_t = \mathbf{E}_t \left[Y + \int_t^\tau f(s, c_s, Y_s, Z_s) ds \right]. \tag{2}$$

Intuitively, the investor's current value of utility is a function of current and future consumption and her future utility defined recursively from the final reward.

A special case is time-separable von Neuman-Morgenstern utility, that is, if $f = f(c, y) = u(c) - \beta y$ for $\beta > 0$, then (1) is reduced to

$$Y_t = \mathbf{E}_t \left[Y + \int_t^\tau e^{-\beta(s-t)} u(c_s)\, ds \right].$$

Another interesting case is SDU that exhibits ambiguity aversion. Let us consider SDU with the following intertemporal aggregator

$$f(c, y, z) = u(c) - \beta_t y - \alpha_t \cdot |z|,$$

[5] The assumptions on the aggregator f is provided in Section 3 of Choi, Koo, and Kwak (2003). The utility function used in Section 4 satisfies the assumptions. See Duffie and Epstein (1992), El-Karoui and Mazliak (1996), Pardoux and Peng (1999) Schroder and Skiadas (1999) for existence of a solution to the BSDE.

where β_t is a positive $\mathcal{F}_t$-adapted process and $\alpha_t = (\alpha_t^1, \alpha_t^2, \ldots, \alpha_t^m)$ the vector of positive $\mathcal{F}_t$-adapted processes. Here, $|z|$ denotes the m-dimensional vector with i-th component $|z_i|$. The utility process satisfies the following BSDE

$$dY_t = -(u(c_t) - \beta_t Y_t - \alpha_t \cdot |Z_t|)dt + Z_t^* \cdot dW_t, \quad Y_\tau = Y$$

This SDU was introduced by Chen and Epstein(2002). β_t is a subjective discount factor at time t and α_t^i is an ambiguity aversion parameter that shows aversion to ambiguity in the risk premium of the i-th risky factor W_t^i in the market.

2.3. The Optimization Problem

A consumption-portfolio plan of the investor is given by a pair (π, c) with a $\mathcal{F}$-progressively measurable portfolio process $\pi(\cdot) = (\pi_1(\cdot), \pi_2(\cdot), \ldots, \pi_m(\cdot))^*$ and a $\mathcal{F}$-progressively measurable consumption process $c(\cdot) \geq 0$ almost surely satisfying

$$\int_0^T c(t)\, dt + \int_0^T \|\pi(t)\|^2\, dt < \infty$$

for any $T \in [0, \infty)$. Each $\pi_i(t)$ represents the amount of the investor's wealth invested in the i-th stock at time t. Let X_t be the total wealth at time t. Then, $\pi_0(t) = X_t - \sum_{i=1}^m \pi_i(t)$ is the amount invested in the money market. We allow short-selling of assets. With initial wealth x, let $X_t = X_t^{(c,\pi,x)}$ denote the investor's wealth process corresponding to a given consumption plan (π, c). If the market has no friction, the wealth process of an agent with initial wealth x has the following dynamics:

$$dX_t^{(c,\pi,x)} = (r_t X_t^{(c,\pi,x)} + \pi_t^* \sigma_t \theta_t - c_t)dt + \pi_t^* \sigma_t dW_t. \tag{3}$$

A consumption-portfolio plan (c, π) is called *admissible* with initial wealth x if $c(t) \geq 0$ and $X_t^{(c,\pi,x)} \geq 0$ almost surely for all $t \in [0, T]$.

For the linear wealth process (3) the investor's wealth at time t satisfies

$$\zeta(t) X_t^{(c,\pi,x)} = x - \int_0^t \zeta(s) c(s)\, ds + \int_0^t \zeta(s) \pi^*(s) \sigma(s)\, d\tilde{W} \tag{4}$$

For every (π, c) the third term on the right-hand side is continuous, $\tilde{\mathcal{P}}$-local martingale bounded below and thus a super-martingale by Fatou's lemma. Then, the optional sampling theorem implies

$$\mathbf{E}\left[H(\tau) X^{(c,\pi,x)}(\tau) + \int_0^\tau H(s) c(s)\, ds\right] \leq x \tag{5}$$

for every $\tau \in S$ where S denotes the set of all $\mathcal{F}$-stopping times.

We now state the investor's optimization problem. The investor maximizes her utility Y_0 in (1) over the consumption-portfolio-stopping plan $(c,\pi,\tau) \in \mathcal{B}(x)$, where $\mathcal{B}(x)$ denotes the set of triples (c,π,τ) such that

$$\mathbf{E}\left[\int_0^\tau |f(t,c_t,Y_t,Z_t)|\,dt + |h(X_\tau^{(c,\pi)})|\right] < +\infty$$

with $0 \le X_t^{(c,\pi,x)}$ $(0 \le t \le \tau)$. The final reward is given as $Y = h(X_\tau)$. The value function $V(x)$ is defined by

$$V(x) = \sup_{(c,\pi,\tau)\in\mathcal{B}(x)} Y_0. \tag{6}$$

Therefore, the investor's problem is the choice of the optimal stopping time τ and consumption and portfolio process $\{c_t,\pi_t\}_{t=0}^{\tau}$. However, the forward and backward comparison theorems imply that the nonnegative constraint of the wealth process is equivalent to the nonnegative constraint of the wealth at the stopping time(See El-Karoui, Peng, and Quenez (1997, 2001)). Thus, we can reformulate the problem to the choice of the optimal stopping time τ and the wealth ξ at τ and $\{c_t\}_{t=0}^{\tau}$. We will need the following definition in order to do so.

Definition 2.1. Let ξ is a nonnegative $\mathcal{F}_\tau$-measurable random variable. A stopping-consumption plan (ξ,c,τ) is called feasible for initial wealth x if $X_0^{(\xi,c,\tau)} \le x$, where the pair $\{(X_t^{(\xi,c,\tau)},\pi_t)\}_{t=0}^{\tau}$ is a solution to the following BSDE

$$dX_t = (r_tX_t + \pi^*\sigma_t\theta_t - c_t)dt + \pi_t^*\sigma_t dW_t, \quad 0 \le t \le \tau, \quad X_\tau = \xi. \tag{7}$$

We denote by $\mathcal{A}(x)$ the set of the feasible stopping-consumption plans for initial wealth x. We also denote by $Y^{(\xi,c,\tau)}$ the utility value of $(\xi,c,\tau) \in \mathcal{A}(x)$, i.e., the pair $\{(Y_t^{(\xi,c,\tau)},Z_t)\}_{t=0}^{\tau}$ is a solution to the following BSDE

$$dY_t = -f(t,c_t,Y_t,Z_t)dt + Z_t^* dW_t, \quad 0 \le t \le \tau, \quad Y_\tau = h(\xi). \tag{8}$$

Now by the backward comparison theorem the value function (6) can be rewritten as

$$V(x) = \sup_{(\xi,c,\tau)\in\mathcal{A}(x)} Y_0^{(\xi,c,\tau)}. \tag{9}$$

3. A DUAL CHARACTERIZATION OF THE VALUE FUNCTION

In this section we briefly sketch the existence result of Choi, Koo, and Kwak (2003). The general setting in the problem is Non-Markovian, and thus the problem is solved by a martingale method or a duality approach. They adopted the forward-backward SDE technique and the dynamic maximum principle (see El Karoui, Peng and Quenez (1997, 2001)) to prove the existence of the value function in the mixture of an optimal consumption and portfolio selection problem and an optimal stopping problem with SDU. They also have the sufficient condition to do this procedure. (Theorem 1) We skip the assumptions on the standard driver f, but the existence results of this section are valid for the analysis in Section 4. Readers can check the utility function used in Section 4 satisfies the assumptions on the standard drivers provided in Section 3 of Choi, Koo, and Kwak (2003)

We first fix a stopping time and solve a dual minimization problem, and second, find an optimal stopping time. Let $\tau \in S$ be a fixed stopping time and let $\Pi_\tau(x)$ be the set of feasible consumption plans (ξ, c) for which $(\xi, c, \tau) \in \mathcal{A}(x)$. Let us consider the following consumption and portfolio selection problem that has a fixed stopping time τ:

$$V(x;\tau) = \sup_{(\xi,c)\in\Pi_\tau} Y_0^{(\xi,c,\tau)}. \tag{10}$$

To get the solution, we consider the following dual optimization problem: for $\nu > 0$,

$$\tilde{V}(\nu;\tau) \stackrel{\triangle}{=} \sup_{(\xi,c)\in\mathcal{L}_\tau\times\mathcal{D}_\tau} J_\tau(\xi, c; \nu), \tag{11}$$

[6]where the functional J_τ is defined by

$$J_\tau(\xi, c; \nu) = Y_0^{(\xi,c,\tau)} - \nu X_0^{(\xi,c,\tau)}. \tag{12}$$

By the standard convex duality argument we have

$$V(x;\tau) = \inf_{\nu>0}(\tilde{V}(\nu;\tau) + \nu x). \tag{13}$$

Since we have the optimal value function (13) corresponding to each τ, i.e the budget constraint is binding for each τ, we finally have

$$V(x) = \sup_{\tau\in S_T} V(x;\tau). \tag{14}$$

[6]See Choi, Koo, and Kwak (2003) for the definition of the space $\mathcal{L}_\tau \times \mathcal{D}_\tau$.

The above procedure is intuitively appealing and easy to understand. However, there are two weakness in doing (13) and (14) equentially. First, as Karatzas and Wang (2000) have pointed out it is difficult to compute the value function explicitly by the above procedure even in simple cases (e.g., the investor has log utility) since usually $V(x;\tau)$ has a complicated form. Secondly, the most important feature of the duality approach is to make an dynamic budget constraint as an static budget constraint. (see Cox and Huang (1989)) However, if we consider an investment problem together with an optimal stopping problem, then we must generate a infinitely many static budget constraints and each one of them depends on a given stopping time τ. Consequently, a Lagrange multiplier ν may differ by each τ, which in turn to lose the advantage of the martingale method. This difficulty can be resolved by reducing the problem to the family of optimal stopping problems for each ν.

Now we present a different procedure that is more amenable to explicit calculation of the value function. For a fixed Lagrange multiplier $\nu \in (0,\infty)$ we define a family of *dual optimal stopping problems* as follows:

$$\tilde{V}(\nu) \stackrel{\triangle}{=} \sup_{\tau\in\mathcal{S}} \tilde{V}(\nu;\tau). \tag{15}$$

We denote by $\hat{S}_\nu$ the set of stopping times attaining supremum in (15) for each given ν. We can easily see that

$$\inf_{\nu>0}(\tilde{V}(\nu;\tau)+\nu x) \le \tilde{V}(\lambda;\tau)+\lambda x, \quad \forall\lambda>0.$$

Taking sup with respect to τ,

$$\sup_{\tau\in\mathcal{S}}\inf_{\nu>0}(\tilde{V}(\nu;\tau)+\nu x) \le \sup_{\tau\in\mathcal{S}}\tilde{V}(\lambda;\tau)+\lambda x, \quad \forall\lambda>0.$$

Hence, we have

$$V(x) \le \inf_{\lambda>0}\sup_{\tau\in\mathcal{S}}(\tilde{V}(\lambda;\tau)+\lambda x) = \inf_{\lambda>0}(\tilde{V}(\lambda)+\lambda x). \tag{16}$$

If the inequility in (16) is indeed an equality, then we can compute the value function by using the right-hand sides of (16). However, the inequality in (16) may be strict. In the following we will characterize the conditions under which the inequality in (16) becomes an equality. We will need the following definition.

Definition 3.1. For each given $\nu>0$ we define a set $\mathcal{G}_\nu$ by

$$\mathcal{G}_\nu = \{X_0^{(\xi_\nu,c_\nu,\hat{\tau}_\nu)} | \quad \hat{\tau}_\nu \quad \text{is optimal in} \quad (15), \quad i.e.\ \tilde{V}(\nu)=\tilde{V}(\nu;\hat{\tau}_\nu)\},$$

where (ξ_ν, c_ν) is optimal plan for (11) corresponding to ν, i.e (ξ_ν, c_ν) satisfies (A.3) and (A.4) that are provided in Appendix A. Let $\mathcal{G} = \cup_{\nu>0}\mathcal{G}_\nu$.

Finally, we now state the following theorem that is an extension of Karatzas and Wang (2000) and gives conditions under which the inequality in (16) becomes an equality.

THEOREM 1. *For any $x \in \mathcal{G}$, there exists an optimal consumption, portfolio and stopping plan that achieves the supremum value $V(x)$ in (9) and we have*

$$V(x) = \inf_{\nu>0}(\tilde{V}(\nu) + \nu x). \tag{17}$$

Conversely, suppose there exist $x \in (0,\infty)$ and (ξ^*, c^*, τ^*) satisfying (17). Then, we have $x \in \mathcal{G}$ provided that

$$\lim_{\lambda\downarrow 0} D^+\tilde{V}(\lambda) = -\infty. \tag{18}$$

COROLLARY 1. *For any $x \notin \mathcal{G}$, we have the strict equality("duality gap'):*

$$V(x) < \inf_{\nu>0}(\tilde{V}(\nu) + \nu x).$$

COROLLARY 2. *If $\tilde{V}(\nu)$ is differentiable for all $\nu > 0$, then for every $x \in (0,\infty)$ there exists an optimal plan and (17) is valid.*

Theorem 1 provides a necessary and sufficient condition under which the order of two operations, sup with respect to τ and inf with respect to ν, can be changed. Corollary 2 provides the validity of our analysis in Section 4.

4. ACTIVE AND PASSIVE PORTFOLIO MANAGEMENT

In this section we provide explicit solutions to the optimal consumption, portfolio selection and stopping problem of an investor who considers to stop managing her portfolio actively and switch to passive management. The investor suffers utility loss when she manages her portfolio actively, because she has to spend time and make efforts to study factors affecting the market and monitor the market price movements and re-balance her portfolio as the market condition changes. The utility cost can be avoided if she stops active management and adopts a passive portfolio strategy of putting her money into a third party manager or a bank account.

The investor is averse to ambiguity about the market as in Chen and Epstein(2002) (See Appendix B for the brief sketch of idea on recursive multiple priors), and has SDU with the following intertemporal aggregator:

$$f(c, y, z) = u(c) - l(c, y, z) - \beta y - \alpha \cdot |z|,$$

where u is a real-valued concave function, l is a real-valued function representing utility loss due to costs of active management, β is a positive constant and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ is a vector consisting of positive constants such that

$$0 \leq \alpha_i \leq \theta_i \qquad i = 1, 2, \ldots, m.,$$

where θ is the market-price-of-risk vector. Here, $|z|$ denotes the vector whose i-th component is $|z_i|$. The investor's risk aversion is described by the concavity of function u, and her aversion toward ambiguity in the i-th risk factor W_t^i is provided by α_i. Then, the utility process evolves according to

$$dY_t = -(u(c_t) - l(c_t, Y_t, Z_t) - \beta Y_t - \alpha \cdot |Z_t|)dt + Z_t^* \cdot dW_t, \tag{19a}$$

$$Y_\tau = h(X_\tau). \tag{19b}$$

One choice of the investor's utility after stopping active management may be given by an aggregator

$$f(c, y, z) = u(c) - \beta y - \alpha \cdot |z|$$

that is equal to the aggregator before switching except for the fact that there is no utility loss. Since the investor does not manage the investment portfolio actively, she does not incur utility loss. However, we do not model the market environments for passive management explicitly in this paper except for the case where the investor's passive management consists of investing only in the risk-free asset. Rather we will assume that the investor knows her value function that will be available after switching to passive management. Therefore, in (4.1) we have assumed that the reduced form of the investor's value function after stopping active management will be provided by $h(X)$.

4.1. The Optimization Problem

In this section [7] $m = 1$, all market coefficients are assumed to be constants (i.e. $r_t = r, \mu_t = \mu, \sigma_t = \sigma$, and $\theta_t = \theta$). Also, we assume that $l(c, y, z)$ to be a constant l for simplicity.

The problem is to solve

$$V(x) = \sup_{(c,\pi,\tau)\in\mathcal{B}(x)} Y_0 \tag{20}$$

with the dynamic budget constraint

$$dX(t) = (rX(t) + \pi_t\sigma\theta - c_t)\,dt + \pi_t\sigma dW_t.$$

To apply the martingale method, we reformulate the problem as Section 3. (20) can be rewritten as

$$V(x) = \sup_{(\xi,c,\tau)\in\mathcal{A}(x)} Y_0^{(\xi,c,\tau)}$$

subject to the static budget constraint $X^{(\xi,c,\tau)} \leq x$, which is equivalent to

$$\mathbf{E}\left[H(\tau)\xi + \int_0^\tau H(s)c(s)\,ds\right] \leq x.$$

Subtracting βY_t, multiplying by $e^{-\beta t}$ in both sides of (19), and applying Ito's lemma, we finally have

$$d(e^{-\beta t}Y_t) = -e^{-\beta t}(u(c_t) - l)dt + e^{-\beta t}(\alpha|Z_t|dt + Z_t dW_t). \tag{21}$$

By section 5 of Chen and Epstein (2002) it can be shown that $|Z_t| = Z_t$ for all $0 \leq t \leq \tau$ at optimum for each fixed τ. [8]

By Girsanov theorem we define a new probability measure P^α

$$P^\alpha(A) \stackrel{\triangle}{=} \mathbf{E}[\Gamma[T]1_A],$$

where a exponential martingale process

$$\Gamma(t) \stackrel{\triangle}{=} \exp\left\{-\alpha W(t) - \frac{1}{2}\alpha^2 t\right\}$$

[7]The number of risky assets are assumed to be 1 in this section, but it's easy to extend all the results to the case where there exist m risky assets. Moreover, the results can be extended to the case where the market coefficients are not just constants, but are bounded. See Choi, Koo, and Kwak(2003)

[8]If $\theta > 0$, then Z_t has same sign as V_t. See Section 5 of Chen and Epstein (2002).

and a Brownian motion $\tilde{W}^{\alpha}$ under P^{α}

$$\tilde{W_t}^{\alpha} = \alpha t + W_t.$$

Integrating both sides of equation (21) from t to τ and taking expectation with respect to P^{α}, we have the following equation equivalent to (19):

$$Y_t = \mathbf{E}_{\alpha}\left[\int_t^{\tau} e^{-\beta(s-t)}(u(c_s) - l)\,ds + e^{-\beta(\tau-t)}h(X_{\tau})|\mathcal{F}_t\right], \qquad (22)$$

where the $\mathbf{E}_{\alpha}$ denotes the expectation with respect to P^{α}-measure. Bayes' rule implies that (22) is equivalent to

$$Y_t^{(c,\xi,\tau)} = \mathbf{E}_t\left[\int_t^{\tau} e^{-\beta(s-t)}\Gamma(s)(u(c_s) - l)\,ds + e^{-\beta(\tau-t)}\Gamma(\tau)h(\xi)\right].$$

We now have a functional $J_{\tau}(\xi, c; \nu)$ for $\nu > 0$ as in (12)

$$\begin{aligned} J_{\tau}(\xi, c; \nu) = \mathbf{E}&\left[\int_0^{\tau} e^{-\beta s}\Gamma(s)(u(c_s) - l)\,ds + e^{-\beta\tau}\Gamma(\tau)h(\xi)\right] \\ -\nu\mathbf{E}&\left[\int_0^{\tau} H(s)c(s)ds + H(\tau)\xi\right]. \end{aligned}$$

From (11) we have

$$\begin{aligned} \tilde{V}(\nu;\tau) = \mathbf{E}&\left[\int_0^{\tau} e^{-\beta s}\Gamma(s)\tilde{u}(\nu e^{\beta s}H(s)\Gamma^{-1}(s))\,ds\right] \\ &+ \mathbf{E}\left[e^{-\beta\tau}\Gamma(\tau)(\tilde{h}(\nu e^{\beta\tau}H(\tau)\Gamma^{-1}(\tau)) + \frac{l}{\beta})\right] - \frac{l}{\beta} \end{aligned}$$

where $\tilde{u}$ and $\tilde{h}$ are the convex conjugates of u and h.[9] Under the P^{α}-measure, we rewrite the above equation as

$$\begin{aligned} \tilde{V}(\nu;\tau) = \mathbf{E}_{\alpha}&\left[\int_0^{\tau} e^{-\beta s}\tilde{u}(\nu e^{\beta s}H(s)\Gamma^{-1}(s))\,ds\right] \\ &+ \mathbf{E}_{\alpha}\left[e^{-\beta\tau}(\tilde{h}(\nu e^{\beta\tau}H(\tau)\Gamma^{-1}(\tau)) + \frac{l}{\beta})\right] - \frac{l}{\beta} \end{aligned} \qquad (23)$$

[9]The convex conjugate $\tilde{g}$ of a function g is defined by

$$\tilde{g}(y) = \sup_{x>0}(g(x) - xy)$$

By (A.3) and (A.4) or by a usual duality method, the optimal plan $(\hat{c}_\nu, \hat{\xi}_\nu)$ corresponding to ν for (11) is given by

$$\hat{c}_\nu(t) = -\tilde{u}'(\nu e^{\beta t}\Gamma^{-1}(t)H(t)), \quad 0 \le t \le \tau, \tag{24}$$

$$\hat{\xi}_\nu = -\tilde{h}'(\nu e^{\beta t}\Gamma^{-1}(\tau)H(\tau)). \tag{25}$$

we consider a more concrete case and analyze the optimal stopping time. In the example u and h will take the following form,

$$u(x) = \frac{x^{1-\gamma}}{1-\gamma}, \quad h(x) = \frac{x^{1-\gamma}}{K_R^{\gamma}(1-\gamma)},$$

where γ is a risk-aversion parameter for intermediate consumption and final wealth respectively with $0 < \gamma < 1$.[10] K_R is determined by the investor's choice of passive management at the stopping time. For example, if she decides to put her money to a mutual fund, then K_R is dependent on the characteristic of the fund. Then, the convex functions are

$$\tilde{u}(y) = \frac{\gamma}{1-\gamma} y^{-\frac{1-\gamma}{\gamma}}, \quad \tilde{h}(y) = \frac{\gamma}{K_R(1-\gamma)} y^{-\frac{1-\gamma}{\gamma}}$$

Here we assume

$$\tilde{K} \triangleq r + \frac{\beta - r}{\gamma} + \frac{(\gamma-1)\theta^2}{2\gamma^2} > 0.$$

This is a condition under which the classical Merton problem has a solution when $\alpha = 0$ (See Merton (1969)).

4.2. Free Boundary Value Problem

First, we try to obtain a solution by solving (13) and (14). From (23) we have

$$\tilde{V}(x;\tau) = \inf_{\nu>0}\left(\frac{\gamma}{1-\gamma}K_\tau \nu^{-\frac{1-\gamma}{\gamma}} + \nu x + \frac{l}{\beta}\mathbf{E}_\alpha[e^{-\beta\tau}] - \frac{l}{\beta}\right),$$

where

$$K_\tau = \mathbf{E}_\alpha\left[\int_0^\tau e^{-\beta s}\left(e^{\beta s}H(s)\Gamma^{-1}(s)\right)^{-\frac{1-\gamma}{\gamma}} ds + \frac{e^{-\beta\tau}}{K_R}\left(e^{\beta\tau}H(\tau)\Gamma^{-1}(\tau)\right)^{-\frac{1-\gamma}{\gamma}}\right].$$

Then, by some computation we have

$$\tilde{V}(x;\tau) = \frac{K_\tau^{\gamma} x^{1-\gamma}}{1-\gamma} + \frac{l}{\beta}\mathbf{E}_\alpha[e^{-\beta\tau}] - \frac{l}{\beta}. \tag{26}$$

[10] We require the condition $0 < \gamma < 1$ to satisfy the technical conditions required for the existence of solution in Choi, Koo, and Kwak (2003). The study of the case where $1 < \gamma$ is left for future research.

But, it is difficult to get an explicit solution to the following optimal stopping problem since (26) has a very complicated form as following:

$$V(x) = \sup_{\tau} \tilde{V}(x;\tau).$$

Now, we use Theorem 3.1. To find the optimal stopping time first, we need to solve a free boundary value problem.

We rewrite (23) as

$$\tilde{V}(\nu;\tau) = \mathbf{E}_\alpha \left[\int_0^\tau e^{-\beta s} \tilde{u}(y_t^\nu)\, ds + e^{-\beta\tau} (\tilde{h}(y_\tau^\nu) + \frac{l}{\beta}) \right] - \frac{l}{\beta}, \tag{27}$$

where we define a new process $y_t^\nu = \nu e^{\beta t} H(t) \Gamma^{-1}(t)$. Observe that y_t^ν is a solution of the following SDE with an initial condition $y_0 = \nu$:

$$dy_t = y_t \left[(\beta - r) dt - (\theta - \alpha)\, d\tilde{W}_t^{\alpha} \right]. \tag{28}$$

Note (28) is defined under P^α-measure.

The optimal stopping time can be found by solving the following optimal stopping problem

$$v(t,y) = \sup_{\tau > t} \mathbf{E}_\alpha^{y_t = y} \left[\int_t^\tau e^{-\beta s} \tilde{u}(y_t^\nu)\, ds + e^{-\beta\tau} (\tilde{h}(y_\tau^\nu) + \frac{l}{\beta}) \right], \tag{29}$$

where we adopt the expectation symbol $\mathbf{E}_\alpha^{y_t = y} = \mathbf{E}_\alpha^y$.

Now we apply the method of variational inequality. Consider the differential operator

$$\mathcal{L} = \frac{\partial}{\partial t} + \frac{1}{2}(\theta - \alpha)^2 y^2 \frac{\partial^2}{\partial y^2} + (\beta - r) y \frac{\partial}{\partial y}.$$

acting on a function $w : \Omega \longrightarrow \mathbb{R}$, $\Omega = (0,\infty) \times \mathbb{R}$. The solution to the next variational inequality (free boundary value) problem will be a solution to our optimal stopping problem.

Variational Inequality 1. *Find a positive number $\bar{y} > 0$ and a function $w(\cdot,\cdot) \in C^1(\Omega) \cap C^2((0,\infty) \times (\mathbb{R} \setminus \{\bar{y}\}))$ satisfying*

(V.1) $\mathcal{L}w + e^{-\beta t}\tilde{u}(y) = 0, \quad 0 < y < \bar{y}$

(V.2) $\mathcal{L}w + e^{-\beta t}\tilde{u}(y) \le 0, \quad \bar{y} \le y$

(V.3) $w(t,y) \ge e^{-\beta t}(\tilde{h}(y) + \frac{l}{\beta}), \quad 0 < y \le \bar{y}$

(V.4) $w(t,y) = e^{-\beta t}(\tilde{h}(y) + \frac{l}{\beta}), \quad \bar{y} \ge y$

(V.5) $w(t,y) > 0, \quad y > 0$

for all $t > 0$.

PROPOSITION 1. *Let* λ *be the positive root of the quadratic equation*

$$\frac{1}{2}(\theta-\alpha)^2\lambda^2 + (\beta - r - \frac{1}{2}(\theta-\alpha)^2)\lambda - \beta = 0. \tag{30}$$

Set

$$\bar{y} = \left(\frac{K_R K}{(K_R - K)}\frac{\lambda(1-\gamma)l}{(1-\gamma+\lambda\gamma)\beta}\right)^{-\frac{\gamma}{1-\gamma}} \tag{31}$$

and consider the function

$$\phi(y) = \begin{cases} Cy^\lambda + \frac{1}{K}\frac{\gamma}{1-\gamma}y^{-\frac{1-\gamma}{\gamma}}, & 0 < y \le \bar{y} \\ \frac{\gamma}{K_R(1-\gamma)}y^{-\frac{1-\gamma}{\gamma}} + \frac{l}{\beta}, & \bar{y} \le y \end{cases}$$

where C *and* K *are given by*

$$C = \left(\frac{(1-\gamma)l}{(1-\gamma+\lambda\gamma)\beta}\right)\bar{y}^{-\lambda}. \tag{32}$$

and

$$K = r + \frac{\beta - r}{\gamma} + \frac{(\gamma-1)(\theta-\alpha)^2}{2\gamma}.$$

Suppose $K < K_R$ *and*

$$\frac{\lambda\gamma K}{(1-\gamma+\lambda\gamma)\beta} \le 1 \tag{33}$$

Then, $w(t,z) = e^{-\beta t}\phi(y)$ *is the solution to Variational Inequality 1.*

Proof. See Appendix C. ∎

Remark 4.1. The quadratic equation (30) has one positive and one negative roots. Also, $\lambda > 1$ is easily checked.

Note that $\tilde{K} < K$ if the market is ambiguous to the investor, i.e, $\alpha \neq 0$. Thus, $K > 0$. If $K > K_R$, then algebraically $\bar{y} < 0$. This implies the free boundary does not exist and the optimal stopping time is trivially determined, i.e, $\tau = 0$.(See Theorem 2.) By a reason similar to the above, $K = K_R$ yields the trivial result.

This result illustrates the case when the investor has an option to put her money to a manager to whom the market is less ambiguous. Let $K_{\alpha_i}, i = 1, 2$ denotes the K-value corresponding to α_i. $\alpha_1 > \alpha_2$ is equivalent to $K_{\alpha_1} > K_{\alpha_2}$. So, if $K = K_{\alpha_1}$ and $K_R = K_{\alpha_2}$, then $\tau = 0$, in other words, as soon as the investor can hire a manager who knows the market better than she does and charges no management fees, she has no reason to manage actively. However, since the manager ordinarily charges fees to the investor, it is not correct to say that $K > K_R$ even if the hired manager knows the market better than the investor

By the above reason we do not consider the case $K \geq K_R$ in Proposition 1 since it yields a trivial result.

Note that the free boundary value (31) is obtained from the smooth-pasting principle. We have the following verification theorem.

THEOREM 2. *Suppose that the pair $(\bar{y}, w(t, y))$ solves the Variational Inequality 1. Then, $w(t, y)$ coincides with the value function $v(t, y)$ of (29) with the optimal stopping time is given by*

$$\tau_\nu = \inf\{\ s > t \quad | \quad y_s \geq \bar{y} \quad \}. \tag{34}$$

Proof. The proof is given by applying Theorem 10.4.1 of Oksendal (1998) directly. ▮

4.3. The Value Function and the Optimal Plan

By the result of Theorem 2 we can get the optimal solution. Since $\tilde{V}(\nu)$ is the value function $v(t, y)$ at $t = 0, y = \nu$, we have

$$\tilde{V}(\nu) = \phi(\nu) - \frac{l}{\beta},$$

where ϕ is provided in Proposition 1. The optimal stopping time is

$$\tau_{\nu^*} = \inf\{\ t > 0 \mid \quad y_t \geq \bar{y} \quad \}. \tag{35}$$

Note that $\tilde{V}(\cdot)$ is differentiable and

$$\tilde{V}'(\nu) = \begin{cases} C\lambda\nu^{\lambda-1} - \frac{1}{K}\nu^{-\frac{1}{\gamma}}, & 0 < \nu \leq \bar{y} \\ -\frac{1}{K_R}\nu^{-\frac{1}{\gamma}}, & \bar{y} \leq \nu \end{cases}$$

The optimal value $V(x)$ is obtained at the value $\nu^* > 0$ fitting $\tilde{V}'(\nu^*) = -x$ by the Envelope condition. Thus, we have the following result.

THEOREM 3. *Set $\bar{x} = \frac{1}{K_R}\bar{y}^{-\frac{1}{\gamma}}$. Then, we obtain the value function to problem (20)*

$$V(x) = \begin{cases} C(\nu^*)^\lambda + \frac{\gamma}{1-\gamma}\frac{1}{K}(\nu^*)^{-\frac{1-\gamma}{\gamma}} + \nu^* x - \frac{l}{\beta}, & x \geq \bar{x} \\ \frac{1}{K_R^\gamma(1-\gamma)}x^{1-\gamma}, & 0 < x \leq \bar{x}, \end{cases} \tag{36}$$

where ν^* is the solution to the following algebraic equation

$$-C\lambda\nu^{\lambda-1} + \frac{1}{K}\nu^{-\frac{1}{\gamma}} = x. \tag{37}$$

Remark 4.2. It can be easily seen that one-to-one correspondence between $\nu^* \in (0, \bar{y})$ and $x \in (\bar{x}, \infty)$ at (37).

From (24) and (35) we have the optimal consumption plan.

$$\hat{c}_{\nu^*}(t) = (y_t^{\nu^*})^{-\frac{1}{\gamma}}, \quad 0 \leq t \leq \tau_{\nu^*},$$

where $y_t^{\nu^*}$ denotes the solution to SDE (28) with an initial condition $y_0 = \nu^*$. To obtain the optimal portfolio process we consider the Hamilton-Jacobi-Bellman (HJB) equation(Proposition 2). Since we already have the free boundary value $\bar{x}$ by solving Variational inequality 1 and duality relation between the lagrange multiplier ν and initial wealth x, we only need to consider the HJB equation defined on the domain $\{x : \bar{x} < x\}$.

PROPOSITION 2. *The function (36) with (37) satisfies the following boundary value problem for HJB equation of dynamic programming,*

$$\beta V(x) = \max_{(c\geq 0,\pi)}\left[\frac{1}{2}\pi^2\sigma^2 V''(x) + (rx + \pi(\theta-\alpha) - c)V'(x) + u(c) - l\right], \quad \bar{x} < x \tag{38}$$

with the boundary condition $V(\bar{x}) = h(\bar{x})$. The optimal controls in (38) are given

$$c^*(x) = (u')^{-1}(V'(x)) = (\nu^*)^{-\frac{1}{\gamma}} \quad \text{and} \tag{39}$$

$$\pi^*(x) = -\frac{(\theta-\alpha)V'(x)}{\sigma V''(x)} = -\frac{(\theta-\alpha)\nu^*}{\sigma\,(d\nu^*/dx)}, \tag{40}$$

where ν^* is the solution of (37).

Proof. See Appendix D. ∎

Remark 4.3. Note that from (37) we have

$$V'(x) = C\lambda(\nu^*)^{\lambda-1}\frac{d\nu^*}{dx} - \frac{1}{K}\frac{d\nu^*}{dx} + x\frac{d\nu^*}{dx} + \nu^* = \nu^*$$

and thus

$$V''(x) = \frac{d\nu^*}{dx} = -\left(C\lambda(\lambda-1)(\nu^*)^{\lambda-2} + \frac{1}{\gamma K}(\nu^*)^{-\frac{1-\gamma}{\gamma}}\right)^{-1}$$

in Proposition 2. Thus, (40) is reduced to

$$\pi^*(x) = \frac{\theta-\alpha}{\sigma}\left(\frac{1}{\gamma K}(\nu^*)^{-\frac{1}{\gamma}} + C\lambda(\lambda-1)(\nu^*)^{\lambda-1}\right).$$

Our setting is time-homogeneous Markovian, thus substituting $y^{\nu^*}(t)$ into (37) instead ν yields the optimal wealth process

$$X^*(t) = -C\lambda(y_t^{\nu^*})^{\lambda-1} + \frac{1}{K}(y_t^{\nu^*})^{-\frac{1}{\gamma}}, \quad 0 \le t \le \tau_{\nu^*} \tag{41}$$

with $X^*(0) = x$.

THEOREM 4. *The value function $V(x)$ of (36) to the problem (20) attains at (π^*, c^*, τ^*) such that*

$$c_t^* = (y_t^{\nu^*})^{-\frac{1}{\gamma}}, \quad 0 \le t \le \tau^*$$

and

$$\pi_t^* = \frac{\theta-\alpha}{\sigma}\left(\frac{1}{\gamma K}(y_t^{\nu^*})^{-\frac{1}{\gamma}} + C\lambda(\lambda-1)(y_t^{\nu^*})^{\lambda-1}\right), \quad 0 \le t \le \tau^*$$

with

$$\tau^* = \inf\{\ t > 0 \mid \quad X_t^* \le \bar{x} \quad \}, \tag{42}$$

where $X^*(t)$ are the optimal wealth process (41).

Proof. See Appendix D. ∎

Equation (42) has several meanings. We observe that $l = 0$ implies $\bar{x} = 0$. In this case the optimal stopping time does not exist, since $X_t > 0$

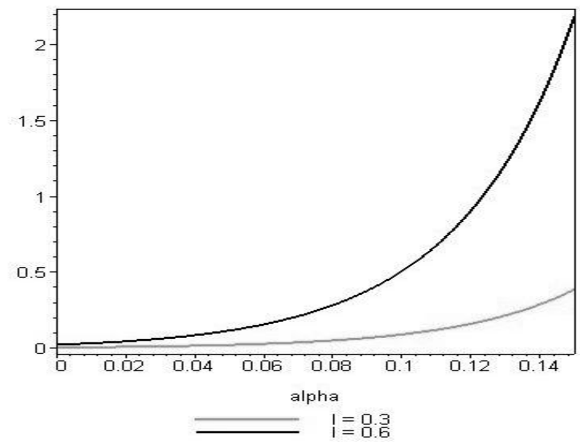

FIG. 1. The graph of the critical wealth level as a function of α: values of α from 0 to 0.15 are shown on the horizontal axis. The lower curve is for $l = 0.3$ and the upper curve is for $l = 0.6$. ($\mu = 0.12, \sigma = 0.20, r = 0.04, \beta = 0.10$, and $\gamma = 0.6$. K_R value is that of consumption-saving problem.)

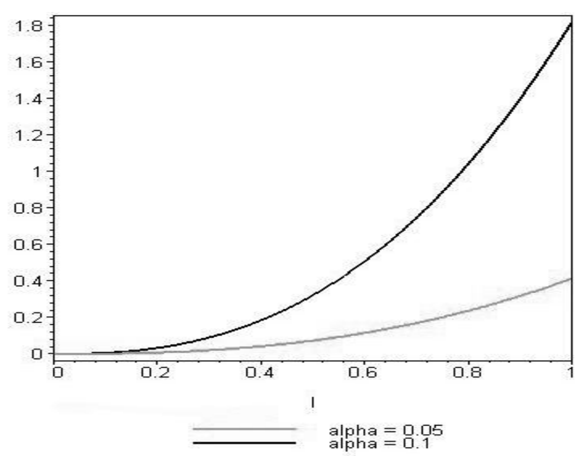

FIG. 2. The graph of critical wealth level as a function of l: values of l from 0 to 1 are shown on the horizontal axis. The lower curve is for $\alpha = 0.05$ and the upper curve is for $\alpha = 0.1$. (Parameter values are same as those in Figure 1)

almost surely at the optimum for the classical infinite horizon problem with no control of stopping time.(Of course, we are considering the case when $K < K_R$. See the statements following Remark 4.1.) The investor never

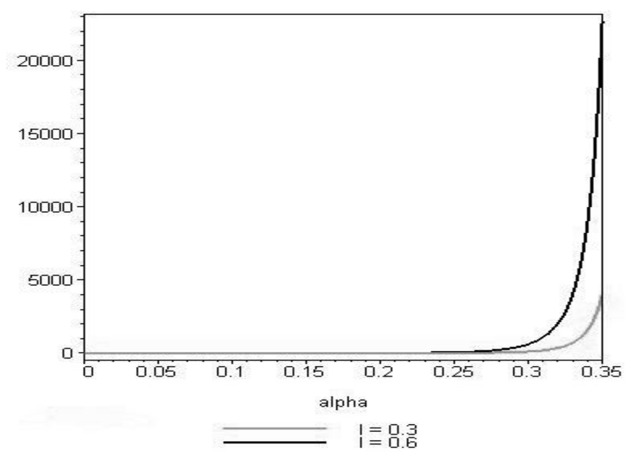

FIG. 3. The graph of the critical wealth level as a function of α: values of α from 0 to 0.35 are shown on the horizontal axis. We enlarge the scale of α-axis. Parameter values are the same as those in previous figures. Note that $\theta = 0.4$ with the parameter values. As $\alpha \to \theta$, we can check that the critical wealth level approaches infinity.

stops. But, the presence of utility loss from active management (i.e, the condition $l > 0$) prevents an investor whose initial wealth is less than $\bar{x}$ from managing actively. It is optimal for a rich investor to manage actively until her wealth level drops down to $\bar{x}$. Our model is quite simple in the sense that l is constant, but it gives an interesting result that an investors with small wealth prefers passive management and an investor with large wealth prefers active management.

As a simple example, let us consider the case where the investor decides to put her money to a bank account after stopping active management, i.e, the investor faces the consumption-saving problem after stopping. In this case, we have

$$K_R = r + \frac{\beta - r}{\gamma} > 0. \tag{43}$$

Then, $K < K_R$ holds whenever $\theta - \alpha > 0$. In this case, it is optimal for an investor having money less than $\bar{x}$ stops active management immediately and puts her money in a bank account. That is, an investor with small wealth does not invest in risky assets at all. This simple example may provide an explanation of why many people do not participate in the stock market (see ,e.g., Bartaut and Haliassos 1995).

From equation (42) we also observe that ambiguity has a crucial effect on the investor's decision. From (31),

$$\bar{x} = \frac{1}{K_R}\left(\frac{K_R K}{(K_R - K)}\frac{\lambda(1-\gamma)l}{(1-\gamma+\lambda\gamma)\beta}\right)^{\frac{1}{1-\gamma}}. \tag{44}$$

We can regard $\bar{x}$ as an increasing function of K for $0 < K < K_R$. On the other hand, K becomes larger as α becomes larger. Thus, if α is larger, then the critical level is higher and the investor decides to manage actively only when she is endowed with more wealth.

Figures 1, 2, and 3 show how the critical wealth level varies with respect to α or l. Figures 1 and 2 illustrate that the critical wealth level is an increasing function of α and l. Figure 3 illustrates that as α approaches the market risk premium θ the critical wealth level diverges to infinity, which shows the it is very sensitive to ambiguity.

Remark 4.4. Consider the classical problem of an infinitely-lived investor endowed with initial wealth x with no control of stopping time. In this case one can easily see by using the classical results together with the previous analysis in this section that the value function will be

$$V_\infty(x) = \frac{1}{K^\gamma}\frac{x^{1-\gamma}}{1-\gamma} - \frac{l}{\beta}.$$

The pair (π^∞, c^∞) of optimal consumption and portfolio processes is

$$c^\infty(t) = (y_t^{\tilde{\nu}})^{-\frac{1}{\gamma}}, \quad 0 \le t < \infty \quad \text{and}$$

$$\pi^\infty(t) = \frac{\theta-\alpha}{\sigma}\frac{1}{\gamma K}(y_t^{\tilde{\nu}})^{-\frac{1}{\gamma}}, \quad 0 \le t < \infty,$$

where $\tilde{\nu} = (Kx)^{-\gamma}$ or $x = \frac{1}{K}(\tilde{\nu})^{-\frac{1}{\gamma}}$. The optimal wealth process is

$$X^\infty(t) = \frac{1}{K}(y_t^{\tilde{\nu}})^{-\frac{1}{\gamma}} \quad 0 \le t < \infty.$$

We compare the results from Theorem 4 and the classical solution in Remark 4.4. Assume that there are two active investors (we call them investor A and investor B); A has the option to stop active management but B does not. Initially they have same amount of wealth x that is greater than $\bar{x}$. Note that

$$c^*(0) - c^\infty(0) = KC\lambda(\nu^*)^{\lambda-1} > 0$$

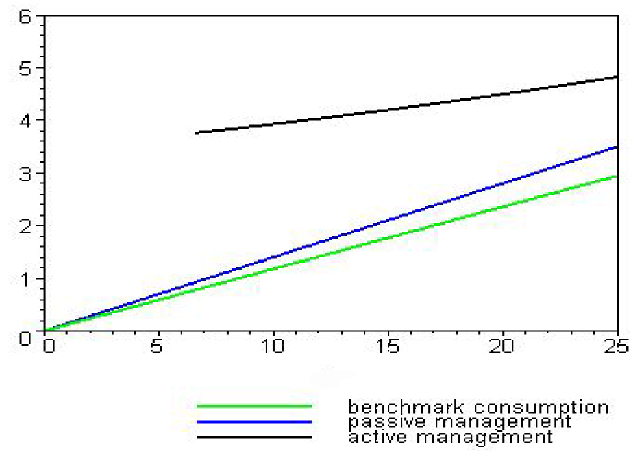

FIG. 4. The graph of optimal consumption as a function of wealth

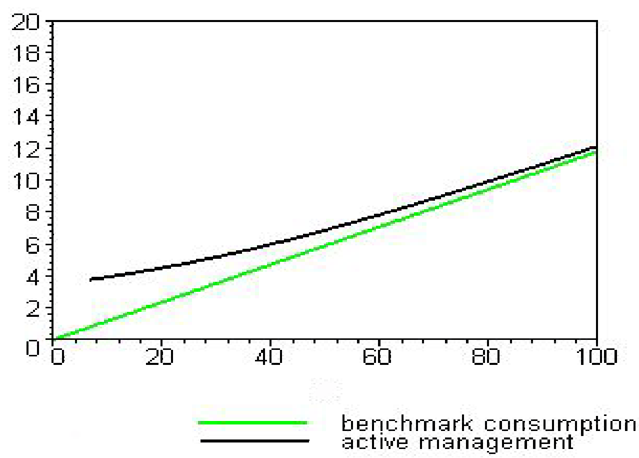

FIG. 5. The graph of optimal consumption as a function of wealth at a large scale

by (37). This implies that investor A consumes more than agent B does.[11] Also,

$$\pi^*(0) - \pi^\infty(0) = \frac{\theta - \alpha}{\sigma}\left(C\lambda(\lambda-1) + \frac{C\lambda}{\gamma}\right)(\nu^*)^{\lambda-1} > 0.$$

Thus, investor A invests more money in the risky asset than agent B does.

Figure 4 shows graphs of optimal consumptions to wealth. The market coefficients are given as follows: $\mu = 0.12$, $\sigma = 0.20$, $r = 0.04$, $\beta = 0.10$, and $\gamma = 0.6$. K_R value is that of the consumption-saving problem. $\alpha = 0.2$ and $l = 0.5$ are given for an agent who actively manages. Then, the threshold wealth level is approximately $\bar{x} \approx 6.6$. The lowest curve[benchmark consumption] represents optimal consumption of investor B who never stops active management. The middle one[passive management] represents consumption of an investor who manages only passively (i.e, an investor who chooses only optimal consumption in a consumption-saving problem .) The highest one[active management] represents optimal consumption of investor A. When the wealth of the investors are small (but greater than the critical wealth level), then investor A seems to consume relatively much more than agent B does. But, their consumption behavior is almost the same when the investors are wealthy relative to the threshold level as seen in Figure 5. Intuitively, this limiting behavior is intuitively appealing, since the probability of her wealth touching the critical level is very small when she is rich investor A behaves much the same as investor B does. The consumption function of agent A seems to be almost linear and there is a jump at the critical wealth level.

Figure 6 shows the optimal dollar amounts invested in the risky asset as functions of wealth. The lower curve[benchmark investment] is the optimal investment in the risky asset of investor B. One can see that the upper curve[actively managed investment] has a nonlinearity around the critical wealth level (it is U-shaped). It illustrates that investors' risk-taking behavior is almost the same when they are rich enough, but the two investors exhibit quite different behavior when they have relatively small wealth. At the small wealth (but greater than the critical wealth level) agent A takes risk much more risk. In our model utility loss from active management is assumed to be constant. This implies that investor A incurs the same utility loss regardless of her wealth. In such a case investor A feels that utility loss due to active management is large compared to profits coming from active management when her wealth is only slightly larger than the critical level. Then she is confronted with a situation that encourages her risk taking; if the result of risk taking is successful then wealth will increase and the cost of active management will become small relative to the benefit of active

[11] The result is due to our assumption that $0 < \gamma < 1$

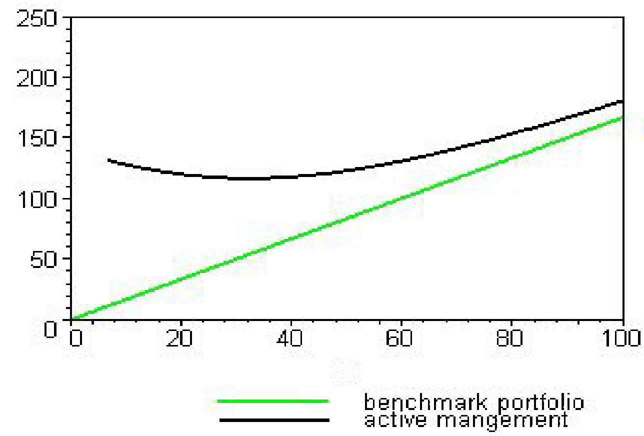

FIG. 6. The graph of optimal investment in the risky asset as a function of wealth

management, if the result is a failure then she can simply choose to switch to passive management and save cost of active management. But, if she becomes wealthier, then utility loss becomes small relative to the benefit from active management, so her amount of risk-taking starts to resemble that of investor B.

It is not clear that optimal investment in the risky asset exhibits the same behavior when the utility loss function $l(c, y, z)$ is not constant. More elaborate model will be needed. We conjecture that the U-shaped behavior will be less pronounced or disappear if l increases as wealth increases.

5. CONCLUSION

We have studied an optimal consumption-portfolio selection problem with stochastic differential utility in which an investor can choose to stop active portfolio management and switch to passive management. By employing stochastic differential utility that exhibits utility loss from active management and aversion to ambiguity about the financial market, we have shown that there exists a critical wealth level such that the investor manages her portfolio actively if and only if her wealth is above this critical level. We have derived an explicit form of the critical wealth level and shown the following facts: the critical wealth level is an increasing function of the ambiguity aversion parameter, the critical wealth level is an increasing function of the utility cost of active management, and while managing actively, the investor invests more in risky assets than other investors who do not have the option of switching to passive management. Typically, the investor takes much higher than others when current wealth is close to the critical wealth level.

APPENDIX: OPTIMALITY PRINCIPLE

In this Appendix we provide the optimality principle. See El-Karoui, Peng, and Quenez (2002) for the full analysis and Choi, Koo, and Kwak (2003) for the stopping-time-inserted version. Consider the following adjoint processes. Let $(\xi^0, c^0) \in \Pi_\tau(x)$ be given and let (Y^0, Z^0) and (X^0, π^0) be the utility and the wealth process associated with (ξ^0, c^0). Let Γ_t be the solution to the following BSDE

$$d\Gamma_t = \Gamma_t(f_y^0(t)dt + (f_z^0(t))^* dW_t), \quad 0 \leq t \leq \tau, \quad \Gamma_0 = 1 \tag{A.1}$$

and $\tilde{H}_t$ be the solution to

$$d\tilde{H}_t = \tilde{H}_t(b_x^0(t)dt + (b_\pi^0)^* dW_t), \quad 0 \leq t \leq \tau, \quad \tilde{H}_0 = 1, \tag{A.2}$$

where

$$f_c^0(t) = f_c(t, c_t^0, Y_t^0, Z_t^0), \quad f_y^0(t) = f_y(t, c_t^0, Y_t^0, Z_t^0), \quad \text{and}$$

$$f_z^0(t) = f_z(t, c_t^0, Y_t^0, Z_t^0)$$

($b_x^0(t)$ and $b_\pi^0(t)$ are defined similarly). Note that $\tilde{H}$ is reduced to the pricing kernel H in the standard linear wealth case. Now we have the following theorem called Stochastic Maximum Principle for the optimal plan.

THEOREM 5. *(ξ^0, c^0) is optimal for (11) if and only if*

$$\Lambda_\tau^0 h'(\xi^0) = \nu \tilde{H}_\tau^0 \quad a.\ s., \tag{A.3}$$

$$\Lambda_t^0 f_c(t, c_t^0, Y_t^0, Z_t^0) = \nu \tilde{H}_t^0 b_c(t, c_t^0, Y_t^0, Z_t^0), \quad 0 \leq t \leq \tau, \quad dt \otimes d\mathcal{P} - a.\ s.. \tag{A.4}$$

APPENDIX: RECURSIVE MULTIPLE-PRIORS UTILITY

We sketch the set up of the utility function introduced in Section 4, which is a special case of recursive multiple-priors. Reader can refer Chen and Epstein (2002) for details. Here, we outline a brief ideas on α-ignorance in ambiguity aversion.

Consider $(\Omega, \mathcal{F}, P)$ with usual condition. Fix a consumption process c in C where C is a suitable consumption space. Let V_t^Q denote the SDU utility process for c computed relative to beliefs given by Q-measure which is equivalent to P-measure such that

$$V_t^Q = \mathbf{E}_Q \left[\int_t^T f(c_s, V_s^Q)\, ds \,|\, \mathcal{F}_t \right], \tag{B.1}$$

Define a utility process $V_t(c)$ for each consumption process as follows:

$$V_t = \min_{Q \in \mathbf{P}} V_t^Q, \tag{B.2}$$

where $\mathbf{P}$ is a set of priors on state space $(\Omega, \mathcal{F}_T)$ that is to be specified later. $\mathbf{E}_Q$ means the expectation with respect to Q-measure. Chen and Epstein (2002) showed that (B.2) admits a unique solution for each c in C. Abbreviate $V_0(\cdot)$ by $V(\cdot)$ and refer to it as *recursive multiple-priors utility.*

Notice that the SDU process V_t^Q defined by (B.1) can be expressed alternatively as the unique solution to the BSDE

$$dV_t^Q = -f(c_t, V_t^Q)\, dt + \sigma_t^Q \cdot W_t, \quad V_T^Q = 0.$$

More correctly, a pair (V_t^Q, σ_t) is the solution.

To specify the set of priors, we need to define a *density generator* to be an R^m-valued process $\kappa = (\kappa_t)$ for which the process z_t^κ is a P-martingale, where

$$z_t^\kappa = \exp\{-\frac{1}{2}\int_0^t |\kappa_s|^2\, ds - \int_0^t \kappa_s \cdot W_s\}, \quad 0 \le t \le T.$$

Then, κ generates a probability measure Q^κ on $(\Omega, \mathcal{F})$ which is equivalent to P such that

$$Q^\kappa(A) = \mathbf{E}[1_A z_T^\kappa], \quad \forall A \in \mathcal{F}_T. \tag{B.3}$$

In other words,

$$\frac{dQ^\kappa}{dP} = z_T^\kappa.$$

Thus, given a set Ξ of density generators, the corresponding set of priors is defined by

$$\mathbf{P}^\Xi = \{\, Q^\kappa \,|\, \kappa \in \Xi \quad \text{and} \quad Q^\kappa \ \text{ is defined by (B.3)}\,\}.$$

A important feature of this specification of $\mathbf{P}^\Xi$ is that it delivers that V_t of (B.2) is a unique solution to the following BSDE:

$$dV_t = [-f(c_t, V_t) + \max_{\kappa \in \Xi} \kappa_t \cdot \sigma_t\,]\, dt + \sigma_t \cdot dW_t, \quad V_T = 0.$$

(See Theorem 2.2 in Chen and Epstein(2002).) One of common examples of multiple-priors is α-ignorance specification. (They called it κ-ignorance, but we do it α-ignorance matching to our notation.)

Fix a parameter $\alpha = (\alpha_1, \ldots, \alpha_m)$, $\alpha_i \in \mathbb{R}_+$ and take a set of density generators such that

$$\Xi = \{\, (\kappa_t) \mid \sup\{|\kappa_t^i| : 0 \le t \le T\} \le \alpha_i, \quad i = 1, \ldots, m\,\}.$$

We can easily see that

$$\max_{\kappa \in \Xi} \kappa_i \cdot \sigma_t = \alpha \cdot |\sigma|,$$

where $|\sigma|$ denotes m-dimensional vector with i-th component $|\sigma_t|$. Consequently, the utility process V_t of (B.2) solves

$$dV_t = [-f(c_t, V_t) + \alpha \cdot |\sigma_t|\,]\,dt + \sigma_t \cdot dW_t, \quad V_T = 0.$$

APPENDIX: PROOF OF PROPOSITION 1

In this section we solve Variational Inequality 1. First, we consider the following partial differential equation.

$$\frac{\partial w}{\partial t} + \frac{1}{2}(\theta - \alpha)^2 y^2 \frac{\partial^2 w}{\partial y^2} + (\beta - r) y \frac{\partial w}{\partial y} + e^{-\beta t}\tilde{u}(y) = 0, \quad 0 < y < \bar{y} \quad \text{(C.1)}$$

where the boundary condition

$$w(t, \bar{y}) = e^{-\beta t}(\tilde{h}(\bar{y}) + \frac{l}{\beta}).$$

So far $\bar{y}$ is not determined yet and we only consider PDE C.1. If we try a solution with

$$w(t, y) = e^{-\beta t}\phi(y),$$

where $\phi : \mathbb{R} \longrightarrow \mathbb{R}$, then we get the following ordinary differential equation

$$\frac{1}{2}(\theta - \alpha)^2 y^2 \phi''(y) + (\beta - r) y \phi'(y) - \beta\phi(y) + \tilde{u}(y) = 0, \quad 0 < y < \bar{y} \quad \text{(C.2)}$$

with $\phi(\bar{y}) = \tilde{h}(\bar{y}) + \frac{l}{\beta}$. The general solution to (C.2) is determined by the characteristic equation (30) and the particular solution is easily obtained. If we try $Ay^{-\frac{1-\gamma}{\gamma}}$ to (C.2), then we have $A = \frac{\gamma}{K(1-\gamma)}$ by comparing the coefficients. Let the solution ϕ have a form

$$\phi(y) = Cy^{\lambda} + \bar{C}y^{\bar{\lambda}} + \frac{\gamma}{K(1-\gamma)} y^{-\frac{1-\gamma}{\gamma}},$$

where λ and $\bar{\lambda}$ are positive and negative root to the characteristic equation (30), respectively, with some constant C and $\bar{C}$. Here, we take $\bar{C} = 0$. By applying the boundary condition, we have

$$C = \left(\frac{(K - K_R)\gamma}{KK_R(1-\gamma)} \bar{y}^{-\frac{1-\gamma}{\gamma}} + \frac{l}{\beta} \right) \bar{y}^{-\lambda}. \tag{C.3}$$

Now, we are ready to prove Proposition 1.

Proof of Proposition 1
($\Longrightarrow$) By the principle of smooth fit, we have

$$C\lambda \bar{y}^{\lambda-1} - \frac{1}{K} \bar{y}^{-\frac{1}{\gamma}} = -\frac{1}{K_R} \bar{y}^{-\frac{1}{\gamma}}. \tag{C.4}$$

From (C.4) together with (C.3), we obtain (31) and (32). To check $(V.2)$, we see for $\bar{y} < y$

$$\mathcal{L}w(t,y) + e^{-\beta t}\tilde{u}(y) = e^{-\beta t} \left(\frac{(K_R - K)\gamma}{K_R(1-\gamma)} y^{-\frac{1-\gamma}{\gamma}} - l \right)$$

by direct calculation. Then, (33) implies that $\mathcal{L}w(t,y) + e^{\beta t} \le 0$ for $\bar{y} < y$ since $y^{-\frac{1-\gamma}{\gamma}}$ is decreasing function on $\bar{y} < y$. Thus, $(V.2)$ is verified. To show $(V.3)$ is equivalent to show

$$m(y) \triangleq Cy^{\lambda} + \frac{(K_R - K)\gamma}{KK_R(1-\gamma)} y^{-\frac{1-\gamma}{\gamma}} - \frac{l}{\beta} \ge 0, \quad 0 < y \le \bar{y}$$

Note that the equality holds at $y = \bar{y}$. For $0 < y \le \bar{y}$,

$$\begin{aligned} m'(y) &= \frac{1}{y} \left[\lambda C y^{\lambda} - \frac{K_R - K}{KK_R} y^{-\frac{1-\gamma}{\gamma}} \right] \\ &= \frac{1}{y} \left[\lambda C \bar{y}^{\lambda} \left(\frac{y}{\bar{y}} \right)^{\lambda} - \frac{K_R - K}{KK_R} \bar{y}^{-\frac{1-\gamma}{\gamma}} \left(\frac{y}{\bar{y}} \right)^{-\frac{1-\gamma}{\gamma}} \right] \\ &= \frac{\lambda(1-\gamma)l}{(1-\gamma+\lambda\gamma)\beta y} \left[\left(\frac{y}{\bar{y}} \right)^{\lambda} - \left(\frac{y}{\bar{y}} \right)^{-\frac{1-\gamma}{\gamma}} \right] \\ &< 0 \end{aligned}$$

The third equality is obtained by applying (31) and (32). Hence, $m(\cdot)$ is decreasing on $0 < y \le \bar{y}$ and this means that $m(\cdot)$ attains the minimum at $y = \bar{y}$ for $0 < y \le \bar{y}$. This proves $(V.3)$. ∎

APPENDIX: PROOF OF PROPOSITION 2 AND THEOREM 4

Proof of Proposition 2

($\Longrightarrow$) Plugging (39) and (40) into HJB equation (38), we have the following nonlinear ODE. For $\bar{x} < x$,

$$\beta V(x) = -\frac{1}{2}(\theta-\alpha)^2\frac{(V'(x))^2}{V''(x)} + \frac{\gamma}{1-\gamma}(V'(x))^{-\frac{1-\gamma}{\gamma}} + rxV'(x) - l. \quad \text{(D.1)}$$

For each $x > \bar{x}$, ν^* is defined by (37). By Remark 4.3, we have

$$\frac{(V'(x))^2}{V''(x)} = -C\lambda(\lambda-1)(\nu^*)^\lambda - \frac{1}{\gamma K}(\nu^*)^{-\frac{1-\gamma}{\gamma}}.$$

Then, the right hand side of (D.1) reduces that

$$\begin{aligned}
\text{(RHS)} &= C\left[\frac{1}{2}(\theta-\alpha)^2\lambda(\lambda-1) - r\lambda\right](\nu^*)^\lambda \\
&+ \left[\frac{1}{2\gamma K}(\theta-\alpha)^2 + \frac{\gamma}{1-\gamma} + \frac{r}{K}\right](\nu^*)^{-\frac{1-\gamma}{\gamma}} - l \\
&= C\beta(1-\lambda)(\nu^*)^\lambda + \left[\frac{1}{2\gamma K}(\theta-\alpha)^2 + \frac{\gamma}{1-\gamma} + \frac{r}{K}\right](\nu^*)^{-\frac{1-\gamma}{\gamma}} - l \\
&= C\beta(1-\lambda)(\nu^*)^\lambda + \frac{\beta}{(1-\gamma)K}(\nu^*)^{-\frac{1-\gamma}{\gamma}} - l \\
&= \beta V(x) = \text{(LHS)},
\end{aligned}$$

where the first equality is from the direct computation, the second is from the definition of λ of (30), and the third is from the definition of K. This completes the proof. ▮

Proof of Theorem 4

($\Longrightarrow$) It is sufficient to show that the given consumption and portfolio processes generate the optimal wealth process $X^*(t)$ of (41) when $> \bar{x}$. Recall that $dy^{\nu^*} = y^{\nu^*}[(\beta-r)\,dt - (\theta-\alpha)\,d\tilde{W}_t^\alpha]$. In this proof, we say $y^* = y^{\nu^*}$ to shorten the notation. By applying Ito's lemma to (41), we have

$$\begin{aligned}
dX^*(t) &= [-C\lambda(\lambda-1)(y_t^*)^{\lambda-2} - \frac{1}{\gamma K}(y_t^*)^{-\frac{1}{\gamma-1}}]dy_t^* \\
&+ \frac{1}{2}[-C\lambda(\lambda-1)(\lambda-2)(y_t^*)^{\lambda-3} + \frac{1+\gamma}{\gamma^2 K}(y_t^*)^{-\frac{1}{\gamma}-2}](dy_t^*)^2 \\
&= [-C\lambda(\lambda-1)(y_t^*)^{\lambda-1} - \frac{1}{\gamma K}(y_t^*)^{-\frac{1}{\gamma}}][(\beta - r)\,dt - (\theta-\alpha)\,d\tilde{W}_t] \\
&+ \frac{1}{2}[-C\lambda(\lambda-1)(\lambda-2)(y_t^*)^{\lambda-1} + \frac{1+\gamma}{\gamma^2 K}(y_t^*)^{-\frac{1}{\gamma}}](\theta-\alpha)^2 dt \\
&= r\left[-C\lambda(y_t^*)^{\lambda-1} + \frac{1}{K}\right] dt \\
&+ (\theta-\alpha)^2\left[C\lambda(\lambda-1)(y_t^*)^{\lambda-1} + \frac{1}{\gamma K}(y_t^*)^{-\frac{1}{\gamma}}\right] dt \\
&- C\lambda\left[\frac{1}{2}\lambda(\lambda-1)(\theta-\alpha)^2 + \beta(\lambda-1) - r\lambda\right](y_t^*)^{\lambda-1}\,dt \\
&+ \left[\frac{r-\beta}{\gamma K} - \frac{r}{K} + \frac{1-\gamma}{2\gamma^2}(\theta-\alpha)^2\right](y_t^*)^{-\frac{1}{\gamma}}\,dt \\
&+ (\theta-\alpha)\left[C\lambda(\lambda-1)(y_t^*)^{\lambda-1} + \frac{1}{\gamma K}(y_t^*)^{-\frac{1}{\gamma}}\right] d\tilde{W}_t.
\end{aligned}$$

Here, we observe that the third term of the last equality is cancelled by the definition of λ of (30) and the forth term is reduced to -1 by the definition of K. Now, we match the first, second, and fifth terms by

$$-C\lambda(y_t^*)^{\lambda-1} + \frac{1}{K} = X^*(t)$$

and

$$(\theta-\alpha)\left[C\lambda(\lambda-1)(y_t^*)^{\lambda-1} + \frac{1}{\gamma K}(y_t^*)^{-\frac{1}{\gamma}}\right] = \pi^*(t)\sigma.$$

Then, we conclude

$$\begin{aligned}
dX_t^* &= (rX_t^* + \pi_t^*(\theta-\alpha)\sigma - c_t^*)\,dt + \pi_t^*\sigma\,d\tilde{W}_t \\
&= (rX_t^* + \pi_t^*\theta\sigma - c_t^*)\,dt + \pi_t^*\sigma\,dW_t.
\end{aligned}$$

Thus, the optimal wealth is obtained from the strategy (π^*, c^*) at $0 \le t \le \tau^*$. ▮

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