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An Alternative to Prospect Theory

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This paper presents a new approach to decision-making under risk. Preference over risky prospects is defined as a triadic reference-dependent relation in a sense similar to Sugden (2003). Characterized by a set of von Neumann-Morgenstern-style axioms, a new reference-dependent representation theory – called compound utility theory (CUT) – is obtained which accommodates nonlinear preferences (in probabilities) without invoking the probability-transformation assumption of cumulative prospect theory. Given any opportunity set, a unique reference level can be identified which is consistent with CUT and which enables one to study preferences over both *relative changes* and *absolute levels* of wealth simultaneously.

Key Words: Expected utility theory; Cumulative prospect theory; Compound utility theory; Reference-dependence; Nonlinear preference; Triadic preference relation; Utility-reward; Disutility-risk.
JEL Classification Number: D81

1. INTRODUCTION

Consider a choice between three mutually exclusive investment opportunities (or gambles) A, B, and C, as shown in Figure 1. On \$10,000 investment capital, A pays off \$12,000 or \$8,000, and B pays off \$15,000 or \$6,000, with equal chances. Investment C is a 50-50 probability mixture of A and B, so it has all four possible payoffs with equal chances. Which of these gambles do you prefer?

When the question was posed to five classes of finance students at the University of Amsterdam in 2002 and 2003, the majority preferred C as shown in the following table:

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Class	1	2	3	4	5	Total
Number of students preferring C	15	25	16	19	30	105
Number of students in the class	25	50	33	26	33	167
Percentage	60%	50%	48%	73%	91%	63%

The revealed preference for C (henceforth quasiconcave preference in a sense to be made clear) obviously violates expected utility theory or EUT (e.g., von Neumann and Morgenstern, 1947; Savage, 1954) and betweenness theories (e.g., Chew, 1983; Dekel, 1986; Gul, 1991), which predict that if A (B) is preferred to B (A), then A (B) is preferred to any probability mixture of A and B. Thus according to these theories gamble C could never be one's top preference unless all three gambles are equally preferred. Indeed, this casual experiment merely adds a new observation to the long list of empirical anomalies for EUT (e.g., Allais, 1953, 1979; Ellsberg, 1961; Kahneman and Tversky, 1979; and Camerer, 1998, Table 1) and to the more recent evidence against betweenness theories (e.g., Tversky and Kahneman, 1992; Lattimore et al., 1992; Gonzalez and Wu, 1999; Tversky and Fox, 1995; Abdellaoui, 2000; Bleichrodt and Pinto, 2000).

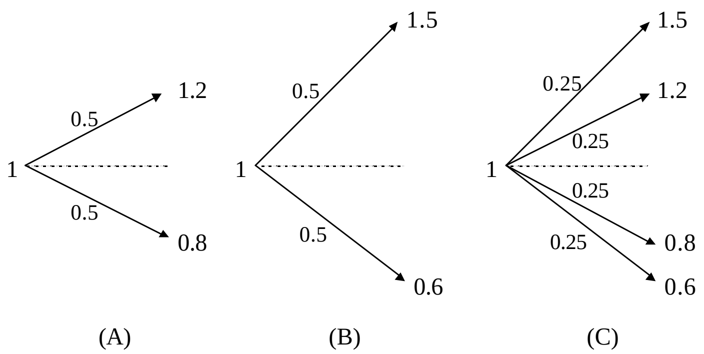

FIG. 1. Three investment opportunities A, B, and C with depicted payoffs ($1 = \$10,000$) and probabilities. Many people seem to consider C the best choice.

A popular explanation for quasiconcave or more generally nonlinear preferences[2] (in probabilities) is that individuals may have different *attitudes toward probabilities.* They may behave *as if* their own (nonadditive) subjective probabilities, called *decision weights*, were used instead of true probabilities while computing expected utility. As Abdellaoui (2002) shows, for instance, the shape of the probability weighting function that transforms true cumulative probabilities into decision weights characterizes attitude toward "probability risk" in much the same way that the shape of the utility function characterizes attitude toward risk in EUT. On the whole, experimental studies find that people appear to overweight small probabilities and underweight moderate to high probabilities (e.g., Tversky and Wakker, 1995 and their references). Among the descriptively more successful theories that address attitudes toward probabilities is cumulative prospect theory or CPT (Tversky and Kahneman,1992), which allows individuals to have two different probability weighting functions on "gains" and "losses" respectively (see, e.g., Camerer, 1998).

In the present state of knowledge, however, the reason why people seem to distort true probabilities in such a systematic manner remains unclear (e.g., Luce, 1996a,b; Safra and Segal, 1998). The human mind may be inclined to attend more to important events (such as large gains or losses); or decision weights may be a deliberate allocation of degree of importance to different outcomes. But these thoughts do not seem to go much beyond EUT. According to EUT, the "weight" for a more important outcome is reflected in a higher (or lower) utility; the "weight" for a more important change of outcome is reflected in a higher (or lower) marginal utility. Therefore, it remains an important open issue as to what *causes* people to have an attitude toward probabilities.

The aim of this paper is to propose an alternative theoretical framework in which nonlinear preferences can be intuitively analyzed and explained without having resort to probability transformations. The theory to be developed is called compound utility theory (CUT) for its numerical representation of two sequential relations: (1) more rewarding and more risky relations, and (2) preference relations over reward and risk. In order to lay a solid foundation for the new theory, I first justify CUT by a set of von Neumann-Morgenstern-style axioms. I then show how *attitude toward utility-reward and disutility-risk* – a new dimension of reward-risk attitude in CUT – can accommodate and explain nonlinear preferences in natural terms.

CUT is closely related to cumulative prospect theory (CPT) for its entailment of a reference point (called reference level) and its emphasis on gains and losses (called better and worse outcomes). Even though the terms used are different and often interpreted differently, the psychological foundation of CUT is similar to that of CPT – namely, that people's perceptions of gains and losses, or pleasure and pain, fundamentally affect their behavior in the face of uncertainty. Decision making under risk typically involves (or ought to involve) planning, judgment, or evaluation before any action or sequence of actions is taken. Therefore, we could conceive a two-stage decision process whereby the decision maker first evaluates the potential gains (u) and losses (d) of each gamble and then chooses one with the most preferred $u - d$ combination.

An important difference between CUT and CPT, however, is that CUT does not assume probability transformations. Instead, CUT enlarges the scope of expected utility theory by allowing people to have different attitudes toward u and d – characterized by their degrees of *disutility aversion*. This new dimension of risk attitude can be conveniently studied through the shapes of indifference curves in the $u - d$ domain of a compound utility function.

Another important difference between CUT and CPT is in the modelling of the reference point. Adopting a very useful idea from Sugden (2003), I define preference as a "triadic" relation so that the traditional binary relation "A is preferred to B" is extended to a triadic relation "A is preferred to B under reference r". Consequently, in the axiomatic development of CUT the reference level is treated more flexibly as a primitive concept without being given any formal interpretation. This "triadic" approach has the advantage of modeling preferences over both *relative changes* and *absolute levels* of wealth simultaneously. In this sense CUT improves upon the "status quo" interpretation of CPT and enlarges its scope of applications.[3] In applying CUT to the special cases where outcomes are monetary payoffs, I also show how a unique reference level can be exogenously identified given any opportunity set.

The rest of the paper is organized as follows. Section 2 is devoted to establishing a mathematical foundation of CUT, where four axioms are shown to be equivalent to the compound-utility (CU) representation of reference-dependent preferences. Preferences are *reference-dependent* in that they may vary with the reference level as the opportunity set varies. Section 3 discusses in more detail the reference level and attitudes toward utility-reward and disutility-risk. A simple way to determine a unique reference level that may vary with the opportunity set is developed. In order to motivate the new theory, the section also gives a more detailed illustration as to how CUT differs from CPT in explaining nonlinear preferences. Section 4 concludes the paper with some further remarks. The appendix contains the proofs of the representation theorem.

2. AXIOMATIC FOUNDATION OF CUT

2.1. Basic concepts and notation

Denote by $\mathbb{X}$ the set of (pure) *outcomes*, and by $\mathcal{P}$ the set of all simple probability measures on $\mathbb{X}$ (i.e., $P : \mathbb{X} \to [0,1]$ such that $P(x) > 0$ for only finitely many $x \in \mathbb{X}$ and $\sum_{x\in\mathbb{X}} P(x) = 1$). Each element $P \in \mathcal{P}$ is a choice object, henceforth called a *gamble*. Given any $P, Q \in \mathcal{P}$, and any $\theta \in [0,1]$, the convex combination $\theta P \oplus (1-\theta)Q$ means a probability mixture of P and Q. Note that $\mathcal{P}$ is closed under finite convex combinations ($P, Q \in \mathcal{P} \Rightarrow \theta P \oplus (1-\theta)Q \in \mathcal{P}$ for all $\theta \in [0,1]$) and contains all degenerate probability measures, δ_x, that assigns probability 1 to the outcome $x \in \mathbb{X}$[4].

Sugden (2003) defines preference as a triadic relation $\succeq$ that is a subset of $\mathcal{P}^3$ in our context. A typical element of this subset is a triplet $(P, Q, Z) \in$

$\mathcal{P}^3$ satisfying $P \succeq Q|Z$, read as "P is weakly preferred to Q, viewed from Z ." The present paper restricts the reference level to be an element of $\mathbb{X}$ rather than an element of $\mathcal{P}$. Thus reference-dependent preference is defined as a triadic relation $\succeq$ in that for all $P, Q \in \mathcal{P}$ and $r \in \mathbb{X}$, $P \succeq Q|r$ is read as "P is weakly preferred to Q under reference r." Notations $\sim$, $\succ$, $\prec$, $\preceq$ and $\nsim$ are defined as usual, i.e., $P \sim Q|r$ if $P \succeq Q|r$ and $Q \succeq P|r$, $P \succ Q|r$ if $P \succeq Q|r$ and not $Q \succeq P|r$, $P \nsim Q|r$ if either $P \succ Q|r$ or $Q \succ P|r$, etc. For pure outcomes x_1 and x_2 in $\mathbb{X}$, define $x_1 \succeq x_2|r$ if and only if $\delta_{x_1} \succeq \delta_{x_2}|r$.

Clearly, if for all $P, Q \in \mathcal{P}$ and $r, \widehat{r} \in \mathbb{X}$, $P \succeq Q|r \Leftrightarrow P \succeq Q|\widehat{r}$, then our triadic preference relation reduces to the usual binary preference relation. Note also that if we restrict attention to a given choice context where the reference level is (exogenously) given and fixed, then $\succeq$ can also be seen as a binary relation with a tacit understanding that it describes a preference relation under the *given* reference r.

We say that $\succeq$ is a (reference-dependent) *preference relation* if it is (i) complete: for all $P, Q \in \mathcal{P}$ and $r \in \mathbb{X}$, either $P \succeq Q|r$ or $Q \succeq P|r$, and (ii) transitive: for all $P, Q, Z \in \mathcal{P}$ and $r \in \mathbb{X}$, ($P \succeq Q|r$ and $Q \succeq Z|r$) implies $P \succeq Z|r$.

Assuming that $\succeq$ is a preference relation, for any $r \in \mathbb{X}$ define $\mathbb{X}_r^+ = \{x \in \mathbb{X} : x \succeq r|r\}$ and $\mathbb{X}_r^- = \{x \in \mathbb{X} : x \preceq r|r\}$. These sets have the natural interpretation of being the sets of *better* and *worse* outcomes with respect to r.

We say that P (first-order) stochastically dominates Q under r, written $P \succ_{SD} Q|r$, if $P \neq Q$ and $\forall y \in \mathbb{X}$, $P(\{x \in \mathbb{X} : x \succeq y|r\}) \geq Q(\{x \in \mathbb{X} : x \succeq y|r\})$ (e.g., Hadar and Russell, 1969). The preference relation $\succeq$ satisfies *stochastic dominance* under $r \in \mathbb{X}$ if for all $P, Q \in \mathcal{P}$, $P \succ Q|r$ whenever $P \succ_{SD} Q|r$.

For every $P \in \mathcal{P}$, its "reward-side equivalence is defined as P_r^+ that substitutes r for all worse outcomes of P. Dually, define P_r^- as the probability measure of the "risk-side equivalence" that substitutes r for all better outcomes of P. These definitions can be formally expressed as (the term $\sum_{x \succeq r|r} P(x)$ is defined as the summation over all $x \succeq r|r$ such that $P(x) > 0$, etc.)

$$
\begin{aligned}
P_r^+ &= \bigoplus_{x \succeq r|r} P(x)\delta_x \oplus [1 - \sum_{x \succeq r|r} P(x)]\delta_r \quad \text{and} \\
P_r^- &= \bigoplus_{x \preceq r|r} P(x)\delta_x \oplus [1 - \sum_{x \preceq r|r} P(x)]\delta_r
\end{aligned}
$$

For example, P_r^+ can be seen as a lottery that assigns probability $P(x)$ for all $x \succeq r|r$ and $1 - \sum_{x \succeq r|r} P(x)$ for r.

The definitions of the "reward-side" and "risk-side" subsets of $\mathcal{P}$ (relative to r under $\succeq$) follow naturally as:

$$\mathcal{P}^U(r,\succeq) \;=\; \{P\in\mathcal{P} : \sum_{x\succeq r|r} P(x)=1\} \;\text{ and}$$
$$\mathcal{P}^D(r,\preceq) \;=\; \{P\in\mathcal{P} : \sum_{x\preceq r|r} P(x)=1\}.$$

It is clear that $\mathcal{P}^U(r,\succeq)$ and $\mathcal{P}^D(r,\preceq)$ are both subsets of $\mathcal{P}$, and each of them is closed under finite convex combinations.

For all P, then, we have $P_r^+\in\mathcal{P}^U(r,\succeq)$ and $P_r^-\in\mathcal{P}^D(r,\preceq)$; and the mapping $P\to(P_r^+,P_r^-)$ is one-to-one. This is because

$$P=\bigoplus_{x\succeq r|r} P_r^+(x)\delta_x \oplus \bigoplus_{x\prec r|r} P_r^-(x)\delta_x=\bigoplus_{x\in\mathbb{X}} P(x)\delta_x.$$

For any given $r\in\mathbb{X}$, a nonnegative real-valued function $U_r(\cdot)$ defined on $\mathbb{X}$ is called a *utility function for better outcomes relative to* r if $U_r(x)=0$ for all $x\preceq r|r$ and $U_r(x_1)\geq U_r(x_2)\Leftrightarrow x_1\succeq x_2|r$ for all $x_1,\ x_2\succeq r|r$. A nonnegative real-valued function $D_r(\cdot)$ defined on $\mathbb{X}$ is called a *disutility function for worse outcomes relative to* r if $D_r(x)=0$ for all $x\succeq r|r$ and $D_r(x_1)\leq D_r(x_2)\Leftrightarrow x_1\succeq x_2|r$ for all $x_1,\ x_2\preceq r|r$. Likewise, a nonnegative real-valued function $\overline{U}_r(\cdot)$ defined on $\mathcal{P}^U(r,\succeq)$ is called a *utility-reward measure* if $P\succeq Q|r\Leftrightarrow \overline{U}_r(P)\geq\overline{U}_r(Q)$ for all $P,Q\in\mathcal{P}^U(r,\succeq)$; and a nonnegative real-valued function $\overline{D}_r(\cdot)$ defined on $\mathcal{P}^D(r,\preceq)$ is called a *disutility-risk measure* if $P\succeq Q|r\Leftrightarrow\overline{D}_r(P)\leq\overline{D}_r(Q)$ for all $P,Q\in\mathcal{P}^D(r,\preceq)$. Note that for all $P\in\mathcal{P}$, the expected values of U_r and D_r are given by

$$E(U_r;P)\;\stackrel{\text{def.}}{=}\;\sum_{x\succ r|r} U_r(x)P(x)=E(U_r;P_r^+);$$
$$E(D_r;P)\;\stackrel{\text{def.}}{=}\;\sum_{x\prec r|r} D_r(x)P(x)=E(D_r;P_r^-)$$

We say that $\succeq$ has an expected utility-reward and disutility-risk representation if

G 1. *(i) For some $r\in\mathbb{X}$, there exist U_r and D_r from $\mathbb{X}$ to $\mathbb{R}_+$, satisfying $U_r(x)=0$ for all $x\preceq r|r$ and $D_r(x)=0$ for all $x\succeq r|r$, such that for all $P,Q\in\mathcal{P}$*

$$P_r^+\succeq Q_r^+|r\Leftrightarrow E(U_r;P)\geq E(U_r;Q)\;\; \textit{and} \quad P_r^-\succeq Q_r^-|r\Leftrightarrow E(D_r;P)\leq E(D_r;Q).$$

(ii) Fixing r of (i), U_r and D_r are unique in that any other U_r^0 and D_r^0 having the same stated property in (i) must be positive ratio scales of U_r and D_r, i.e., satisfying $U_r^0 = aU_r$ and $D_r^0 = bD_r$ for some $a, b > 0$, and that all positive ratio scales of U_r and D_r satisfy the properties in (i).

For any r, U_r, and D_r satisfying G1, define the induced reward-risk set by

$$\Phi_r = \{(u, d) \in \mathbb{R}_+^2 : (u, d) = (\overline{U}_r(P), \overline{D}_r(P)) \text{ for some } P \in \mathcal{P}\}, \quad (1)$$

where $(\overline{U}_r(P), \overline{D}_r(P)) \equiv (E(U_r; P), E(D_r; P))$.

Assume that r, U_r, and D_r of G1 exist, then $\succeq$ is said to have a compound utility (CU) representation if

G 2. *(i) There exists a continuous function $V_r : \Phi_r \to \mathbb{R}$, strictly increasing in u and strictly decreasing in d, such that*

$$P \succeq Q|r \Leftrightarrow \widehat{V}_r(P) \geq \widehat{V}_r(Q) \quad \text{for all } P, Q \in \mathcal{P}. \quad (2)$$

where $\widehat{V}_r(P) \equiv V_r(\overline{U}_r(P), \overline{D}_r(P))$ for all $P \in \mathcal{P}$.

(ii) V_r is unique up to a strictly increasing transformation in that for all V_r^0 having the same property as V_r, there exists an increasing function $f : \mathbb{R} \to \mathbb{R}$ such that $V_r^0 = f \circ V_r$ (i.e., $V_r^0(u, d) = f(V_r(u, d))$ on Φ_r) and f is strictly increasing on $\{v : v = V_r(u, d)$ for some $(u, d) \in \Phi_r\}$. Conversely, for any strictly increasing function $g : \mathbb{R} \to \mathbb{R}$, $g \circ V_r$ has the same property as V_r.

We say that compound utility theory holds if both G1 and G2 hold.

2.2. Axioms

Now consider the following four axioms. For notational convenience, we write $P \succ Q \succ Z|r$ instead of ($P \succ Q|r$ and $Q \succ Z|r$), etc.

A 1 (preference relation). *$\succeq$ is complete and transitive.*

A 2 (continuity). *For all $P, Q, Z \in \mathcal{P}$ and $r \in \mathbb{X}$, if $P \succ Q \succ Z|r$, then there exists $\psi \in (0, 1)$ and $\xi \in (0, 1)$ such that $\psi P \oplus (1 - \psi)Z \succ Q \succ \xi P \oplus (1 - \xi)Z|r$.*

B 1 (partial independence). *For some $r \in \mathbb{X}$, for all $P, Q, Z \in \mathcal{P}^U(r, \succeq)$ (dually, $P, Q, Z \in \mathcal{P}^D(r, \preceq)$), and for all $\theta \in (0, 1]$, if $P \succ Q|r$ then $\theta P \oplus (1 - \theta)Z \succ \theta Q \oplus (1 - \theta)Z|r$.*

B 2 (monotonicity). *For all $r \in \mathbb{X}$ satisfying B1, and for all $P, Q \in \mathcal{P}$, if $P_r^+ \succ Q_r^+|r$ and $P_r^- \succeq Q_r^-|r$, or if $P_r^+ \succeq Q_r^+|r$ and $P_r^- \succ Q_r^-|r$, then $P \succ Q|r$.*

Axioms A1, A2 and B1 are adapted from Jensen (1967). Under each reference level r, these axioms are essentially the same as Jensen's except that B1, being assumed only on the subsets $\mathcal{P}^U(r, \succeq)$ and $\mathcal{P}^D(r, \preceq)$ of $\mathcal{P}$, is weaker than the traditional von Neumann-Morgenstern independence axiom. Axiom B2 has the intuitive interpretation that if a gamble is deemed by the decision-maker to be more rewarding as well as less risky than another gamble, the former should be preferred to the latter. B2 helps ensure stochastic dominance and serves as a "link" between the reward-side P_r^+ and the risk-side P_r^- of gambles P. Note that B2 alone postulates only a sufficient condition for $P \succ Q|r$; it is also possible that $P_r^+ \prec Q_r^+|r$ or $P_r^- \prec Q_r^-|r$ yet $P \succ Q|r$. But if $P \succ_{SD} Q|r$, then we must have $P_r^+ \succeq Q_r^+|r$ and $P_r^- \succeq Q_r^-|r$ with strict preference in at least one of the two cases; hence $P \succ Q|r$ by B2.

In light of evidences against the independence axiom of EUT, axiom B1 may need some defence. Since the behavioral significance of reference levels has been well established in cognitive science and behavioral economics, the existence of some outcome level that has the special role of a reference seems quite acceptable. At the axiomatization stage, it seems also desirable to keep B1 general with respect to the reference level. This would give the theory more flexibility in application, as individuals may have different tendencies in forming their *aspiration levels*. "Reasonable" reference levels could be postulated in subsequent stages when the theory is applied to more specific choice contexts.

When a reference level is identified, by one way or another, the intuitive ground of B1 could be defended by interpreting the preference relation $\succeq$, when restricted to be on $\mathcal{P}^U(r, \succeq)$ and $\mathcal{P}^D(r, \preceq)$, as the "more rewarding" (i.e., having more utility-reward) and "less risky" (i.e., having less disutility-risk) relations that reflect one's "judgment" rather than "choice". The distinction between judgment and choice becomes meaningful in CUT when elements of $\mathcal{P}^U(r, \succeq)$ or $\mathcal{P}^D(r, \preceq)$ are interpreted as the *attributes* (see, e.g., the lower-partial moment measure of risk, Bawa, 1975; Fishburn, 1977; see also Schmidt, 2003) of the gambles rather than gambles themselves. In judging or evaluating the attributes – risk and reward – of the alternative gambles, it might be in one's interest to seek advice from specialists and to obey some "normative" principles such as stated in B1. Even though some people may not obey the "behavioral" assumption of the traditional independence axiom when making an actual choice (e.g., when they have a nonlinear attitude toward utility-reward and disutility-risk; see Section 3.3).

2.3. Compound representation theorem

The next theorem establishes an axiomatic foundation of CUT.

THEOREM 1. *The set of Axioms {A1, A2, B1, B2} is equivalent to the set of statements {G1,G2}.*

Figure 2 shows a compound utility function that is quasiconcave on Φ_r.

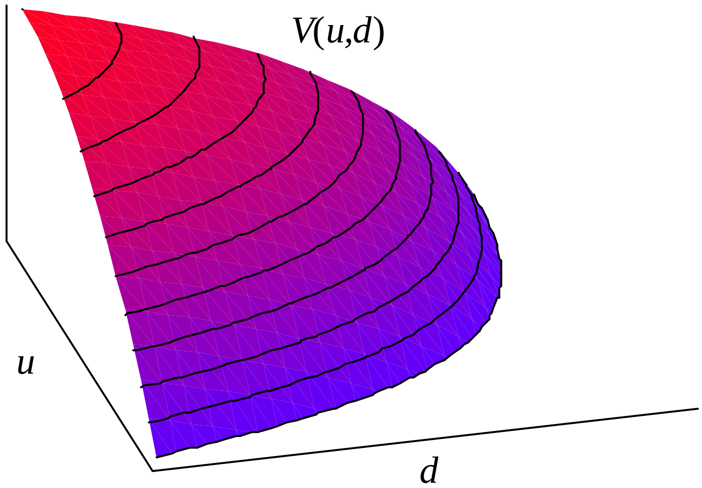

FIG. 2. An example of compound utility function $V_r(u,d)$, where the preference is quasiconcave in utility-reward $u(=\overline{U}_r)$ and disutility-risk $d(=\overline{D}_r)$.

Remarks:

(i) In the proof of Theorem 1, the existence of a CU function is shown by constructing a function $V_r(u,d)$ on Φ_r which satisfies a "border linear condition" that $V_r(u,0) = u$, $V_r(0,d) = -d$, and $V_r(0,0) = 0$. Being linear on the two *borders* of Φ_r, however, V_r is not necessarily quasilinear on the entire set of Φ_r. Moreover, unlike EUT which is single dimensional in EU maximization, it is not always convenient to impose these linear conditions on the border of Φ_r for the preference function V_r.

It suffices to note that any $V_r(u,d)$ is equivalent, in the sense of Theorem 1, to a function $V_r^0(u,d)$ which satisfies this border linear condition. To see this, define two continuous and strictly increasing functions f and g by $f^{-1}(u) = V_r(u,0)$ and $g^{-1}(-d) = V_r(0,d)$), and without loss of generality assume that $V_r(0,0) = 0$. Then it is easy to verify that

$$V_r^0(u,d) \stackrel{\text{def.}}{=} \begin{cases} f(V_r(u,d)) \text{ if } V_r(u,d) > 0 \\ g(V_r(u,d)) \text{ if } V_r(u,d) \leq 0 \end{cases}$$

is a well defined CU function satisfying Theorem 1 as well as $V_r^0(u,0) = u$, $V_r^0(0,d) = -d$, and $V_r^0(0,0) = 0$. But V_r^0 is not necessarily a piece-wise linear function of u and d because $V_r(u,d) > 0$ does not imply $d = 0$, nor does $V_r(u,d) \leq 0$ imply $u = 0$. And V_r^0 need not have a simpler form than V_r.

(ii) Extension of Theorem 1 to cases where the gambles may be non-simple probability measures mainly concerns statement G1, and can be done by straightforward applications of standard results in the literature (e.g., Fishburn, 1982, Chapter 3). Thus the proof of G2, the main new result in Theorem 1, does not rely on the assumption that gambles are simple probability measures. For instance, in application of the theory it is safe to assume that CUT holds for continuously distributed payoffs and that U_r and D_r are continuous functions of x. Furthermore, thanks to the similarity between CUT and EUT, the CU functions can be empirically elicited using standard procedures once the reference is identified.

(iii) It is worth remarking that Theorem 1 is purely deductive theory which establishes a mathematical equivalence between two sets of formal statements {A1, A2, B1, B2} and {G1,G2}. The theory in itself has no empirical content unless primitive concepts are given empirical meanings. In particular, r is stated as an uninterpreted element of the outcome set where the derived utility, disutility, and compound utility functions are parameterized on its value. The advantage of such flexibility is to give the theory more applicability, and it marks an important departure of CUT from prospect theory. For one thing, CUT could be applied to investigate how people differ in forming their aspiration levels and how such differences may affect their behavior under risky situations.

3. ANALYSIS AND APPLICATION

The CU function has four major components: the reference level r, the utility and disutility functions U_r and D_r, and the preference function V_r for utility-reward and disutility-risk. The shapes of U_r and D_r are related to one's attitudes toward *"better-outcome risk"* as well as *"worse-outcome risk"* under reference r. These attitudes could be characterized in much the same way the shape of a von Neumann-Morgenstern utility function characterizes risk attitude in EUT. For instance, with attention restricted to $\mathbb{X}_r^+$ and $\mathbb{X}_r^-$, respectively, the Arrow-Pratt measures (Pratt, 1964, Arrow,

1965) of risk aversion can be adapted straightforwardly to the analysis of these risk attitudes in CUT.

When outcomes x are interpreted as gains or losses (hence $r \equiv 0$), the difference $U_r - D_r$ reduces to the CPT value function v of gains and losses defined on $\mathbb{R}$; with U_r typically found to be concave on $\mathbb{X}_r^+$ and $-D_r$ convex on $\mathbb{X}_r^-$ (e.g., Kahneman and Tversky, 1979).

The manner in which V_r links $\overline{U}_r$ and $\overline{D}_r$ in CUT, however, also permits a discontinuous change in risk attitudes across the reward-risk subspaces of $\mathcal{P}$ such as "loss-averse" type of behavior (e.g., Tversky and Kahnemann, 1991). Thus the CU function can have the property of second-order risk aversion when the function U_r, D_r, and V_r are smooth, as well as the property of first-order risk aversion when some of these functions have kinks (see Segal and Spivak, 1990). Owing to space limit, I focus here on the new elements: the reference level and the degree of disutility aversion.

3.1. Determination of context-dependent reference levels

Theories involving the notion of "reference" might be classified into two categories. In the first category reference is a "status quo" concept, which is typically treated as a natural zero. This is the approach of CPT (Kahneman and Tversky, 1979; Tversky and Kahneman, 1992; Wakker and Tversky, 1993; Chateauneuf and Wakker, 1999) and ranked-weighted utility theory (Luce and Fishburn, 1991; Marley and Luce, 2001). The second category views reference as a "forward looking" concept that depends somehow on the opportunity set of gambles. Among others, in regret theory (Bell, 1982; Loomes and Sugden, 1986; Sugden, 1993) potential *regret* and *rejoicing* from a gamble are measured relative to another gamble. Similarly, Gul (1991) assumes that every lottery ticket P has its own reference level(s), $r(P)$, defined in terms of its certainty equivalent (when it exists). After a choice P has been made, outcomes that are higher than $r(P)$ produce *elation* and those lower than $r(P)$ produce *disappointment*. Luce et al. (1993) suggest an approach that takes the certainty equivalent of each gamble as the basic primitive concept, and postulate conditions for a preference interval to contain the reference level. Machina (2000) compares the implications of payoff kinks for various existing models. Koszegi and Rabin (2002) model reference as an endogenously determined choice (or gamble) that gives a "personal equilibrium." Since the reference level in CUT is modelled as a primitive parameter, its interpretation is a modeler's choice and may fall in both categories (as in Sugden, 2003).

In what follows, I define a context-dependent reference level that can be *uniquely* identified by any investigator using only exogenous information. Although the subsequent analysis can be extended to more general situations, for simplicity assume from now on that $\mathbb{X} = \mathbb{R}$. The pure outcomes in $\mathbb{X}$, interpreted as monetary payoffs, are thus ordered by $\geq$ and independent of the reference – that is, for all $x, y \in \mathbb{X}$, $x \geq y$ if and only if $\delta_x \succeq \delta_y | r$ for some (and thus for all) $r \in \mathbb{X}$. In this context, stochastic dominance (SD) can also be defined without a reference: $P \succ_{SD} Q$ if and only if $P \succ_{SD} Q | r$ for some (and thus for all) $r \in \mathbb{X}$.

Now let any subset $\Lambda \subseteq \mathcal{P}$ be given and define the *SD-efficient* subset of Λ by

$$\Lambda^e = \{P \in \Lambda : \nexists Q \in \Lambda \text{ such that } Q \succ_{SD} P\}$$

Note that $\Lambda^e \neq \emptyset$ whenever $\Lambda \neq \emptyset$, and that Λ^e contains at most one degenerate gamble.

A decision problem can be described by a non-empty opportunity set Λ ($\subseteq \mathcal{P}$) that includes all *feasible* gambles. If "no choice" is feasible, then Λ contains also one's current wealth or status quo. Since stochastic dominance is a basic criterion for rational choice under risk, there is little loss of generality to restrict attention to feasible gambles that are stochastically undominated. Let $I(P)$ be the smallest interval containing the support of P, and define subset $X_\Lambda \subseteq \mathbb{X}$ by

$$X_\Lambda = \bigcap_{P \in \Lambda^e} I(P)$$

It is clear that X_Λ contains a single element x whenever Λ^e contains a degenerate gamble δ_x, or else X_Λ is an interval.

Assumption 1. *For any subset $\Lambda \subseteq \mathcal{P}$, the reference level*

$$r = (\inf\{x | x \in X_\Lambda\} + \sup\{x | x \in X_\Lambda\})/2 \tag{3}$$

satisfies axioms B1 and B2.

We may interpret r defined in (3) as an "indicative price" of the opportunity set Λ. This price is exogenously given by the opportunity set in that it is independent of one's preference $\succeq$. A precise meaning of such a price depends on how the opportunity set is specified. For example, the opportunity set $\Lambda = \{A, B, C\}$ in Figure 1 has a price of \$10,000 because one must pay this price in order to obtain one of the payoffs of A, B, or C. If outcomes are defined as gross (resp. net) investment returns, then the price of $\{A, B, C\}$ is 1 (resp. 0). More generally, the "price" of an investment opportunity set in a perfectly competitive capital market is the *opportunity cost* of capital or the (gross) risk-free interest rate. This is because arbitrage would prevent the existence of investment opportunities that stochastically dominate, or are dominated by, the risk-free asset in the opportunity set (see, e.g., Zou, 2003 for more discussions).

Assumption 1 may also be motivated by viewing Λ as an "instant endowment" in a similar sense made in the studies of "endowment effect" (e.g., Tversky and Kahnemann, 1991, p.1041 and their references). Thus an individual's "reference wealth r" when "endowed" with opportunity set Λ is likely to incorporate the price or value (which may be negative, e.g., in an insurance context) of Λ as well as his current wealth w. The status quo wealth w is the reference level in CUT whenever maintaining the status quo is an option and no gambles can be found in the opportunity set that stochastically dominate or are dominated by w.

In general, we have identified a *unique* reference level under which all feasible and stochastically undominated gambles involve potential gains and losses. The remainder of the paper will assume that an opportunity set $\Lambda \subseteq \mathcal{P}$ and its associated reference level r defined in (3) are given, and that $\succeq$ is represented by a compound utility function $\widehat{V}_r = V_r(\overline{U}_r(\cdot), \overline{D}_r(\cdot))$.

3.2. Disutility aversion

CUT has a nice "separation property." Namely, the attitude toward risk - in the traditional sense that is measured by the shape of utility/disutility functions - is entirely separated from the attitude toward utility-reward and disutility-risk that is measured by the shape of the preference function $V_r(u, d)$ on Φ_r. This latter attitude is related to the slope of the indifference curves. From now on, assume that V_r is continuously differentiable in u and d (on the border of Φ_r the derivatives are understood as the right limit as u and/or d approaches 0). To ease notation the subscript r is omitted whenever there is no ambiguity.

Define function $\rho : \Phi_r \to \mathbb{R}^+$ (or $\overline{\rho} : \mathcal{P} \to \mathbb{R}^+$) by

$$\rho(u, d) = -\frac{\partial V_r(u, d)}{\partial d} \Big/ \frac{\partial V_r(u, d)}{\partial u} \quad (\text{or } \overline{\rho}(P) = \rho(\overline{U}_r(P), \overline{D}_r(P)))$$

I shall interpret ρ (or $\overline{\rho}$) as a measure of *disutility aversion* of V_r.

Note that, by the monotonicity property of V_r, $0 < \rho < \infty$ on Φ_r. A ρ is continuous on Φ_r by the assumption that V_r is continuously differentiable. Graphically, see Figure 3.3, $\rho(u, d)$ is the slope of the indifference curve at point (u, d). This ratio tells how much of a marginal increase in u is required to compensate for a marginal increase in d in order for the individual to be indifferent at the location (u, d). Thus $\rho(u, d)$ gives a local measure of the individual's degree of disutility-risk aversion relative to utility-reward (henceforth disutility aversion).

Alternatively, the dual measure $1/\rho$ can be interpreted as one's marginal willingness of increasing d in order for a marginal increase in u. These two measures are one-to-one, and a higher level of ρ indicates either a stronger aversion to a marginal increase in d or a weaker desire for a marginal increase in u.

3.3. CUT as a descriptive model

Behavioral patterns that cannot be explained by the shape of utility functions are traditionally examined in a three-outcome probability simplex (e.g., the Marschak-Machina triangle, Machina, 1982, p. 305). Violations of EUT occur if the revealed indifference curves are not parallel straight lines in the simplex. The u-d domain Φ_r of preference function V_r nests such simplexes as special cases, and offers a more general framework for the analysis of nonlinear preferences in terms of preferences over utility-reward and disutility-risk.

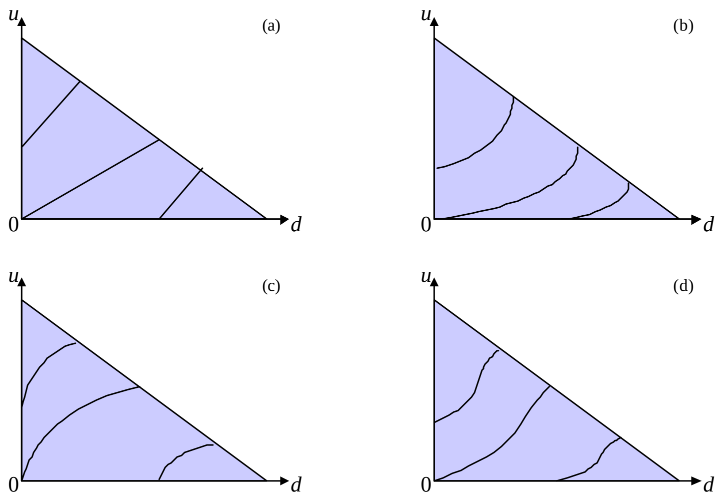

FIG. 3. The triangle is the set Φ_r of $u-d$ combinations of the choice-objects. Indifference curves in Φ_r represent (a) betweenness preferences, (b) quasiconcave preferences, (c) quasiconvex preferences, and (d) squiggle preferences. CUT accommodates all these preferences.

A very special case of CUT is EUT where ρ is everywhere constant on Φ_r. The betweenness models are another special case (Figure 3.3(a)) where ρ is constant along each indifference curve on Φ_r. In general, the functional form of $\rho(u,d)$ allows us to characterize and accommodate other preference patterns (e.g., Camera and Ho, 1994). For instance, "increasing (decreasing) disutility aversion along the indifference curves" explains and characterizes quasiconcave (quasiconvex) preference as depicted in Figure 3.3(b) (Figure 3.3(c)). The shape of indifference curves depicted in Figure 3.3(d) is called the squiggle hypothesis (e.g., Bernasconi, 1994) where disutility aversion first increases and then decreases along the indifference curves.

Other phenomena such as simultaneous insurance and gambling can be accommodated in CUT by interpreting r to be one's current (context-dependent) wealth level, and assuming $U_r(x)$ to be convex over some interval of higher wealth levels as suggested by Friedman and Savage (1948, Fig. II & III). Since CUT allows the utility/disutility functions as well as their definition domains to move with one's wealth level r, traditional objections to Friedman and Savage's explanations do not arise in CUT (see, e.g., Machina, 1982, Section 2.3).

3.4. Comparing CUT with CPT: An Illustration

In the context of decision under risk, CPT assumes that preferences can be represented by the "expectation" of a two-piece value function v defined on gains and losses respectively, where expectation E_π is computed with one's decision weights π rather than true probabilities p. Let $\mathbb{X}^+$ denote the set of gains and $\mathbb{X}^-$ the set of losses relative to one's status quo (or natural zero), then CPT assumes that one maximizes

$$E_\pi(v) = \sum_{x_i \in \mathbb{X}^+} \pi_i^+ v(x_i) + \sum_{x_i \in \mathbb{X}^-} \pi_i^- v(x_i) \tag{4}$$

where π_i^+ and π_i^- are the decision weights associated with state i. For consistency with stochastic dominance, CPT assumes that $E_\pi(v)$ takes a rank-dependent form in which $\pi_i's$ are related to the cumulative probabilities through the individual's probability weighting function (see, e.g., Tversky and Kahneman,1992).

The opportunity set $\{A, B, C\}$ in Figure 1 can be seen as involving four states $i = 1, 2, 3, 4$ with equal probability $p_i = 0.25$. Taking into account the \$1 (= \$10,000) investment let us write $v_1 = v(-0.4)$ and $v_2 = v(-0.2)$ for losses, and $v_3 = v(0.2)$ and $v_4 = v(0.5)$ for gains. It can then be shown that for the three gambles,

$$\begin{aligned} E_\pi(v; A) &= (\pi_1^- + \pi_2^-)v_2 + (\pi_3^+ + \pi_4^+)v_3 \\ E_\pi(v; B) &= (\pi_1^- + \pi_2^-)v_1 + (\pi_3^+ + \pi_4^+)v_4 \\ E_\pi(v; C) &= \pi_1^- v_1 + \pi_2^- v_2 + \pi_3^+ v_3 + \pi_4^+ v_4 \end{aligned}$$

It follows from preferring C that

$$\begin{aligned} \text{(i) } E_\pi(v; C) &> E_\pi(v; B) \Leftrightarrow \pi_2^-(v_2 - v_1) > \pi_3^+(v_4 - v_3) \\ \text{(ii) } E_\pi(v; C) &> E_\pi(v; A) \Leftrightarrow \pi_4^+(v_4 - v_3) > \pi_1^-(v_2 - v_1) \\ \text{(i) and (ii) hold } &\Leftrightarrow \pi_2^- \pi_4^+ > \pi_1^- \pi_3^+ \end{aligned} \tag{5}$$

In other words, $\pi_i \neq p_i = 0.25$ for some i. One explanation for this revealed preference could be that the individual somehow overweights the probability of receiving $x_4 = \$15,000$ or underweights the probability of receiving $x_1 = \$6,000$.

A limitation of CPT is to assume that preferences depend only on the changes in wealth and not on the wealth levels. Although this assumption simplifies the modelling of the reference point or status quo – which can then be assumed to be a "natural zero" – it may cause ambiguity in application. For instance, assume in Figure 1 that the $10,000 investment has been made and the subject is now offered the chance to re-evaluate the risky payoffs of A, B, and C. Then according to CPT, all the payoffs would now be judged as *gains* with respect to zero. But the investment decision has been made treating payoffs below $10,000 as losses. As a result, the "natural zero" interpretation of reference point is likely to lead to inconsistent predictions of behavior.[5] Kahneman and Tversky (1979) call this a reference-shifting problem and admit that the "location of the reference point, and the manner in which choice problems are coded and edited emerge as critical factors in the analysis of decisions (p.288)."

Another limitation of CPT (as well as other rank-dependent models) is to explain nonlinear preferences *only* through decision weights or probability weighting functions. To illustrate, suppose we fix the four states and their probabilities p_i $(= 0.25)$ in Figure 1 and let the outcomes vary (without changing their ranks). Then CPT cannot accommodate the choice patterns where C is most preferred at some outcome levels yet is least preferred at some other outcome levels. This is because $\pi_i' s$ are fixed by the probability weighting functions (which depend only on p and the ranks of outcomes), thus the inequality in (5) cannot change sign. Such a choice pattern, however, is consistent with CUT when the individual's preference over u and d is quasiconcave at some levels of u and d and quasiconvex at some other levels of u and d (see Figure 3.3(d)).

Assuming that CUT holds, then attitude toward utility-reward u and disutility-risk d can be directly analyzed by the indifference curves $\{(u, d) : V_r(u, d) = \text{constant}\}$ on the u-d plane. The preference for C in our experiment now reveals two things. First, the decision maker's reference level for the opportunity set $\{A, B, C\}$ must be lower than the greatest outcome and higher than the smallest outcome; otherwise CUT reduces to EUT and gamble C would never be chosen. Second, for reference levels $r \in (0.6, 1.5)$,[6] the preference function $V_r(u, d)$ cannot be quasilinear in (u, d) (i.e., indifference curves are straight lines) on the u-d plane. It is easy to verify that C lies half-way on the straight line connecting A and B. Therefore a quasilinear function $V_r(u, d)$ is only consistent with C being the second choice (unless all gambles are equivalent). Figure 3.4(a) depicts the indifference curves of a preference function V_r that is quasiconcave in (u, d) and that is consistent with C being the best choice.

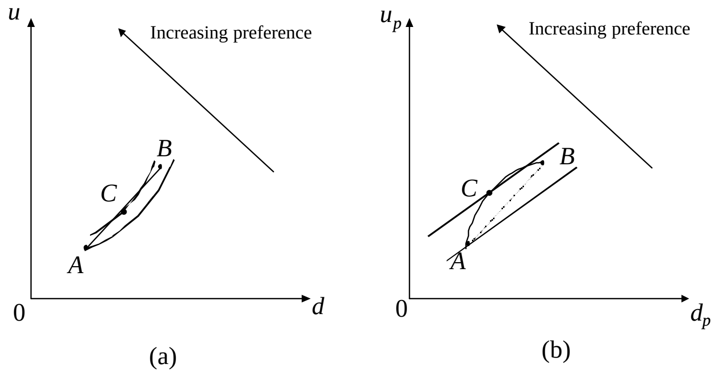

FIG. 4. Two views of the same effect that C is preferred to B and B is preferred to A. (a) CUT says that the individual has increasing disutility aversion along the indifference curves (indicated by the thick convex curves) where utility-reward u and disutility-risk d are measured by expected utility for better outcomes and expected disutility for worse outcomes, respectively. Any probability mixture of A and B must be plotted on the straight line connecting A and B. (b) CPT (and the rank dependent approach in general) says that the utility-reward u_π and disutility risk d_π are measured by the individual's subjective decision weights rather than true probabilities so that C – the "(nonadditive) subjective probability mixture" of A and B – may be higher than the (dotted) straight line connecting A and B. Indifference curves on the $u_\pi - d_\pi$ plane are the parallel straight lines.

Quite obviously, all preference functions that take an "expectation" (i.e., additive or integrable) form can be written as the difference between two parts: a "better part" and a "worse part" with respective to some reference level. In particular, the CPT preference in (4) can be equivalently written as

$$E_\pi(v) = u_\pi - d_\pi \quad (u_\pi = \sum_{x_i \in \mathbb{X}^+} \pi_i^+ v(x_i), \quad d_\pi = -\sum_{x_i \in \mathbb{X}^-} \pi_i^- v(x_i)).$$

Interpreting u_π and d_π as the "rank-dependent" reward and risk measures and viewing $E_\pi(v)$ as a linear function of u_π and d_π, we can plot the indifference curves $\{(u_\pi, d_\pi) : E_\pi(v) = \text{constant}\}$ on the plane spanned by u_π and d_π. These indifference curves are the parallel 45° straight lines. In order to accommodate preference for C, then, the position of C necessarily deviates from the point half-way on the straight line connecting A and B (see Figure 3.4(b)).

The two figures (a) and (b) in Figure 3.4 highlight the different views of CUT and CPT regarding the effects of nonlinear preferences. Whereas both theories explain the same observed (quasiconcave) behavior at a descriptive level, CUT appears to offer a more transparent *explanation* for such *effects*: "For some people the *undesirability* of disutility-risk may increase more than in proportion to increases in the magnitude of disutility-risk. Consequently, some people may demand increasingly higher marginal compensations of utility-reward for increases in disutility-risk."

4. CONCLUDING DISCUSSION

This paper has presented a simple, perhaps novel, approach to decision making under risk. The main contribution is a general representation theory (called compound utility theory or CUT) that is both normative (in partly keeping the independence axiom of EUT) and easy to use. The theory unifies – conceptually and to a certain extent structurally – two hitherto separate mainstream approaches: one in the tradition of EUT where preference functions are linearly additive in probabilities, the other in the tradition of risk and return trade-offs where preference functions depend on exogenously defined measures of reward and risk (e.g., the mean-variance model or the downside-risk models).

As an alternative to cumulative prospect theory (CPT), the new theory demonstrates that "anomalies" in the EU paradigm can be explicable within a simple rational model without having to invoke psychological assumptions about distortion of (true) probabilities in people's perception of risk. As long as the decision maker does not perceive all possible outcomes of choice as better (or as worse) outcomes, CUT is able to explain a broad range of choice behavior.

A distinctive feature of CUT is its notion of *utility-reward* and *disutility-risk* (or reward and risk for short) measured by the expected utility for better outcomes and expected disutility for worse outcomes respectively. Assuming the independence axiom for rational judgment of reward and risk, the theory preserves the normative appeal and analytical simplicity of EUT. The descriptive capacity of the theory, however, is greatly enhanced by allowing the preference over reward and risk to be convex, concave, or varying as reward and risk change.

For monetary outcomes, a theory of reference level determination is developed as well, which ensures a unique, context dependent, reference level for applying CUT. Different from the "natural zero" assumption of reference point in CPT, this reference level is determined by the opportunity set and always entails potential gains and losses. Consequently, unlike CPT where only the relative changes of wealth affect preferences, CUT allows preferences to be affected by absolute levels as well as relative changes of wealth.

Attitude toward utility-reward and disutility-risk derives from the intuitive premise that potential gains and losses, or pleasure and pain, that any gamble entails are more fundamental attributes than the gamble itself. They affect our choice, but they can also be evaluated independently of our choice. CUT suggests that judgment and choice are decisions of two kinds. Judgment is likely to benefit from a *normative theory* that helps ensure consistency in our evaluation of reward and risk, whereas choice may be affected by the *feeling* or *psychology* of individuals. This intuition enables CUT to achieve a nice balance between specificity (expected utility/disutility representation of reward and risk) and generality (varying degree of disutility aversion or shapes of the indifference curves).

Worth remarking is the fact that CUT is not always more general than CPT. Conceivably, behaviors under some complex situations may still call for attitude toward probabilities to explain. A true generalization of CPT (as well as CUT) could be one that incorporates decision weights of rank-dependent models into the evaluation of utility-reward and disutility-risk. A "rank-dependent compound utility" could then take the form of, say,

$$V_r(u_\pi, d_\pi) = V_r(E_\pi(U_r), E_\pi(D_r))$$

where the "expectation" E_π is computed using decision weights π instead of true probabilities (cf. eq. (4)).

Generalizing CUT to dynamic decision problems where commitment to a sequential strategy cannot be made at the start may be more promising, however. Important issues such as dynamic consistency, path dependence, changing preferences or tastes, etc., could then be addressed (e.g., Kreps and Porteus, 1979; Epstein and Zin, 1989). It is my hope that the reader concurs that the strong intuitive ground of CUT promises more interesting findings down the road.

NOTES

1. This paper grew substantially from a discussion paper entitled "Proposing a Compound Utility Approach to Decision Making under Risk" which I presented at the 2003 meetings of the Econometric Society and the European Economic Association in Stockholm. I wish to thank several participants there, as well as Rod Aya, Sugato Bhattacharyya, Eddie Dekel, Audrey Hu, David de Meza, Laixiang Sun, Kin Lam, Florian Wagener, Peter Wakker, Zaifu Yang and colleagues and seminar participants at the University of Amsterdam, Hong Kong Baptist University, University of Hong Kong, and the Tinbergen Institute for helpful discussions, comments, and suggestions.

2. Nonlinear preferences figure in, e.g., rank-dependent theories (e.g., Quiggin, 1982; Yaari, 1987), subjective probability theories (e.g., Gilboa, 1987; Schmeidler, 1989), mixture symmetry and quadratic utility theory (Chew et al., 1991), ranked-weighted theories (Luce and Fishburn, 1991; Marley and Luce, 2001), cumulative prospect theory (Tversky and Kahne-

man, 1992; Wakker and Tversky, 1993; Chateauneuf and Wakker, 1999), and multiple-priors theories (Jaffray, 1989; Gilboa and Schmeidler, 1989). For more recent contributions, see, e.g., reviews by Starmer (2000), Sugden (2000), Schmidt (2000), and Bell and Fishburn (2000). See also Harless and Camerer (1994) and Hey and Orme (1994) on comparing theories of choice under risk and uncertainty.

3. Empirical evidence supporting the view that the reference (or target) levels may be affected by the choice context is abundant (e.g., Luce et al., 1993 and their references). These levels may be influenced by the opportunity set, social norms, market sentiment, one's past experience and current wealth, and so on (e.g., Tversky and Kahnemann, 1991). Note that Sugden's concept of reference is more general in scope than the reference level in CUT because he allows the reference to be either a deterministic outcome or an *act* (gamble) with uncertain consequences (outcomes). On the other hand, as Sugden acknowledges, his model is linear in probabilities and therefore cannot accommodate nonlinear preferences such as the Allais paradoxes.

4. A distinct benefit of focusing on the set of simple probability measures, apart from its simplicity, is that the outcome set $\mathbb{X}$ is general. We shall assume that $\mathbb{X}$ consists of at least three distinct elements. The choice-objects $P \in \mathcal{P}$ may in fact be interpreted as either objective or subjective, provided they are probability measures in the usual sense (e.g., Arrow, 1951, pp. 405-6). Note that by assuming that $\mathcal{P}$ includes *all* simple probability measures on $\mathbb{X}$ we force ourselves (as in EUT) to be concerned with both feasible and unfeasible (purely hypothetical) gambles. Even though actual decisions are made under specific choice contexts, inclusion of hypothetical gambles allows us to imagine – and therefore assume – a richer set $\mathcal{P}$.

5. Sagi (2002) outlines similar potential choice "inconsistencies" in a large class of reference dependent theories including CPT.

6. Indeed, the investment capital 1 sets a natural target level (assuming the risk-free rate is zero) in this context. Note that one may add the decision maker's current wealth to every number in the gambles without affecting the subsequent analysis.

APPENDIX A

Proof of Theorem 1: The necessity part of Theorem 1, {G1,G2} $\Rightarrow$ {A1,A2,B1,B2}, is straightforward hence is omitted. The sufficiency part, {A1,A2,B1,B2} $\Rightarrow$ {G1,G2}, is proved here by way of lemmas when clarity is enhanced. Since for any $r \in \mathbb{X}$ given and fixed, there is no ambiguity writing $P \succeq Q$ instead of $P \succeq Q|r$, {A1,A2,B1} $\Rightarrow$ {G1(i)} is standard when gambles are restricted to $\mathcal{P}^U(r, \succeq)$ and $\mathcal{P}^D(r, \preceq)$ respectively (e.g., see

Kreps, 1988, Theorem (5.15) for an easier reference). We assume therefore that G1(i) holds and start with the proof of G1(ii).

Assume {A1,A2,B1}. For any $r \in \mathbb{X}$ satisfying B1, let another utility function U_r^0 having the same property as U_r as stated in G1(i) be given. For all $x_1 \succeq x_2 \succ r|r$ it can then be shown that there exists a unique $\theta \in (0,1]$ such that $x_2 \sim \theta x_1 \oplus (1-\theta)r|r$ (e.g., Kreps, 1988, Lemma (5.6)). Thus, from G1(i), $U_r(x_2) = \theta U_r(x_1) > 0$ and $U_r^0(x_2) = \theta U_r^0(x_1) > 0$. Dividing the latter by the former on both sides of the two equations yields

$$\frac{U_r^0(x_2)}{U_r(x_2)} = \frac{U_r^0(x_1)}{U_r(x_1)} = a > 0 \text{ independent of } x_1, x_2,$$

which implies the uniqueness property $U_r^0(x) = aU_r(x)$ for all $x \succ r|r$. Clearly, $U_r^0(r) = aU_r(r) = 0$ is also true. The remaining proof concerning D_r is similar hence omitted. This completes the proof of G1.

Now we show {A1,A2,B1,B2} $\Rightarrow$ {G1,G2}: Assume {A1,A2,B1,B2} holds. Fix (U_r, D_r) and define $(\overline{U}_r, \overline{D}_r) : \mathcal{P} \to \mathbb{R}_+^2$ by

$$(\overline{U}_r(P), \overline{D}_r(P)) = (E(U_r; P), E(D_r; P)).$$

LEMMA 1. *For all $P, Q \in \mathcal{P}$ and $r \in \mathbb{X}$, (i) if $P \sim Q|r$, then $\overline{U}_r(P) > \overline{U}_r(Q) \Leftrightarrow \overline{D}_r(P) > \overline{D}_r(Q)$. (ii) If $\overline{U}_r(P) = \overline{U}_r(Q)$ and $\overline{D}_r(P) = \overline{D}_r(Q)$ then $P \sim Q|r$.*

Proof: Let $P, Q \in \mathcal{P}$ and $r \in \mathbb{X}$ be given.

(i) If $\overline{U}_r(P) > \overline{U}_r(Q)$ and $\overline{D}_r(P) \leq \overline{D}_r(Q)$ then by G1 and B2 $P \succ Q|r$. Similarly, if $\overline{U}_r(P) \leq \overline{U}_r(Q)$ and $\overline{D}_r(P) > \overline{D}_r(Q)$ then $P \prec Q|r$. (ii) Suppose $\overline{U}_r(P) = \overline{U}_r(Q)$ and $\overline{D}_r(P) = \overline{D}_r(Q)$ but $P \succ Q|r$. Define Z for any $\theta \in (0,1)$ as follows

$$Z = \theta[\bigoplus_{x \succ r} P(x)\delta_x] \oplus [\bigoplus_{x \prec r} P(x)\delta_x] \oplus [1 - \theta \sum_{x \succ r} P(x) - \sum_{x \prec r} P(x)]\delta_r$$

It is easy to verify that $Z \in \mathcal{P}$, and that $Z_r^+ \sim \theta P_r^+ \oplus (1-\theta)\delta_r|r$ and $Z_r^- \sim P_r^-|r$.

Consequently, $\overline{U}_r(Z) = \theta\overline{U}_r(P) < \overline{U}_r(Q)$ and $\overline{D}_r(Z) = \overline{D}_r(P) = \overline{D}_r(Q)$, which implies $P \succ Q \succ Z|r$. By A2, then, there exists $\psi \in (0,1)$ such that $\psi P \oplus (1-\psi)Z \succ Q|r$. However, since $\overline{U}_r(\psi P \oplus (1-\psi)Z) = \psi\overline{U}_r(P) + (1-\psi)\overline{U}_r(Z) < \overline{U}_r(Q)$ and $\overline{D}_r(\psi P \oplus (1-\psi)Z) = \overline{D}_r(Q)$, B2 implies $\psi P \oplus (1-\psi)Z \prec Q|r$. This contradiction shows that we must have $P \sim Q|r$. $\square$

In light of Lemma 1 it is meaningful to define a triadic relation $\succ_{ud}$ (and analogously, $\sim_{ud}$ and $\succeq_{ud}$) on Φ_r such that for all $r \in \mathbb{X}$ and $(u_i, d_i) =$

$(\overline{U}_r(P_i), \overline{D}_r(P_i)) \in \Phi_r$ where $P_i \in \mathcal{P}$, $i = 1, 2$,

$$(u_1, d_1) \succ_{ud} (u_2, d_2)|r \text{ if and only if } P_1 \succ P_2|r. \tag{A.1}$$

Define next the indifference sets in $\mathcal{P}$ and indifference sets in Φ_r, respectively, as

$$I_r(P) = \{Q \in \mathcal{P} : Q \sim P|r\} \quad \text{and} \quad I_r(u, d) = \{(\tilde{u}, \tilde{d}) \in \Phi_r : (\tilde{u}, \tilde{d}) \sim_{ud} (u, d)|r\}$$

It is easily seen that fixing any $r \in \mathbb{X}$, the collection of distinct $I_r(P)$'s (respectively, $I_r(u, d)$'s) form a partition of $\mathcal{P}$ (respectively, Φ_r). Under B2, the indifference sets in Φ_r are upward sloping (see Figure 3.3) and may take various shapes.

Let $I_r(\mathcal{P})$ denote the set of all $I_r(P) \subset \mathcal{P}$ and $I_r(\Phi_r)$ the set of all $I_r(u, d) \subset \Phi_r$. Define $f : I_r(\mathcal{P}) \to I_r(\Phi_r)$ such that for all $I_r(P) \in I_r(\mathcal{P})$, $f(I_r(P)) = I_r(\overline{U}_r(P), \overline{D}_r(P))$. Then f is an order-isomorphism from $I_r(\mathcal{P})$ onto $I_r(\Phi_r)$. That is, f is a one-to-one correspondence between $I_r(\mathcal{P})$ and $I_r(\Phi_r)$, and satisfies further that for all $I_r(P), I_r(Q) \in I_r(\mathcal{P})$, $f(I_r(P)) \succ_{ud} f(I_r(Q))|r$ holds if and only if $I_r(P) \succ I_r(Q)|r$. The meaning of these relations are self-evident. (The inverse function is given by $f^{-1}(I_r(u, d)) = \{P \in \mathcal{P} : ((\overline{U}_r(P), \overline{D}_r(P)) \sim_{ud} (u, d)|r\}$ for all $I_r(u, d) \subset \Phi_r$.)

Since Φ_r is a convex subset of $\mathbb{R}^2_+$ (indeed, a triangle if $\overline{U}_r$ and $\overline{D}_r$ are bounded) and since two isomorphic completely ordered sets are essentially identical, A1-A2 must hold for $\succeq_{ud}$ on Φ_r as well.

Lemma 2. *For any $P \succ \delta_r|r$, there exists a unique $\alpha_r(P) \in (0, 1]$ such that*

$$P \sim \alpha_r(P)P_r^+ \oplus (1 - \alpha_r(P))\delta_r|r. \tag{A.2}$$

Dually, for any $P \prec \delta_r|r$, there exists a unique $\alpha_r(P) \in (0, 1]$ such that

$$P \sim \alpha_r(P)P_r^- \oplus (1 - \alpha_r(P))\delta_r. \tag{A.3}$$

Proof: Suppose $P \succ \delta_r|r$ (the case with $P \prec \delta_r|r$ is analogous). If $P \sim P_r^+|r$, then by G1 (A.2) holds if and only if $\alpha_r(P) = 1$. If $P \nsim P_r^+$, then $P_r^+ \succ P$ by B2 and we define sets $Y_1(r)$ and $Y_2(r)$ as follows.

$$\begin{aligned} Y_1(r) &= \{\theta | \theta P_r^+ \oplus (1 - \theta)\delta_r \succ P|r; \ \ \theta \in [0, 1]\} \\ Y_2(r) &= \{\theta | \theta P_r^+ \oplus (1 - \theta)\delta_r \prec P|r; \ \ \theta \in [0, 1]\} \end{aligned}$$

A2 implies that these sets are not empty. For all $\psi \in Y_1$ and $\phi \in Y_2$, we must have $\psi > \phi$, which follows from A1 (transitivity) and G1. Thus there exists α_r such that $\inf\{\theta|\theta \in Y_1(r)\} \geq \alpha_r \geq \sup\{\theta|\theta \in Y_2(r)\}$. By A2, this α_r does not belong to either $Y_1(r)$ or $Y_2(r)$, thus $P \sim \alpha_r P_r^+ \oplus (1-\alpha_r)\delta_r|r$. G1 and transitivity of $\sim$ further imply that this α_r is unique, which defines $\alpha_r(P)$. □

By order-isomorphism, Lemma 2 has an equivalent statement that for all $(u,d) \in \Phi_r$ there exists a unique $\alpha_r(u,d) \in (0,1]$ such that $I_r(u,d) \sim_{ud} I_r(\alpha_r(u,d)u,0)|r$ if $(u,d) \succ_{ud} (0,0)|r$ and $I_r(u,d) \sim_{ud} I(0,\alpha_r(u,d)d)|r$ if $(u,d) \prec_{ud} (0,0)|r$. This suggests a simple way to define $V_r : \Phi_r \to \mathbb{R}$ as follows (cf. the differentiable-path approach in Mehta, 1998).

$$V_r(u,d) = \begin{cases} \alpha_r(u,d)u & \text{if } (u,d) \succ_{ud} (0,0)|r \\ 0 & \text{if } (u,d) \sim_{ud} (0,0)|r \\ -\alpha_r(u,d)d & \text{if } (u,d) \prec_{ud} (0,0)|r \end{cases}$$

That V_r represents $\succeq_{ud}$ in that for all $(u_1,d_1),(u_2,d_2) \in \Phi_r$, $(u_1,d_1) \succeq_{ud} (u_2,d_2)|r$ if and only if $V_r(u_1,d_1) \geq V_r(u_2,d_2)$ is now obvious by our construction of α_r in Lemma 2. That is, for all $P,Q \in \mathcal{P}$,

$$P \succeq Q|r \Leftrightarrow (\overline{U}_r(P),\overline{D}_r(P)) \succeq_{ud} (\overline{U}_r(Q),\overline{D}_r(Q))|r \qquad (A.4)$$
$$\Leftrightarrow V_r(\overline{U}_r(P),\overline{D}_r(P)) \geq V_r(\overline{U}_r(Q),\overline{D}_r(Q)). \qquad (A.5)$$

which is condition (2) of G2. The monotonicity properties for V_r follow readily from G1 and B2.

Since Φ_r is a subset of $\mathbb{R}^2_+$, continuity of V_r is easy to establish (see Figure **??**). Let any $(u,d) \in \Phi_r$ and $\varepsilon > 0$ be given. For concreteness, assume that $(u,d) \sim_{ud} (v,0) \succ_{ud} (0,0)|r$ where $v = V_r(u,d)$ (the other cases are similar, hence omitted). Define

$$Y_r^+ = \{(\widetilde{u},\widetilde{d}) \in \Phi_r : (\widetilde{u},\widetilde{d}) \succeq_{ud} (v+\varepsilon,0)|r\} = \{(\widetilde{u},\widetilde{d}) \in \Phi_r : V_r(\widetilde{u},\widetilde{d}) \geq v+\varepsilon\}$$
$$Y_r^- = \{(\widetilde{u},\widetilde{d}) \in \Phi_r : (\widetilde{u},\widetilde{d}) \preceq_{ud} (v-\varepsilon,0)|r\} = \{(\widetilde{u},\widetilde{d}) \in \Phi_r : V_r(\widetilde{u},\widetilde{d}) \leq v-\varepsilon\}$$

where the second equality of each equation follows from (A.4)-(A.5).

If v is the maximum u in Φ_r, Y_r^+ is empty and we only need to consider Y_r^-. More generally, assume that (u,d) is an interior point of Φ_r and ε is small enough so that both Y_r^+ or Y_r^- are non-empty. Then, (A.4)-(A.5) ensure that the distance of (u,d) from Y_r^+ (and from Y_r^-) is strictly positive for all $\varepsilon > 0$; for else there would be $(\widetilde{u},\widetilde{d})$ in Y_r^+ or in Y_r^- with $(\widetilde{u},\widetilde{d}) = (u,d)$, implying the contradiction $(v,0) \succeq_{ud} (v+\varepsilon,0)|r$ or $(v,0) \preceq_{ud} (v-\varepsilon,0)|r$. In other words, there exists sufficiently small $\gamma > 0$ such that the set

$$B_\gamma(u,d) = \{(\widetilde{u},\widetilde{d}) \in \Phi_r : \sqrt{(\widetilde{u}-u)^2 + (\widetilde{d}-d)^2} < \gamma\}$$

does not belong to either Y_r^+ or Y_r^-. Consequently, $B_\gamma(u,d) \subset V_r^{-1}(B_\varepsilon^v)$ where $B_\varepsilon^v = (v-\varepsilon, v+\varepsilon) \subset \mathbb{R}$ and $V_r^{-1}(B_\varepsilon^v)$ is the inverse image of B_ε^v

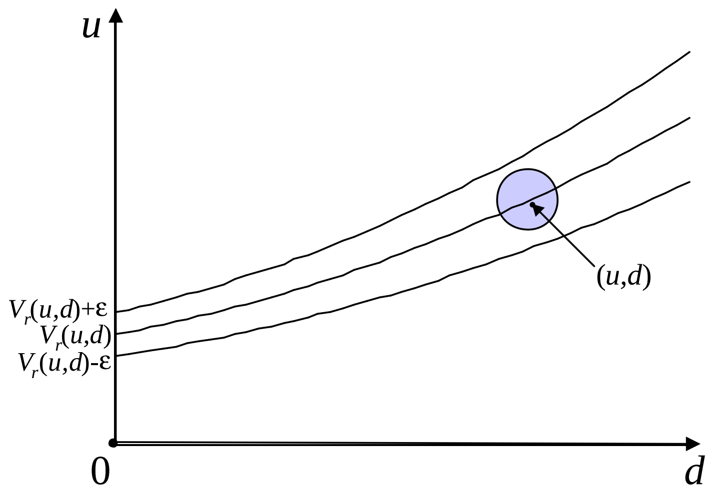

FIG. 5. Continuity of V_r: Given any $(u,d) \in \Phi$, for all $\varepsilon > 0$ there exists a neibourhood of (u,d) (the shaded "disc") such that for all $(\widetilde{u},\widetilde{d})$ in the "disc", $||V_r(u,d) - V_r(\widetilde{u},\widetilde{d})|| < \varepsilon$.

(i.e., the set of all $(\widetilde{u},\widetilde{d}) \in \Phi_r$ with $V_r(\widetilde{u},\widetilde{d}) \in B^v_\varepsilon$). This establishes the continuity of V_r.

Finally, for any strictly increasing function $g : \mathbb{R} \to \mathbb{R}$, $g \circ V_r$ obviously satisfies (A.4)-(A.5). Conversely, let any V_r^0 be given that has the same property as V_r. We have $V_r^0(u,d) = V_r^0(0,0)$ if $(u,d) \sim_{ud} (0,0)|r$, $V_r^0(u,d) = V_r^0(\alpha_r(u,d)u,0)$ if $(u,d) \succ_{ud} (0,0)|r$ and $V_r^0(u,d) = V_r^0(0,\alpha_r(u,d)d)$ if $(u,d) \prec_{ud} (0,0)|r$. Defining $f(v) = V_r^0(v,0)$ on $\{v \geq 0 : v = V_r(u,d)$ for some $(u,d) \in \Phi_r\}$ and $f(v) = V_r^0(0,-v)$ on $\{v \leq 0 : v = V_r(u,d)$ for some $(u,d) \in \Phi_r\}$ yields

$$f(V_r(u,d)) = \begin{cases} f(\alpha_r(u,d)u) = V_r^0(\alpha_r(u,d)u,0) = V_r^0(u,d) & \text{if } (u,d) \succ_{ud} (0,0)|r \\ V_r^0(0,0) & \text{if } (u,d) \sim_{ud} (0,0)|r \\ f(-\alpha_r(u,d)d) = V_r^0(0,\alpha_r(u,d)d) = V_r^0(u,d) & \text{if } (u,d) \prec_{ud} (0,0)|r \end{cases}$$

Thus $f : \mathbb{R} \to \mathbb{R}$ satisfies $f(V_r(u,d)) = V_r^0(u,d)$ and f is strictly increasing on $\{v : v = V_r(u,d)$ for some $(u,d) \in \Phi_r\}$. □

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