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Local Bifurcations and Global Dynamics in a Solow-type Endogenous Business Cycles Model

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Neoclassical growth models are essentially characterized by the formation of a steady state where the main economic aggregates (capital, output, consumption and investment) do not grow, unless some external event takes place (e.g., technological progress or population growth). Hence, the long term result corresponds to a fixed point and any kind of endogenous fluctuations is ruled out. This paper introduces an inefficiency mechanism through which non perfect firms' expectations about future demand conduct to underinvestment, which originates periodic and a-periodic long run cycles; this mechanism is applied over the simplest neoclassical growth model, the Solow equation regarding capital accumulation. Results, containing a large variety of dynamic features, are discussed through the analysis of local bifurcations and through numerical examples that give insights about global dynamics.

Key Words: Endogenous business cycles; Solow growth model; Nonlinear dynamics; Bifurcations; Chaos.
JEL Classification Numbers: C61, E32, O41.

1. INTRODUCTION

Economic literature has produced, over the past decades, multiple explanations concerning business cycles. While some argue that fluctuations in macroeconomic aggregates are the result of market imperfections, others try to conciliate Walrasian market structures with the clear observation that economies do not tend to a fixed point steady state.

The Keynesian view, originally developed by Phelps (1970) and Lucas (1972), explains cycles in a context of market anomalies; there are nominal

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rigidities (i.e., prices and wages do not adjust automatically in response to changes in real phenomena) and coordination failures are likely to occur (i.e., incomplete knowledge about market conditions leads to sequences of generalized high and low confidence periods arising from strategic complementarities that would not be observable under a fully competitive market setup).

On the other hand, real business cycles (RBC) theory, proposed by Kydland and Prescott (1982) and Long and Plosser (1983), argues that macroeconomic fluctuations occur in perfectly competitive markets. In RBC models a disturbance in some real variable (e.g., technology or government purchases) implies changes in the availability of labor, and consequently in the potential to generate income. This process is strongly based on microeconomic foundations in the sense that labor force availability is a result of the choice between leisure utility and additional income that additional hours of work allow to produce.

RBC theory is able to explain fluctuations in economic aggregates over time within the structure of conventional growth models, but this is only possible through the introduction of stochastic perturbations like an innovation process. This is an important shortcoming that resembles the main obstacle of the first (neoclassical) economic growth models: the true primary source shaping economic evolution over time is in fact exogenous to the model, that is, it is seen as something that comes from outside the economic system.

A careful analysis of the implications of the two previous interpretations of the growth process with fluctuations leads us to the conclusion that they both emphasize important aspect of cyclical economic movements, but the two approaches can hardly be put together to form a unified explanation of business cycles. While the Keynesian view has the merit of clearly stating that cycles are the direct result of market structures departing from the perfectly competitive benchmark, the RBC theory has the advantage of integrating macro fluctuations in the known growth paradigms. Putting these two interpretations together and, at the same time, avoiding making exogenous stochastic perturbations a central piece of the business cycles modelling is the task that we propose to undertake in this paper.

The model to develop along the following sections contains the following features:

i) A growth framework will be considered. Our important argument is that without any market imperfection the growth model is able to explain long term smooth trends of growth, that is, it reveals the steady state (fixed point) result that one would obtain in a perfect world of no inefficiencies or information problems;

ii) A purely deterministic structure is assumed. As stated, this is a departure from the RBC models, to which the whole explanation of fluctuations relies on the presence of exogenous shocks;

iii) A market imperfection is added to the growth model. In this way we conciliate the RBC view (by considering a benchmark growth model) with the Keynesian view (by assuming that markets do not allocate resources with full efficiency). The engine of cycles over the growth process is in fact this imperfection, that we consider to have its origin in the firms' expectations about future demand. If firms do not have perfect information and they are risk averse they will underinvest and the rule they adopt to make investment decisions, besides being suboptimal, is also likely to generate business cycles.

A model that encompasses the previous items needs to make use of a framework involving non linear dynamics in discrete time. Continuous time models are also able to generate endogenous deterministic cycles, but it is known from the literature that a discrete time setup is more suitable because low dimensional nonlinear discrete systems frequently lead to complex dynamics. The simplest discrete model able to generate complex dynamics is the one dimensional logistic map, thoroughly studied by May (1976). Our rule of demand expectations, to be defined in the following sections, is similar to a logistic map.

Economic literature has resorted, in the last few decades, to nonlinear discrete dynamics to explain long term cycles involving the growth process. Some of the earlier more significant work at this level includes Stutzer (1980), Day (1982), Boldrin and Montrucchio (1986) and Deneckere and Pelikan (1986) [and also, under overlapping generations modelling structures, Benhabib and Day (1981, 1982) and Grandmont (1985)]. While the first models seemed to include nonlinear features in a rather ad-hoc way, more recent approaches generate nonlinear results based on solid and reasonable economic assumptions.

One of the most relevant strands of literature that incorporates endogenous fluctuations in the optimal growth model is the one developed by Nishimura, Sorger and Yano (1994), Nishimura and Yano (1994, 1995), Mitra (1996, 2001), Nishimura, Shigoka and Yano (1998), Boldrin, Nishimura, Shigoka and Yano (2000), Mitra and Nishimura (2001), Khan and Mitra (2005). Basically, this group of authors has searched for specific conditions under which optimal growth models, without any associated market imperfections, lead to long term stability results other than a fixed point (namely periodic cycles or chaos). The accomplished results indicate that strange dynamics are associated with some special conditions, like unrealistically low discount factors, the adoption of non standard production functions or the need to consider more than one economic sector. Therefore, cycles are not, indeed, a universally observable phenomenon in the competitive growth structure.

A line of work that is closer to the framework that we will propose is the one followed by Christiano and Harrison (1999), Schmitt-Grohé (2000), Guo and Lansing (2002) and Coury and Wen (2005). These authors adopt, as their guiding model, the optimal growth model with intertemporal leisure-labor choices, as in the RBC theory, but they exclude any stochastic component. The source of endogenous fluctuations is, in these models, a production externality. Combining labor market considerations with increasing returns due to intersectoral spillovers, the referred models are capable of generating long term time series displaying cycles of various orders and even chaotic motion. This occurs for specific parameter values, and the main criticism that these models can be confronted with is precisely that cycles are possible only for externality values that are unrealistically high.

The models described in the last paragraph include the features that we have highlighted as necessary to integrate Keynesian and RBC interpretations of cycles: a growth setup is assumed, the structure is entirely deterministic and the source of cycles is a market imperfection. Following a same type of reasoning, the following pages furnish an alternative market imperfection as the source of cycles. The adopted reasoning is inspired on the reflection by Cyert and March (1989) and Dosi, Fagiolo and Roventini (2005) on how firms decide to invest.

The referred authors observe that firms do not make fully rational investment decisions, instead they tend to employ simple routinized rules that reflect a risk averse or prudent behaviour. This mechanism involves non perfect expectations about future demand (prudence generally induces an underestimation of future demand), that are reflected in investment decisions that do not make a complete use of all the available resources. Demand expectations vary in each time period, making the ratio between effective investment and potential investment to vary as well, and this propagates to the whole economy originating long term fluctuations in economic aggregates (output, accumulated capital and consumption).

An important assumption, fundamental to generate endogenous cycles, is that there is always a part of demand that is subject to hump-shaped expectations. This is the case, for instance, of durable goods (e.g., houses): when, in some time period, the demand for durables is low, firms expect it to continue to be low; expected demand will then rise as demand also rises, until a certain point is reached; for too high demand for durables, firms will realize that in the next time periods there will not be a large amount of agents interested in acquiring a new unit of such good, and thus they will expect a lower level of demand.

To simplify the analysis we consider the most basic growth setup, the Solow dynamic equation of capital accumulation, and add to this the non perfect rule of firms' investment. We consider three rules, which do not differ from each other significantly, in order to have a broader view of the implications of this business cycles framework. These rules are piecewise linear equations for the relation between demand and expected demand growth. Piecewise linear maps are one of the most common ways of modelling economic phenomena in order to generate endogenous fluctuations [see, e.g., Bhom and Kaas (2000), Kaas and Zink (2004) and Caball, Jarque and Michetti (2004)].

The analysis of the model will be undertaken in two steps. First, local bifurcations are identified; these are an important guide for the overall behaviour of the dynamic system, but the local analysis is not sufficient. Through some numerical examples one realizes that periodic regular cycles and chaos are a common result, under a global dynamics perspective.

Global dynamics are addressed under a graphical perspective and formal definitions regarding cycles and chaos will not constitute our main concern [see, for a wider understanding of nonlinear phenomena, the textbook of Medio and Lines (2001), and the rigorous work of Sengupta (2003)]. Chaotic motion will be associated to the notion of sensitive dependence on initial conditions (this is just one of various characteristics defining a chaotic system), and this is measured through the computation of Lyapunov characteristic exponents, that reflect the degree of divergence of nearby orbits.

The remainder of the paper is organized as follows. Section 2 describes the model's main features, introducing the demand expectations rule in the Solow growth model. Section 3 investigates the existence of local bifurcations. Section 4 undertakes a graphical analysis of global dynamics in search for endogenous fluctuations. Finally, section 5 concludes.

2. ENDOGENOUS FLUCTUATIONS IN THE SOLOW GROWTH MODEL

Consider a closed economy without government. The relation between income and demand is given by $Y_t \geq I_t + C_t$, where variables Y_t, I_t and C_t respect to real levels of output, investment and consumption. Index t denotes the time period. Let L_t represent the population / labor force of the economy; to simplify the analysis, we assume that the population does not grow over time. We will work with per capita variables and therefore we write the demand equation in intensive form: $y_t \geq i_t + c_t$, with $y_t \equiv Y_t/L_t$, $i_t \equiv I_t/L_t$ and $c_t \equiv C_t/L_t$.

The demand equation is presented as an inequality, meaning that generated income can be used in its full extent for investment and consumption or it may exist a problem of inefficient use of resources, that is, a part of the output may not be used to fulfil consumption needs neither to serve again in the production of new goods. Thus, one must distinguish effective investment, I_t, from potential investment, $J_t \geq I_t$. This last presented variable is the one that guarantees the demand equality, i.e., $Y_t = J_t + C_t$, or, in intensive form, $y_t = j_t + c_t$, with j_t per capita potential investment. In each moment of time, inefficiency implies that an amount of goods $j_t - i_t$, per person, that is not consumed, is not either employed to generate additional goods.

In what follows, our concerns go in two directions: (i) how can we justify a departure from a Walrasian setup with no productive inefficiencies; (ii) how to integrate this inefficiency in a simple model of growth, in order to generate dynamic long term behaviour that departs from standard growth results.

We start by addressing the second issue. In order to focus in inefficiency and generation of cycles we abstract from sophisticated growth scenarios with microeconomic foundations regarding households and firms behaviour. The adopted setup is just the conventional Solow growth model, where a dynamic equation regarding the accumulation of capital explains a process of long term growth that, in the presence of decreasing marginal returns, is determined by exogenous technical progress.

Capital accumulation is defined as $k_{t+1} - k_t = i_t - \delta k_t$, where k_t is the per capita capital stock and $\delta > 0$ is a depreciation rate; we consider a Cobb-Douglas production function, $y_t = Ak_t^{\alpha}$, with $A > 0$ a technology index and $0 < \alpha < 1$ the output-capital elasticity. Finally, a constant marginal propensity to save, $0 < s < 1$, is also assumed, meaning that the potential per capita investment comes $j_t = sy_t$. As stated before, although households save at a rate s, only part of these savings will be directed to productive investment; the remainder is just wasted as firms make non optimal investment decisions.

The Solow model is known to produce, in the absence of exogenous shocks, a steady state where endogenous variables k_t, y_t and c_t do not grow, that is, the steady state is simply a fixed point. This result is modified with our assumption about underinvestment. The question we are interested in addressing is this: which is the mechanism underlying firms' investment decisions and in what sense these can be non optimal? First, we assume that investment decisions depend on firms' expectations about future demand. Define Ψ_t as the ratio between the effective demand level and the level of demand implying optimal investment decisions (i.e., the level of demand allowing for $i_t = j_t$).

The level of demand might be divided in two parts; the first one reflects the demand concerning non durable consumption goods and investment expenditures that give sequence to previously planned projects, while the second refers to consumption of durable goods (e.g., houses) and to potential investment relatively to which there is no previous compromise. Let z_t be the ratio between the effective and the potential levels of the first category of demand; let d_t be the ratio between the effective level and the potential level of the second type of demand. Our main assumption is that independently of the value of z_t, firms expect this kind of demand to be always fulfilled, i.e., $E_t z_{t+1} = 1$, independently of the value of $z_t \in (0, 1)$ (E_t is an expectations operator); relatively to the second part of demand, it is important to establish that firms are risk averse or prudent and therefore they will expect that this kind of demand will locate anywhere between zero and unity according to a rule to be defined below ($0 \le E_t d_{t+1} \le 1$).

We consider perfect foresight (that is, $E_{t-1}\Psi_t = \Psi_t$, $E_{t-1}z_t = z_t = 1$, $E_{t-1}d_t = d_t$), and that the relative weight of type 1 demand in total demand is $\hat{d}$ (e.g., if $\hat{d} = 3$, this means that non durables and planned investment correspond to a share of demand that is three times larger than the share corresponding to durables and new investment projects). Thus, the ratio effective demand / potential demand can be defined as $\Psi_t = \frac{d_t + \hat{d}}{1+\hat{d}}$. Note that $\frac{\hat{d}}{1+\hat{d}} \le \Psi_t \le 1$.

As stated earlier, firms make investment decisions based on their perception about demand and therefore we can establish that $i_t = f(E_{t-1}\Psi_t) \cdot j_t = f(\Psi_t) \cdot j_t$, with $f' > 0$. A specific functional form is taken for the posterior analytical treatment of the model: $f(\Psi_t) = \Psi_t^\theta, \theta > 0$.

Putting together all the elements previously presented, we can represent the Solow model with inefficiency levels of investment through the rule given by dynamic equation (1),

$$k_{t+1} = sAf(\Psi_t) \cdot k_t^\alpha + (1-\delta) \cdot k_t, \quad k_0 \text{ given} \tag{1}$$

For $d_t = 1$ (when the demand expectations coincide with the expectations leading to an economy working without inefficient levels of investment), we have the original Solow model with constant long run levels of variables given by,

$$\overline{k} = \left(\frac{sA}{\delta}\right)^{1/(1-\alpha)} ; \quad \overline{y} = A^{1/(1-\alpha)} \cdot \left(\frac{s}{\delta}\right)^{\alpha/(1-\alpha)} ;$$

$$\overline{i} = \overline{j} = \left(\frac{sA}{\delta^\alpha}\right)^{1/(1-\alpha)} ; \quad \overline{c} = (1-s) \cdot A^{1/(1-\alpha)} \cdot \left(\frac{s}{\delta}\right)^{\alpha/(1-\alpha)}$$

To complete the presentation of the model, it is necessary to discuss how firms formulate expectations about future demand. We have stated that there is only a fraction of demand, of relative size $\frac{1}{1+\hat{d}}$, relatively to which firms make a careful risk averse computation of its actual realization rather than just expecting that all potential demand is fulfilled, and thus we are just concerned with variable d_t. The optimal case would imply $d_{t+1} = 1$, independently of d_t, i.e., in a given moment the growth rate of this ratio had to be $\gamma_t = 1/d_t - 1$. The most pessimistic expectation formation rule would be the one implying $d_{t+1} = 0$, independently of d_t, case in which $\gamma_t = -1$. If firms form expectations in a non optimal way for the class of goods / projects we have highlighted, their decisions will be somewhere in the interval $\gamma_t \in (-1; 1/d_t - 1)$. Figure 1 depicts graphically the region in which this interval is located.

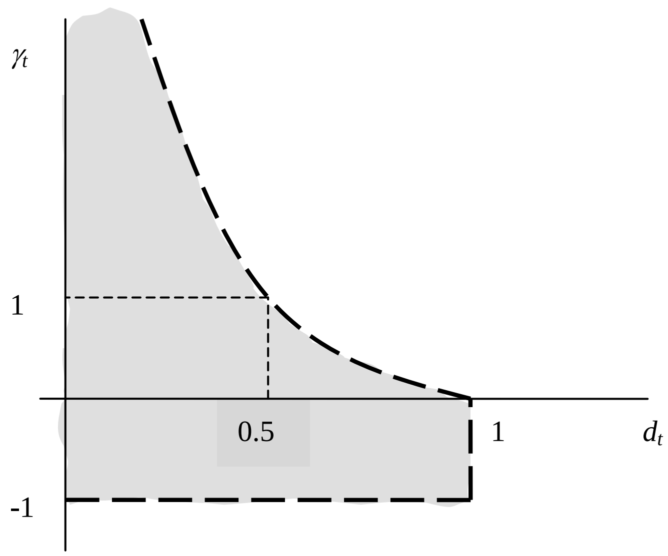

FIG. 1. Relative demand (of type 2) and its growth rate

The way in which firms form expectations about future demand regarding the class of goods / investment projects under consideration will obey to the following specific requirements:

i) the growth rate of d_t is always a finite value. This means that if $d_t = 0$, then $d_{t+1} = 0$, that is, if no durables are consumed in this period and no new investment projects are undertaken, then the same is expected to happen in the next period;

ii) when d_t is high (close to 1), firms must expect the next period level of demand to be very low (γ_t should be negative and equal or near to -1). This point may be supported on the observation that periods of strong demand do not last forever; when demand for durables and new investment projects is too high in a given period, firms will expect that in next periods economic agents will not need to undertake these same expenditures and thus they expect demand to fall;

iii) there is some intermediate level of d_t for which firms expect demand to rise to the optimal level; this is the same as saying that it exists a point $0 < d_{t+1} < 1$ where condition $\gamma_t = 1/d_t - 1$ is satisfied.

The paper considers three alternative rules of demand expectations formation that generate, for some parameter values, endogenous cycles; these rules obey to the requirements specified above, which are taken here as the essence of business cycles: there is a kind of hump-shaped or tent-shaped demand expectations function, that is, firms will expect very low demand if this is very low or very high today (for the second class of demanded goods); for intermediate levels of demand, demand is expected to be high in the next period. This reasoning, under which firms are suspicious about high levels of demand, is the basic ingredient for cycles to occur.

Figures 2 to 4 identify our alternative hypothesis about the formation of demand expectations.

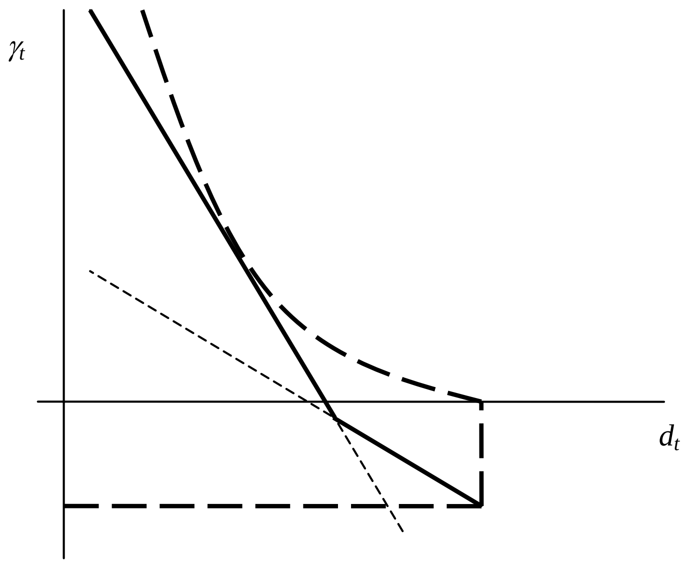

FIG. 2. Evolution of demand expectations (case 1)

The first case that we consider divides the expectation demand evolution relation in two segments. The first is a linearization of $\gamma_t = 1/d_t - 1$ in a given point $0 < \phi_0 \leq 0.5$, that is, $\gamma_t = \frac{2}{\phi_0} - 1 - \frac{1}{\phi_0^2} d_t$. The second segment is also given by a linear function that passes through point $(d_t, \gamma_t) = (1, -1)$;

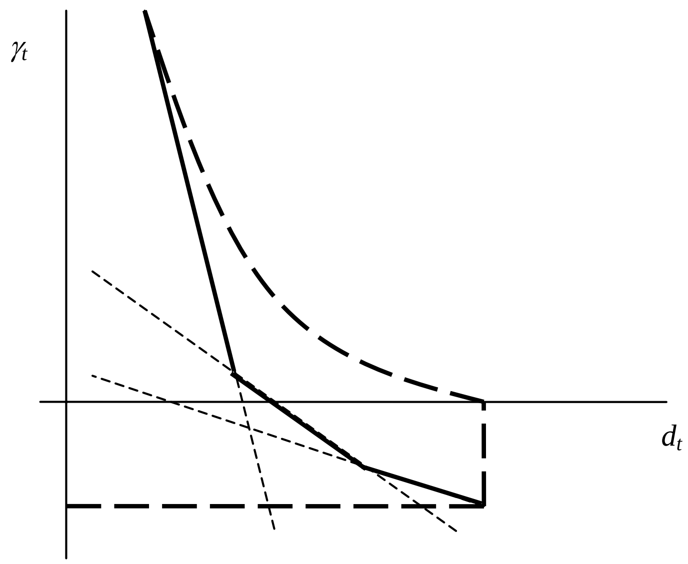

FIG. 3. Evolution of demand expectations (case 2)

we want this second part of the rule to be within the specified bounds and, thus, we take $\gamma_t = \phi_1 - (1+\phi_1)\cdot d_t$, with $-1 \leq \phi_1 \leq 3$. The two defined lines meet at point $d_t = \frac{2\phi_0-(1+\phi_1)\cdot\phi_0^2}{1-(1+\phi_1)\cdot\phi_0^2}$.

Noting that $\gamma_t \equiv d_{t+1}/d_t - 1$, we gather our information about case 1 in dynamic equation (2),

$$d_{t+1} = \begin{cases} \frac{2}{\phi_0}d_t\cdot\left(1-\frac{1}{2\phi_0}d_t\right), & 0 \leq d_t \leq \frac{2\phi_0-(1+\phi_1)\cdot\phi_0^2}{1-(1+\phi_1)\cdot\phi_0^2} \\ (1+\phi_1)\cdot d_t\cdot(1-d_t), & \frac{2\phi_0-(1+\phi_1)\cdot\phi_0^2}{1-(1+\phi_1)\cdot\phi_0^2} \leq d_t \leq 1 \end{cases} \tag{2}$$

d_0 given

In sections 3 and 4, we study the dynamics underlying (2), and the dynamics associated to system (1)-(2); for now, we proceed with the analytical characterization of cases two and three, relatively to which we will also undertake a local analysis of bifurcations and a discussion of global dynamics under specific parameterizations.

The second case regarding demand expectations splits the rule in three branches. The first and the third are similar to the two rules in equation (2). We now introduce an intermediate equation that obeys to the following: i) it is a negatively sloped straight line; ii) it passes on the point $(d_t, \gamma_t) = (0.5, 0)$. This can be written as $\gamma_t = \phi_2 - 2\phi_2 d_t$, with ϕ_2 a given

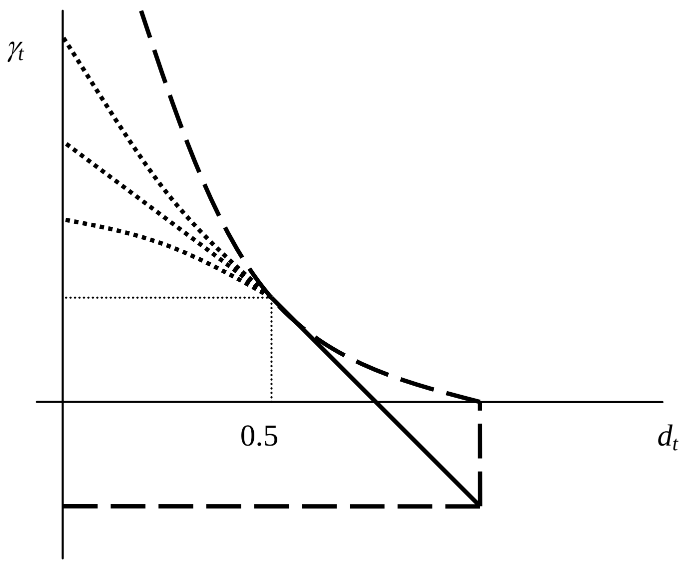

FIG. 4. Evolution of demand expectations (case 3)

parameter. The boundaries of parameters ϕ_0, ϕ_1 and ϕ_2 are given by the will to maintain always the existence of three distinct segments. To allow for the second equation not to disappear, we must impose the following bounds: $\phi_0 \in \left(0, \frac{2-\sqrt{2}}{2}\right)$; $\phi_1 \in (-1, 1)$; $\phi_2 \in \left(1, \frac{2+\sqrt{2}}{2-\sqrt{2}}\right)$. To complete our expectations rule, we have to encounter the intersection points between equations. The first two equations intersect at point $d_t = \frac{2\phi_0-(1+\phi_2)\cdot\phi_0^2}{1-2\phi_0^2\phi_2}$, and the second and the third intersect at $d_t = \frac{\phi_2-\phi_1}{2\phi_2-(1+\phi_1)}$. Our second case corresponds, then, to

$$d_{t+1} = \begin{cases} \frac{2}{\phi_0} d_t \cdot \left(1 - \frac{1}{2\phi_0} d_t\right), & 0 \leq d_t \leq \frac{2\phi_0-(1+\phi_2)\cdot\phi_0^2}{1-2\phi_0^2\phi_2} \\ (\phi_2 + 1 - 2\phi_2 d_t) \cdot d_t, & \frac{2\phi_0-(1+\phi_2)\cdot\phi_0^2}{1-2\phi_0^2\phi_2} \leq d_t \leq \frac{\phi_2-\phi_1}{2\phi_2-(1+\phi_1)} \\ (1+\phi_1) \cdot d_t \cdot (1-d_t), & \frac{\phi_2-\phi_1}{2\phi_2-(1+\phi_1)} \leq d_t \leq 1 \end{cases} \quad (3)$$

d_0 given

The previous two cases begin by linearizing equation $\gamma_t = 1/d_t - 1$ in a given point (the left-hand side of the rule is this linearized equation). Our last case considers a linear right-hand side of the form $\phi_t = 3 - 4d_t$, for $0.5 \leq d_t \leq 1$, and assumes the possibility of a non linear left-hand side. In particular, we take $\gamma_t = [\phi_3 - 2 \cdot (\phi_3 - 1) \cdot d_t]^{\phi_4}$, with $\phi_3 > 1$ and $\phi_4 > 0$, for $0 \leq d_t \leq 0.5$. Note that if $\phi_3 = 3$ and $\phi_4 = 1$, we are restricted to a unique linear relation between d_t and ϕ_t; this relation is not linear for $\phi_4 \neq 1$. Once again, the expectations rule can be presented as a relation between period t demand and period $t+1$ expected demand.

$$d_{t+1} = \begin{cases} [\phi_3 - 2 \cdot (\phi_3 - 1) \cdot d_t]^{\phi_4} \cdot d_t + d_t, & 0 \leq d_t \leq 0.5 \\ 4d_t \cdot (1 - d_t), & 0.5 \leq d_t \leq 1 \end{cases} \tag{4}$$

d_0 given

Synthesizing, the Solow model explains growth under optimal investment decisions (firms invest exactly the correspondent to households' savings). Our main assumption, that is able to generate endogenous fluctuations, is that some of the available savings are simply wasted because firms do not make, in each moment of time a perfect evaluation of investment conditions. Alternatively to an optimal investment rule, firms tend to adopt decisions based on 'rules of thumb'; they just calculate how much to invest guessing how demand will evolve. Because they have imperfect knowledge of market conditions and adopt a risk averse behaviour, these rules produce long term results that are worse in average (in terms of accumulated capital, output and consumption) than the ones given by the benchmark model, and long term levels of the several aggregates are possibly not linear any more.

3. BIFURCATION ANALYSIS

The model built in the previous section introduces an inefficiency factor that is generated by a hump-shaped / tent-shaped map. This factor spreads through the economic system producing eventually economy wide endogenous fluctuations, that is, the long term may be characterized by periodic and a-periodic time series movements; the presence of a fixed point is possible for some parameter values but it hardly can be regarded as a generic result. This evidence will be confirmed in next section's global dynamic analysis. For now, we undertake a local analysis, in the steady state vicinity.

This study allows to identify bifurcation points and areas where the equilibrium result corresponds to a stable node, an unstable node or a saddle-point. However, we should keep in mind that the local analysis is not satisfactory to give an overall picture of the underlying dynamics. The misleading results relate to the fact that locally one is able to identify the points where a transition from a stable to an unstable subspace occurs, but in reality the change in the hyperbolicity of the system may imply a whole new dynamic result that a local analysis is unable to capture. Nevertheless, the local bifurcation analysis is essential to identify the points in which the qualitative change in dynamic behaviour takes place.

The analysis of local bifurcations will be undertaken for each one of the three proposed models.

Consider first the system (1)-(2). Since we are interested in studying dynamics in the neighbourhood of the steady state, we have to begin by determining the condition under which the steady state falls on the first or on the second part of the equation relating to the movement of d_t. We define the steady state as the value $\overline{d}$ for which $d_t = d_{t+1}$ (as well, $\overline{k}$ will be the capital stock satisfying condition $k_t = k_{t+1}$; the same reasoning applies to all the other variables). For d_t, the steady state is,

$$\overline{d} = \begin{cases} 2\phi_0 - \phi_0^2, & 0 \leq \overline{d} \leq \frac{2\phi_0 - (1+\phi_1)\cdot\phi_0^2}{1-(1+\phi_1)\cdot\phi_0^2} \\ \frac{\phi_1}{1+\phi_1}, & \frac{2\phi_0 - (1+\phi_1)\cdot\phi_0^2}{1-(1+\phi_1)\cdot\phi_0^2} \leq \overline{d} \leq 1 \end{cases}$$

Some computation allows to reveal the condition under which one or the other steady state is accomplished. The first steady state (the one over the left hand side of the rule in figure 2) is obtained if $\phi_1 \leq \frac{2\phi_0 - \phi_0^2}{(1-\phi_0)^2}$; otherwise, the second steady state is the one that effectively holds (of course, if the strict equality is observed, the steady state will be located exactly in the intersection point of the two equations). Let us discuss each one of the two cases.

i) $\phi_1 \leq \frac{2\phi_0 - \phi_0^2}{(1-\phi_0)^2}$. Under this condition, the linearization of (1)-(2) produces the following matricial system,

$$\begin{bmatrix} k_{t+1} - \overline{k} \\ d_{t+1} - \overline{d} \end{bmatrix} = \begin{bmatrix} 1-(1-\alpha)\cdot\delta & \theta\delta\frac{\overline{k}}{\overline{d}+\hat{d}} \\ 0 & 2\cdot(1-1/\phi_0) \end{bmatrix} \cdot \begin{bmatrix} k_t - \overline{k} \\ d_t - \overline{d} \end{bmatrix} \tag{5}$$

Denote the Jacobian matrix in (5) by J_1. Bifurcation points, where the system changes qualitative dynamic properties, occur for combinations of parameters for which eigenvalues of J_1 are over the unit circle. Proposition 1 summarizes the bifurcation result in appreciation.

PROPOSITION 1. *For the first Solow model with non perfect expectations, if* $\phi_1 \leq \frac{2\phi_0 - \phi_0^2}{(1-\phi_0)^2}$ *then no bifurcation is observed. Local dynamics are characterized by saddle-path stability for every* ϕ_0 *and* ϕ_1 *within the specified bounds.*

Proof. The absence of bifurcations implies that the three following conditions hold for the boundaries imposed on parameter values: $1 + \mathrm{Tr}(J_1) + \det(J_1) \neq 0$, $1 - \mathrm{Tr}(J_1) + \det(J_1) \neq 0$ and $\det(J_1) \neq 1$. Furthermore, saddle-path stability corresponds to the case where the first expression is negative, the second is positive and the determinant is below one.

We begin by observing the absence of flip bifurcations: $1 + \text{Tr}(J_1) + \det(J_1) = 0 \Rightarrow \phi_0 = 2/3$. Under the constraint imposed on the parameter value, this result is outside bounds; for admissible parameter values, one plus the trace plus the determinant is always a negative quantity. We also observe that $1 - \text{Tr}(J_1) + \det(J_1) = \frac{2-\phi_0}{\phi_0} \cdot (1-\alpha) \cdot \delta > 0$, which rules out the presence of fold, pitchfork and transcritical bifurcations. Finally, the defined parameter bounds imply $\det(J_1) < 1$. Note that a Neimark-Sacker bifurcation would occur for $\phi_0 = \frac{2-2\cdot(1-\alpha)\cdot\delta}{1-2\cdot(1-\alpha)\cdot\delta}$; this is a true condition for $\phi_0 > 1$, a condition that does not serve our setup purposes, as remarked in section 2. ▮

ii) $\phi_1 \geq \frac{2\phi_0 - \phi_0^2}{(1-\phi_0)^2}$. Now, the Jacobian matrix of system (1)-(2) is J_2, the matrix in linearized system (6).

$$\begin{bmatrix} k_{t+1} - \overline{k} \\ d_{t+1} - \overline{d} \end{bmatrix} = \begin{bmatrix} 1 - (1-\alpha)\cdot\delta & \theta\delta\frac{\overline{k}}{\overline{d}+\hat{d}} \\ 0 & 1 - \phi_1 \end{bmatrix} \cdot \begin{bmatrix} k_t - \overline{k} \\ d_t - \overline{d} \end{bmatrix} \qquad (6)$$

Proposition 2 indicates that in this case a qualitative change in the dynamic behaviour of the model is identifiable.

PROPOSITION 2. *For the first Solow model with non perfect demand expectations, if* $\phi_1 \geq \frac{2\phi_0 - \phi_0^2}{(1-\phi_0)^2}$ *then a flip bifurcation takes place at* $\phi_1 = 2$. *This bifurcation separates a region of stable node stability (for* $\phi_1 < 2$*) from a region of saddle-path stability (for* $\phi_1 > 2$*). No other bifurcation is identified.*

Proof. Solving $1 + \text{Tr}(J_2) + \det(J_2) = 0$, we obtain the condition $\phi_1 = 2$. This corresponds to a flip bifurcation, given that under this value of parameter ϕ_1, one observes that $\text{Tr}(J_2) = -(1-\alpha)\cdot\delta \in (-1, 0)$ and $\det(J_2) = -[1-(1-\alpha)\cdot\delta] \in (-1, 0)$. When $\phi_1 > 2$, condition $1 + \text{Tr}(J_2) + \det(J_2) < 0$ is true and, for $\phi_1 < 2$, $1 - \text{Tr}(J_2) + \det(J_2) > 0$ holds.

We also regard that $1 - \text{Tr}(J_2) + \det(J_2) = (1-\alpha)\cdot\delta\cdot\phi_1 > 0$, and that $\det(J_2) < 1$ unless $\phi_1 = -\frac{(1-\alpha)\cdot\delta}{1-(1-\alpha)\cdot\delta}$; this value of ϕ_1 is not admissible under constraint $\phi_1 \geq \frac{2\phi_0 - \phi_0^2}{(1-\phi_0)^2}$, and therefore a Neimark-Sacker bifurcation is ruled out. The obtained results allow to clearly distinguish between a zone of stability ($\phi_1 < 2$) and a zone of saddle-path stability ($\phi_1 > 2$). ▮

Figure 5 synthesizes local dynamic results for our first Solow model with endogenous cycles. The picture separates the regions where each steady state is attainable and also highlights the bifurcation point. As a result, the region of stability is an intermediate region in the space of parameters, which is surrounded by a saddle-path equilibrium that is locally observable but, as we will understand in the following section, is associated with strange dynamics (periodic and chaotic motion).

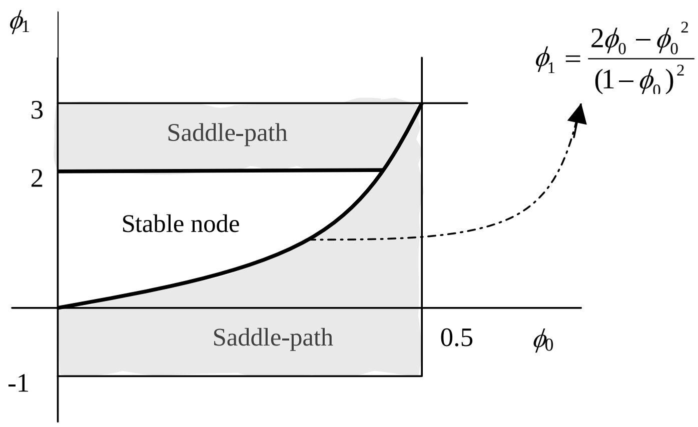

FIG. 5. Local dynamics in the Solow endogenous cycles model (first case)

Consider now the second example [equations (1)-(3)]. In a first moment one needs to know where the steady state is located. The first equation of (3) is the same as the first equation of (2), and thus, in this case, $\overline{d} = 2\phi_0 - \phi_0^2$; this steady state is an admissible value if $\frac{2-\sqrt{2}}{2} < \phi_0 < \frac{2+\sqrt{2}}{2}$; nevertheless, we have imposed the constraint $\phi_0 < \frac{2-\sqrt{2}}{2}$, and thus, this first equation can be excluded from the equilibrium vicinity analysis. A similar conclusion is drawn for the third equation of (3) [that is the same as the second equation in (2)]; the correspondent steady state is $\overline{d} = \phi_1/(1+\phi_1)$, which, again, is impossible to be reached since we have imposed that ϕ_1 is located in the interval $(-1, 1)$; this is not compatible with the boundary $\frac{\phi_2-\phi_1}{2\phi_2-(1+\phi_1)} \leq \overline{d} \leq 1$, for the determined steady state value (in particular, the lower bound implies $\phi_1 \geq 1$). Thus, the steady state can only be located in the second straight line of figure 3. The steady state is simply $\overline{d} = 0.5$. The linearization of the second equation of (3) around $\overline{d} = 0.5$, along with the linearized version of (1), leads to system (7).

$$\begin{bmatrix} k_{t+1} - \overline{k} \\ d_{t+1} - \overline{d} \end{bmatrix} = \begin{bmatrix} 1-(1-\alpha)\cdot\delta & \theta\delta\frac{\overline{k}}{\overline{d}+\hat{d}} \\ 0 & 1-\phi_1 \end{bmatrix} \cdot \begin{bmatrix} k_t - \overline{k} \\ d_t - \overline{d} \end{bmatrix} \tag{7}$$

Let J_3 be the Jacobian matrix in (7). The bifurcation result is given by proposition 3.

Proposition 3.
For the second Solow model with non perfect demand expectations, the observation of condition $\phi_2 = 2$ implies a flip bifurcation. There are not any other bifurcations.

Proof.
As before, one just has to verify that $1 + \text{Tr}(J_3) + \det(J_3) = 0$ for $\phi_2 = 2$, and that for this parameter value it is true that $1 + \text{Tr}(J_3) + \det(J_3) > 0$ and $\det(J_3) < 0$. Indeed, since we have imposed $\phi_2 \in (1; 5.828)$, the presence of a flip bifurcation is confirmed. Because neither $1 + \text{Tr}(J_3) + \det(J_3) = 0$ nor $\det(J_3) = 1$, no other bifurcation occurs. ▮

Note that if $\phi_2 < 2$ a stable node equilibrium holds and when $\phi_2 > 2$ saddle-path stability prevails.

Finally, case 3 relates to the local dynamic analysis of system (1)-(4). The first equation of (4) produces the following equilibrium result: $\overline{d} = \frac{\phi_3}{2\cdot(\phi_3-1)}$. Note that this result is not compatible with the constraint $\overline{d} \leq 0.5$, and therefore we exclude the first equation as a possible scenario for a steady state analysis. The steady state is always found in the second equation and this is simply $\overline{d} = 0.75$. Once again we find a linearized version of the system, now for a J_4 Jacobian matrix,

$$\begin{bmatrix} k_{t+1} - \overline{k} \\ d_{t+1} - \overline{d} \end{bmatrix} = \begin{bmatrix} 1 - (1-\alpha)\cdot\delta & \theta\delta\frac{\overline{k}}{\overline{d}+\hat{d}} \\ 0 & -2 \end{bmatrix} \cdot \begin{bmatrix} k_t - \overline{k} \\ d_t - \overline{d} \end{bmatrix} \tag{8}$$

Proposition 4. *For the third Solow model with non perfect demand expectations, no bifurcation is found to disturb the qualitative nature of the steady state, which corresponds to saddle-path stability.*

The proof in this case is straightforward as we regard that one of the eigenvalues of J_4 is inside the unit circle and the other is -2.

4. GLOBAL DYNAMICS

Bifurcation analysis gives some clues about the dynamic nature of the endogenous business cycles model we have proposed. But it also hides the most important features of the dynamics of economic aggregates, namely the fact that for a large set of parameter values endogenous fluctuations exist. Through several numerical examples we extract, in this section, some meaningful results about the evolution of variables over time. An interesting result that we can infer from the examples in this section is that last section's areas of stability continue to correspond to fixed point regions, while the situations for which we have identified saddle-path stability are now, in terms of global dynamics, associated to routes to chaos, that is, to a process of increasing periodicity that in most cases ends up by involving the presence of sensitive dependence on initial conditions.

The analysis in this section is essentially graphical and from this graphical analysis some specific results are withdrawn.[1]

Dynamic results are basically governed by the demand expectations equations and the values of parameters in these equations. Thus, parameters in the Solow equation have quantitative impact over steady state results but do not influence qualitative dynamics. For this reason, we just take a set of reasonable parameter values: $\{A = 1.2; s = 0.25; \alpha = 0.2; \delta = 0.05; \theta = 0.5; \hat{d} = 3\}$. These values will be considered throughout the numerical analysis. Note that for the specified values, the stable steady state of the benchmark case with optimal investment decisions $(d_t = 1)$ is characterized by the following economic aggregates values: $\overline{k} = 9.39; \overline{y} = 1.878; \overline{i} = \overline{j} = 0.469; \overline{c} = 1.409$. We should keep in mind these values, in order to compare them with the various fluctuations scenarios that we will analyze.

Consider system (1)-(2). Figures 6 and 7 present equation (2) for different parameter values. Independently of parameter values the function has a clear hump shape and the break point between the two equations is easily identifiable. The following figures, 8 to 11, represent the long term time path of the capital variable. As one regards, the shape of this time path varies not only for changes in the parameter values but also for (small) changes in the initial conditions; this last observation is a first evidence of chaos in the sense that there is sensitive dependence on initial conditions.

Figures 12 to 14 present specific time paths for the other per capita aggregates, namely output, investment and consumption. The output path follows the same qualitative behaviour as the capital trajectory (these variables are related through the production function), while consumption and investment paths present a different kind of pattern, although a symmetric evolution of these two variables is observed. Note that, when comparing these long term outcomes with the steady state benchmark case defined above it is interesting to observe that capital, output and investment remain always below the levels implied by perfect investment decisions, but consumption is in some time moments above the level suggested by an efficient use of available resources.

Relatively to the first case, figures 15 and 16 present two attractors; the first relating variables k_t and c_t, the second variables c_t and i_t. Observe

[1]The program used to draw figures 6 to 51 is iDMC (interactive Dynamical Model Calculator). This is a free software program available at www.dss.uniud.it/nonlinear, and copyright of Marji Lines and Alfredo Medio.

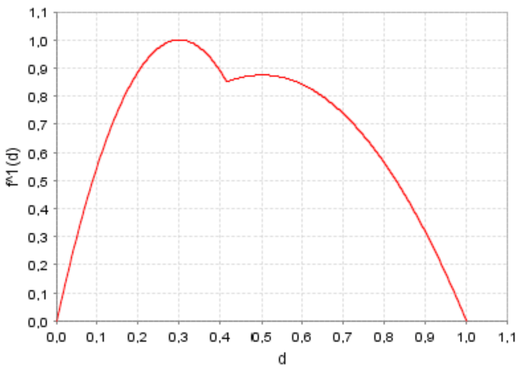

FIG. 6. model 1; phase diagram for difference equation (2) [$\phi_0 = 0.3; \phi_1 = 2.5$]

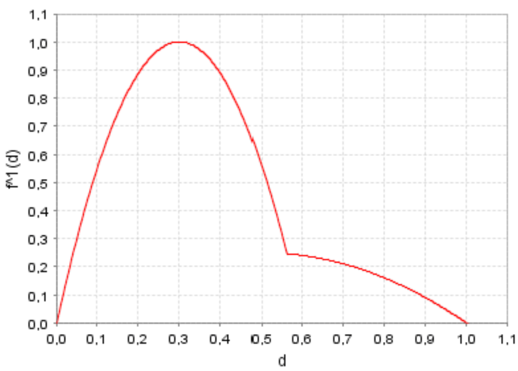

FIG. 7. model 1; phase diagram for difference equation (2) [$\phi_0 = 0.3; \phi_1 = 0$]

that, relatively to the second attractor, the sum of consumption and investment is, of course, never above $\overline{y} = 1.878$ (the benchmark steady state output value). Observe as well that the attractors are confined to a relatively small region of the state space, which is a direct result of having considered that the share of demand subject to varying expected growth is low relatively to the whole set of demanded goods. Thus, in the chosen example fluctuations are not too pronounced.

Several bifurcation diagrams, for each one of the parameters in the equation of d_t and for variables k_t and c_t are presented in figures 17 to 20; in these, one observes that periods of chaotic motion coexist with low periodicity cycles. Putting together the information concerning the two parameters ϕ_0 and ϕ_1, we reach a figure (figure 21) with some similarities with figure 5. In fact, the area where a fixed point exists is the same in the two figures; the remaining area, that locally is defined by saddle-path stability, is in fact

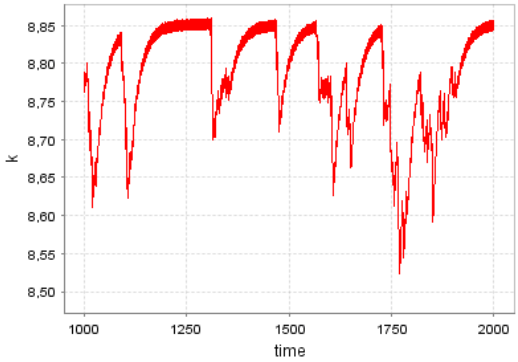

FIG. 8. model 1; trajectory of k_t $[k_0 = 8; d_0 = 0.25; \phi_0 = 0.3; \phi_1 = 2.5;$ transients=1000; iterations=1000]

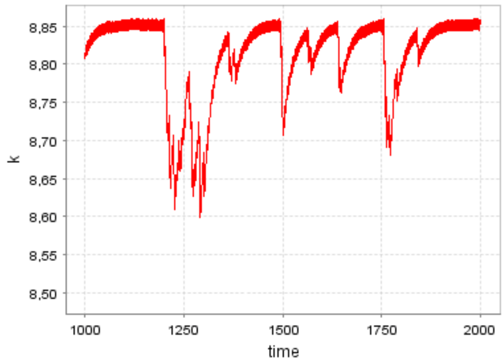

FIG. 9. model 1; trajectory of k_t $[k_0 = 8; d_0 = 0.26; \phi_0 = 0.3; \phi_1 = 2.5;$ transients=1000; iterations=1000]

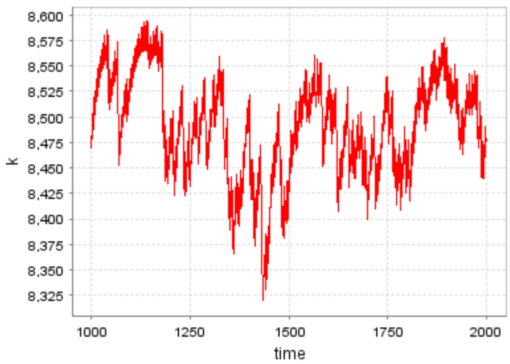

FIG. 10. model 1; trajectory of k_t $[k_0 = 8; d_0 = 0.25; \phi_0 = 0.3; \phi_1 = 0;$ transients=1000; iterations=1000]

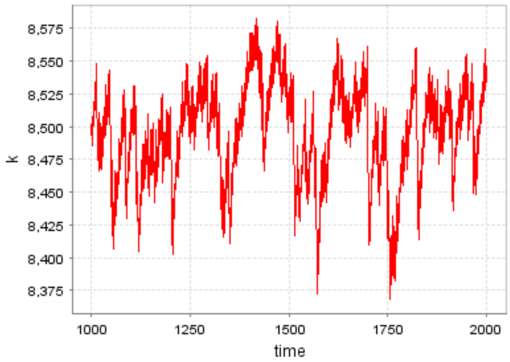

FIG. 11. model 1; trajectory of k_t [$k_0 = 8; d_0 = 0.26; \phi_0 = 0.3; \phi_1 = 0$; transients=1000; iterations=1000]

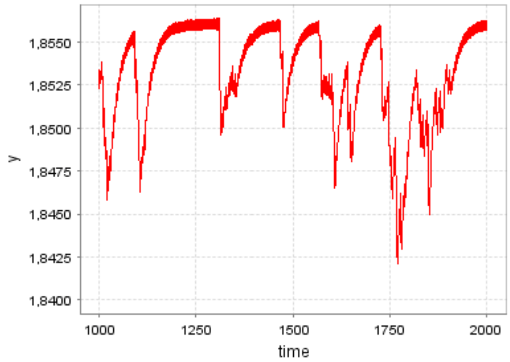

FIG. 12. model 1; trajectory of y_t [$k_0 = 8; d_0 = 0.25; \phi_0 = 0.3; \phi_1 = 2.5$; transients=1000; iterations=1000]

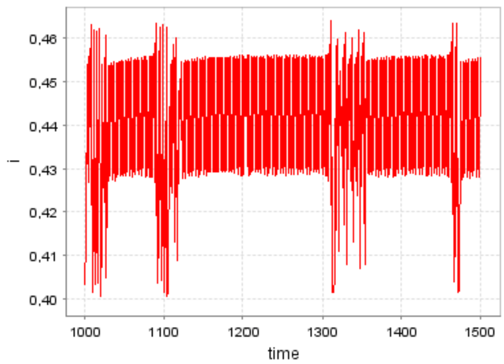

FIG. 13. model 1; trajectory of i_t [$k_0 = 8; d_0 = 0.25; \phi_0 = 0.3; \phi_1 = 2.5$; transients=1000; iterations=500]

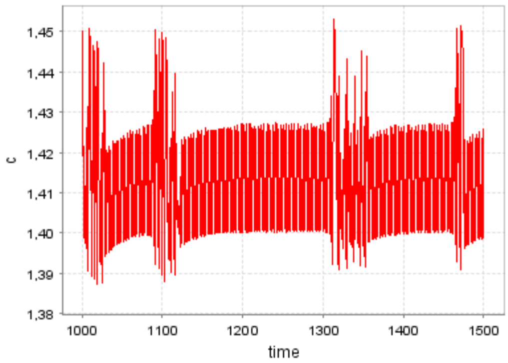

FIG. 14. model 1; trajectory of c_t [$k_0 = 8; d_0 = 0.25; \phi_0 = 0.3; \phi_1 = 2.5$; transients=1000; iterations=500]

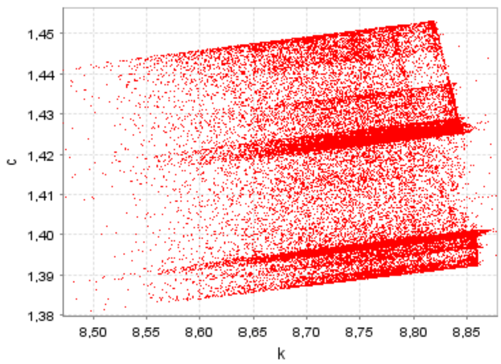

FIG. 15. model 1; attractor - variables k_t, c_t [$k_0 = 8; d_0 = 0.25; \phi_0 = 0.3; \phi_1 = 2.5$; transients=1000; iterations=100,000]

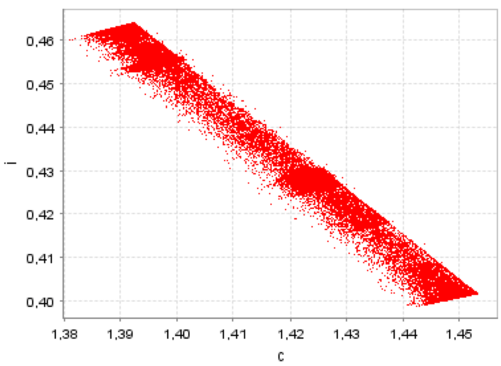

FIG. 16. model 1; attractor - variables c_t, i_t [$k_0 = 8; d_0 = 0.25; \phi_0 = 0.3; \phi_1 = 2.5$; transients=1000; iterations=100,000]

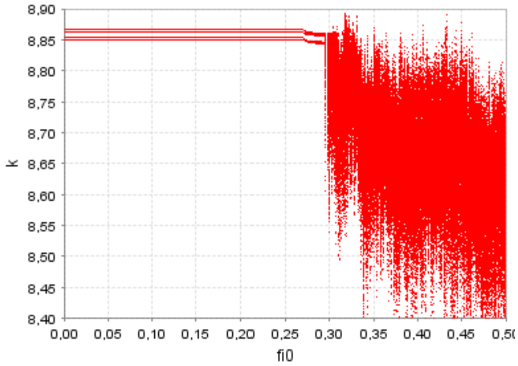

FIG. 17. model 1; bifurcation diagram (k_t) [$k_0 = 8; d_0 = 0.25$; $0 \leq \phi_0 \leq 0.5$; $\phi_1 = 2.5$; transients=1000; iterations=1000]

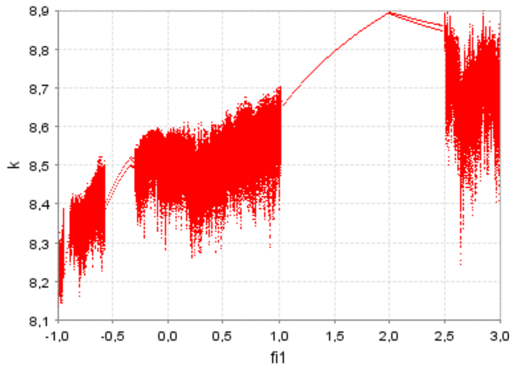

FIG. 18. model 1; bifurcation diagram (k_t) [$k_0 = 8; d_0 = 0.25; \phi_0 = 0.3$; $-1 \leq \phi_1 \leq 3$; transients=1000; iterations=1000]

a region where cycles of several periods are found (figure 21 displays points of cycles till order 35; the area with no represented point will correspond to cycles of order above 35 or cycles with no order at all).

Finally, we consider Lyapunov characteristic exponents as a measure of chaos (to be rigorous these exponents are a measure of exponential divergence of nearby orbits, which in turn can be taken as a non formal definition of chaos). Figures 22 and 23 confirm previous graphical results, namely the ones seen in the bifurcation diagrams. Recall that a positive Lyapunov exponent is synonymous of exponential divergence of orbits, which is absent when both exponents are negative values.

The graphical analysis that can be done for cases 2 and 3 does not differ significantly from the previous discussion. Case 2, corresponding to equations (1)-(3), involves the presence of three equations in (3) and three

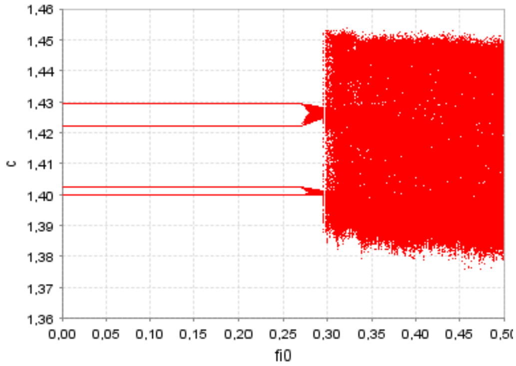

FIG. 19. model 1; bifurcation diagram (c_t) [$k_0 = 8; d_0 = 0.25$; $0 \leq \phi_0 \leq 0.5$; $\phi_1 = 2.5$; transients=1000; iterations=1000]

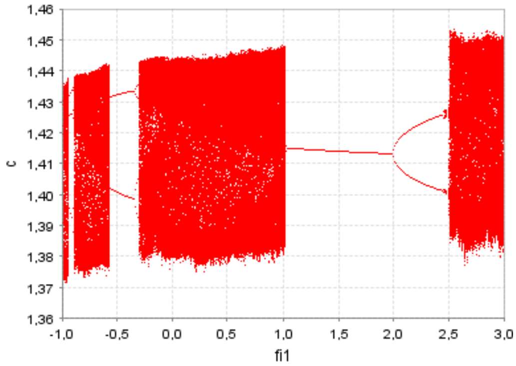

FIG. 20. model 1; bifurcation diagram (c_t) [$k_0 = 8; d_0 = 0.25$; $\phi_0 = 0.3$; $-1 \leq \phi_1 \leq 3$; transients=1000; iterations=1000]

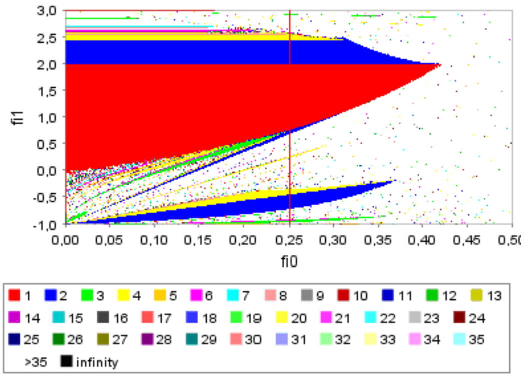

FIG. 21. model 1; periods of various orders in the space of parameters [$k_0 = 8; d_0 = 0.25$; transients=1000]

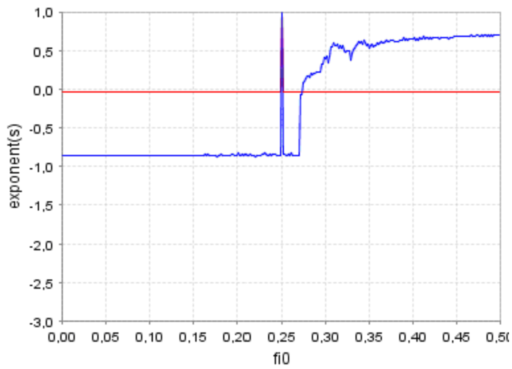

FIG. 22. model 1; Lyapunov exponents [$k_0 = 8; d_0 = 0.25$; $0 \leq \phi_0 \leq 0.5$; $\phi_1 = 2.5$; iterations=1000]

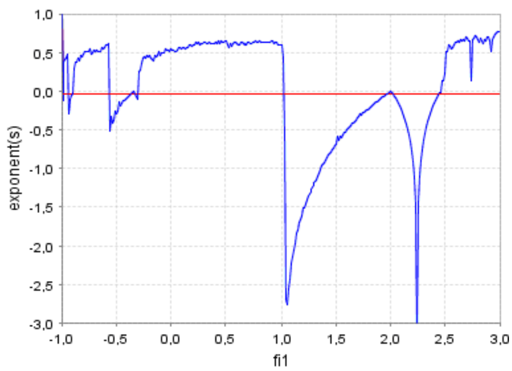

FIG. 23. model 1; Lyapunov exponents [$k_0 = 8; d_0 = 0.25$; $\phi_0 = 0.3$; $-1 \leq \phi_1 \leq 3$; iterations=1000]

parameters determining the dynamic behaviour. Graphs 24 and 25 depict two possible phase diagrams of (3); some time series results, for parameter values implying chaos, are given by figures 26 to 28, and figures 29 and 30 represent attractors that resemble the ones in the first case.

Bifurcation diagrams (figures 31 to 36) and the identification of cycles in the space of parameters (figures 37 to 39) imply a result that is similar to the one in the first case: condition $\phi_2 = 2$ (the bifurcation point) separates an area of fixed point stability from an area where multiple steady state qualitative results arise. Lyapunov characteristic exponents (figures 40 to 42) confirm the bifurcation analysis. Note, in particular, in figure 42, that the bifurcation point coincides with a zero Lyapunov exponent.

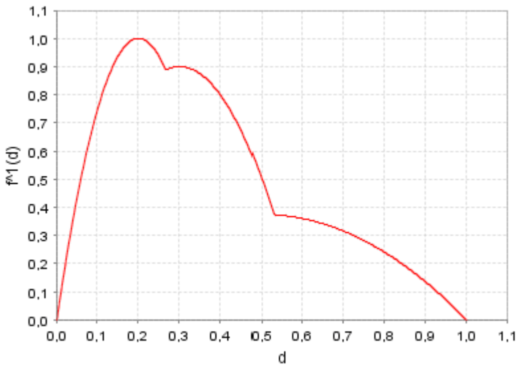

FIG. 24. model 2; phase diagram for difference equation (3) [$\phi_0 = 0.2; \phi_1 = 0.5; \phi_2 = 5$]

Finally, case 3 is characterized through figures 43 to 51. Following the same sequence as before, we present a possible phase diagram, time tra-

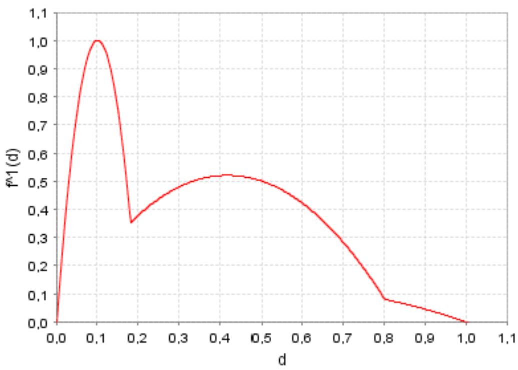

FIG. 25. model 2; phase diagram for difference equation (3) [$\phi_0 = 0.1; \phi_1 = -0.5; \phi_2 = 1.5$]

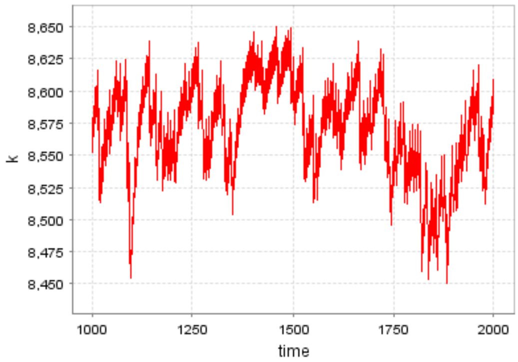

FIG. 26. model 2; trajectory of k_t [$k_0 = 8; d_0 = 0.25$; $\phi_0 = 0.2; \phi_1 = 0.5; \phi_2 = 5$; transients=1000; iterations=1000]

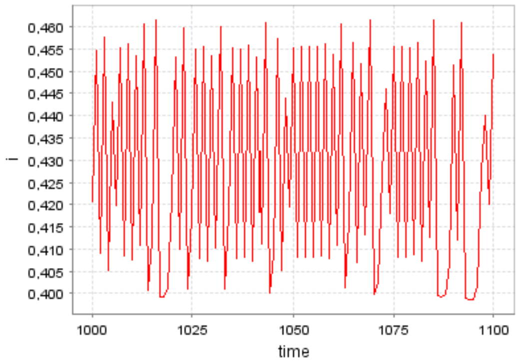

FIG. 27. model 2; trajectory of it [$k_0 = 8; d_0 = 0.25$; $\phi_0 = 0.2; \phi_1 = 0.5; \phi_2 = 5$; transients=1000; iterations=100]

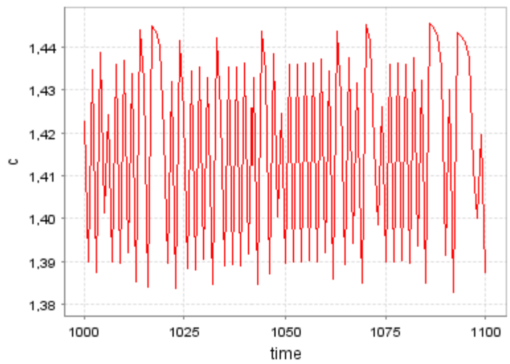

FIG. 28. model 2; trajectory of ct [$k_0 = 8; d_0 = 0.25$; $\phi_0 = 0.2; \phi_1 = 0.5; \phi_2 = 5$; transients=1000; iterations=100]

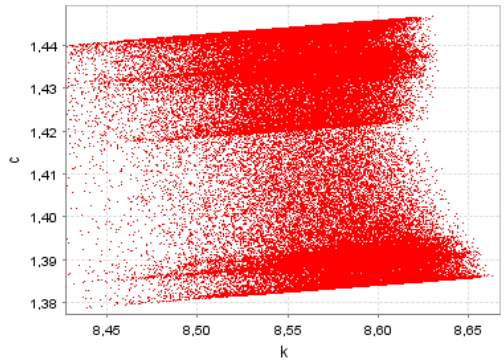

FIG. 29. model 2; attractor - variables k_t, c_t [$k_0 = 8; d_0 = 0.25$; $\phi_0 = 0.2; \phi_1 = 0.5; \phi_2 = 5$; transients=1000; iterations=100,000]

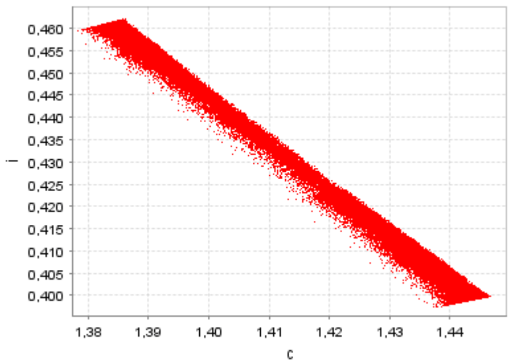

FIG. 30. model 2; attractor - variables c_t, i_t [$k_0 = 8; d_0 = 0.25$; $\phi_0 = 0.2; \phi_1 = 0.5; \phi_2 = 5$; transients=1000; iterations=100,000]

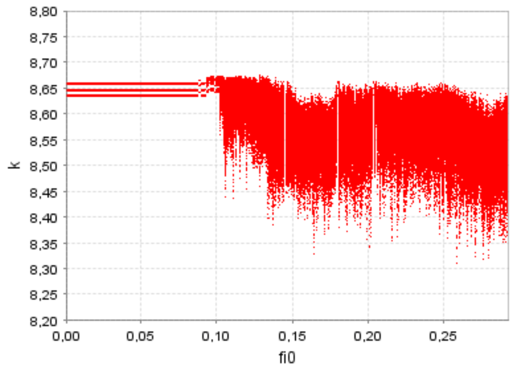

FIG. 31. model 2; bifurcation diagram (k_t) [$k_0 = 8; d_0 = 0.25$; $0 \leq \phi_0 \leq 0.293$; $\phi_1 = 0.5; \phi_2 = 5$; transients=1000; iterations=1000]

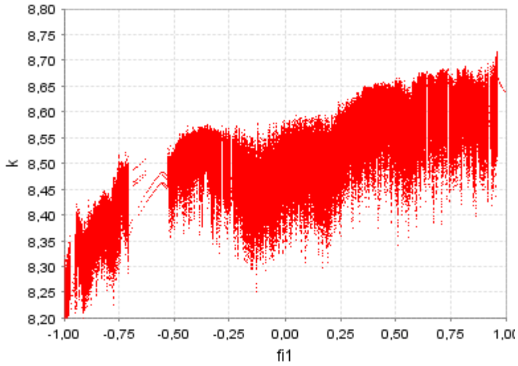

FIG. 32. model 2; bifurcation diagram (k_t) $[k_0 = 8; d_0 = 0.25;\ \phi_0 = 0.2;\ -1 \leq \phi_1 \leq 1;\ \phi_2 = 5;$ transients=1000; iterations=1000]

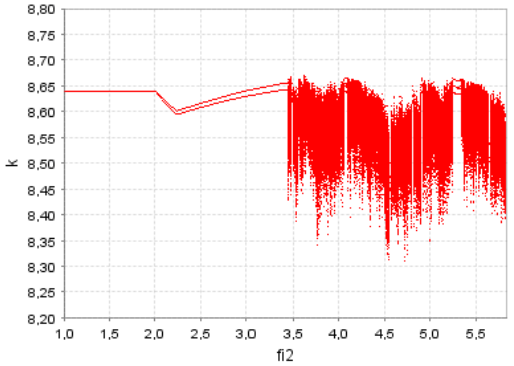

FIG. 33. model 2; bifurcation diagram (k_t) $[k_0 = 8; d_0 = 0.25;\ \phi_0 = 0.2; \phi_1 = 0.5;\ 1 \leq \phi_2 \leq 5.828;$ transients=1000; iterations=1000]

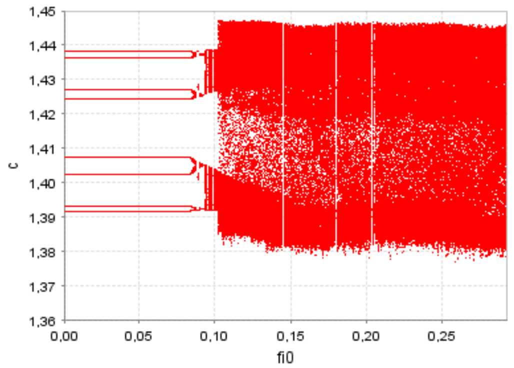

FIG. 34. model 2; bifurcation diagram (c_t) $[k_0 = 8; d_0 = 0.25;\ 0 \leq \phi_0 \leq 0.293;\ \phi_1 = 0.5; \phi_2 = 5;$ transients=1000; iterations=1000]

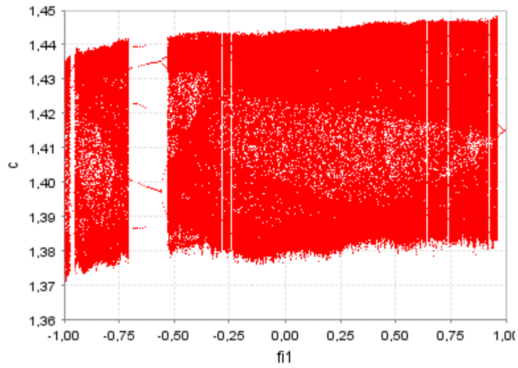

FIG. 35. model 2; bifurcation diagram (c_t) [$k_0 = 8; d_0 = 0.25$; $\phi_0 = 0.2$; $-1 \leq \phi_1 \leq 1$; $\phi_2 = 5$; transients=1000; iterations=1000]

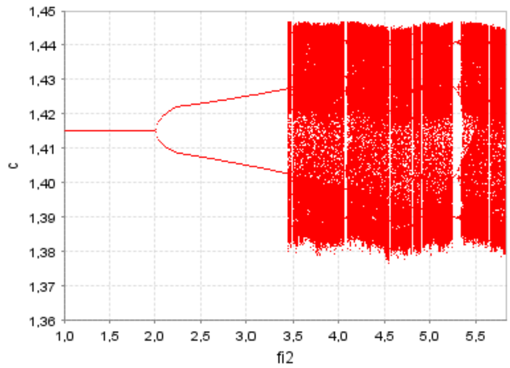

FIG. 36. model 2; bifurcation diagram (c_t) [$k_0 = 8; d_0 = 0.25$; $\phi_0 = 0.2; \phi_1 = 0.5$; $1 \leq \phi_2 \leq 5.828$; transients=1000; iterations=1000]

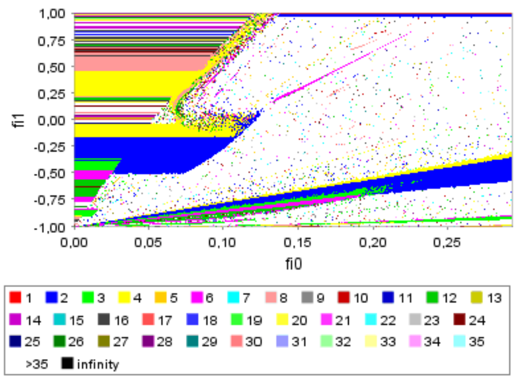

FIG. 37. model 2; periods of various orders in the space of parameters [$k_0 = 8; d_0 = 0.25$; $\phi_2 = 5$; transients=1000]

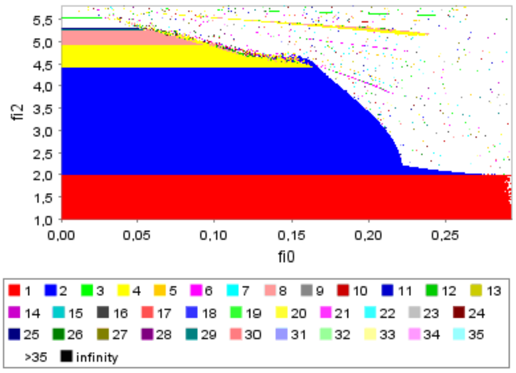

FIG. 38. model 2; periods of various orders in the space of parameters [$k_0 = 8; d_0 = 0.25; \phi_1 = 0.5$; transients=1000]

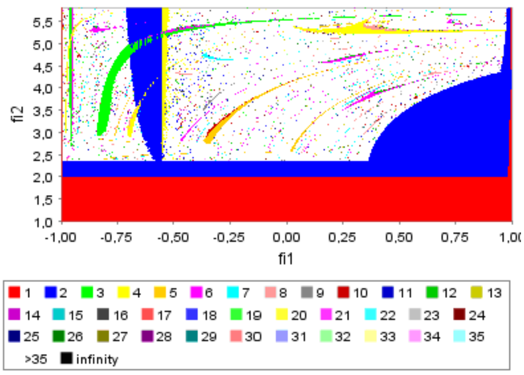

FIG. 39. model 2; periods of various orders in the space of parameters [$k_0 = 8; d_0 = 0.25; \phi_0 = 0.2$; transients=1000]

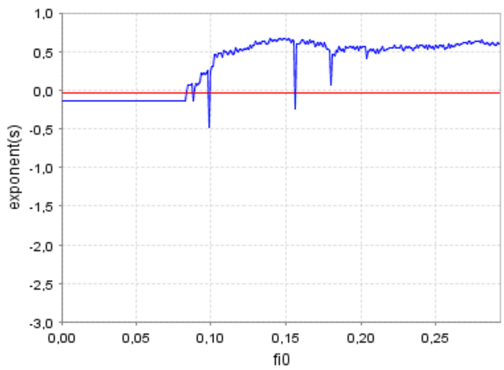

FIG. 40. model 2; Lyapunov exponents [$k_0 = 8; d_0 = 0.25$; $0 \leq \phi_0 \leq 0.293$; $\phi_1 = 0.5; \phi_2 = 5$; iterations=1000]

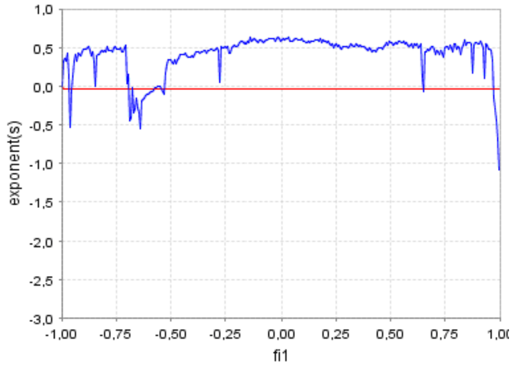

FIG. 41. model 2; Lyapunov exponents [$k_0 = 8; d_0 = 0.25;\ -1 \leq \phi_1 \leq 1;\ \phi_0 = 0.2; \phi_2 = 5$; iterations=1000]

jectories and attractors. Bifurcation diagrams are not displayed since they are uninteresting; as figure 49 shows, there is a large region of the space of parameters in which high periodicity cycles or chaos are observable and an area of instability. Lyapunov exponents, presented in figures 50 and 51, confirm the sudden jump from chaos to instability.

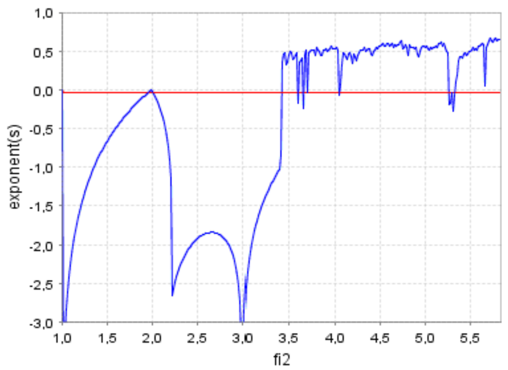

FIG. 42. model 2; Lyapunov exponents [$k_0 = 8; d_0 = 0.25;\ 1 \leq \phi_2 \leq 5.828;\ \phi_0 = 0.2; \phi_1 = 0.5$; iterations=1000]

The graphical analysis that was undertaken allowed to realize that sub-optimal investment decisions are a strong candidate to explain real endogenous business cycles. The rules that we have adopted indicate that a large variety of fluctuations results are possible if we slightly change the perception that the firms have about future demand. A relevant result is that although investment is always below the efficiency level, periods of consumption above 'potential' are common (nevertheless, the average level

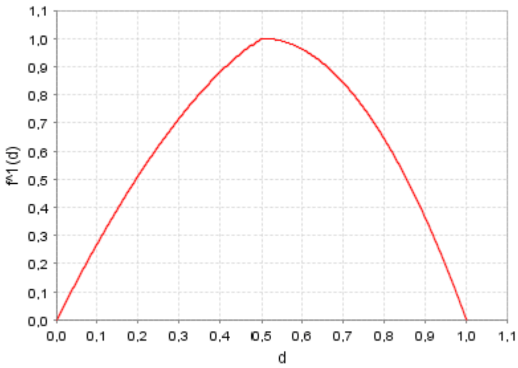

FIG. 43. model 3; phase diagram for difference equation (4) [$\phi_3 = 2.5; \phi_4 = 0.7$]

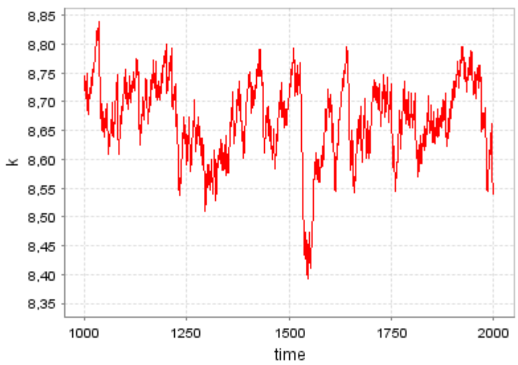

FIG. 44. model 3; trajectory of k_t [$k_0 = 8; d_0 = 0.25;$ $\phi_3 = 2.5; \phi_4 = 0.7$; transients=1000; iterations=1000]

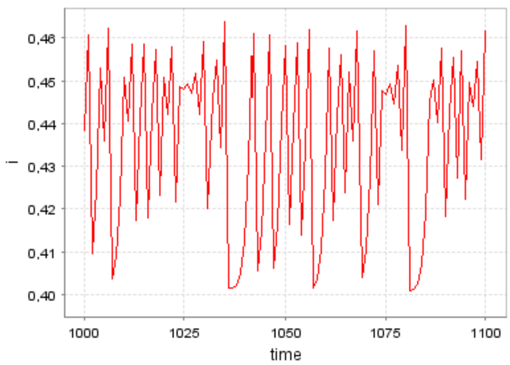

FIG. 45. model 3; trajectory of i_t [$k_0 = 8; d_0 = 0.25;$ $\phi_3 = 2.5; \phi_4 = 0.7$; transients=1000; iterations=100]

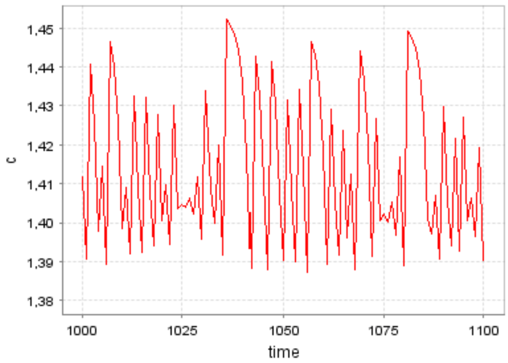

FIG. 46. model 3; trajectory of c_t [$k_0 = 8; d_0 = 0.25$; $\phi_3 = 2.5; \phi_4 = 0.7$; transients=1000; iterations=100]

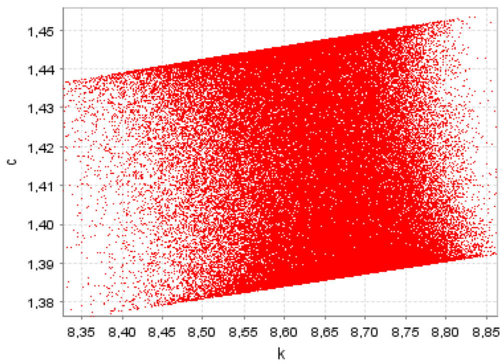

FIG. 47. model 3; attractor (k_t, c_t) [$k_0 = 8; d_0 = 0.25$; $\phi_3 = 2.5; \phi_4 = 0.7$; transients=1000; iterations=100,000]

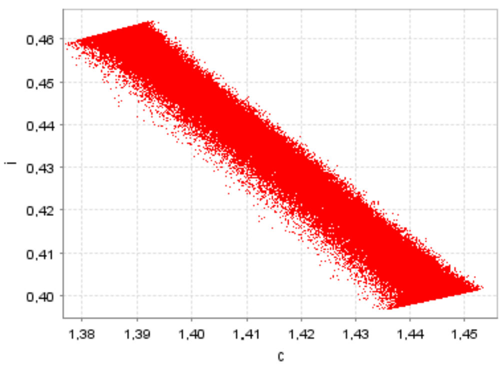

FIG. 48. model 3; attractor (c_t, i_t) [$k_0 = 8; d_0 = 0.25$; $\phi_3 = 2.5; \phi_4 = 0.7$; transients=1000; iterations=100,000]

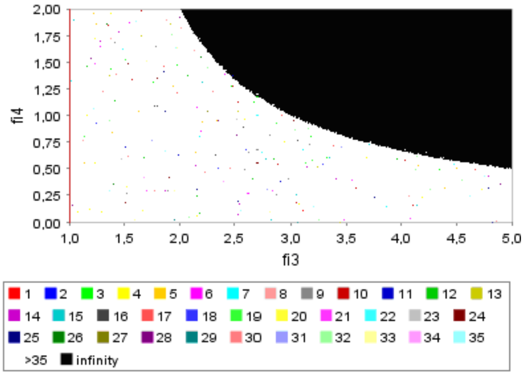

FIG. 49. model 3; periods of various orders in the space of parameters [$k_0 = 8; d_0 = 0.25$; transients=1000]

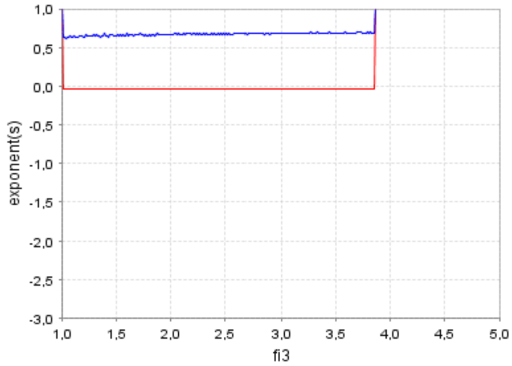

FIG. 50. model 3; Lyapunov exponents [$k_0 = 8; d_0 = 0.25$; $1 \leq \phi_3 \leq 5$; $\phi_4 = 0.7$; iterations=1000]

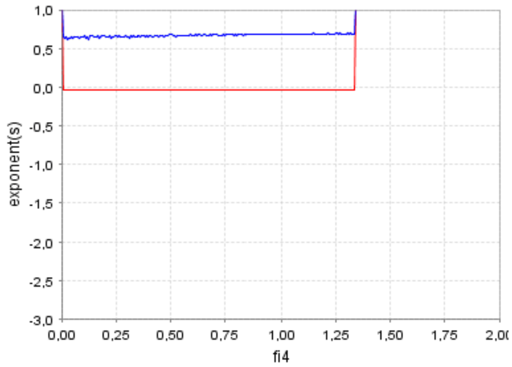

FIG. 51. model 3; Lyapunov exponents [$k_0 = 8; d_0 = 0.25$; $0 \leq \phi_4 \leq 2$; $\phi_3 = 2.5$; iterations=1000]

of consumption in relatively large time periods has to be lower the one encountered in the original Solow model).

5. CONCLUSIONS

Growth models serve the particular purpose of explaining long term economic performance under the assumption of efficient use of resources. As a result, these models generally lead to a constant long term growth (endogenous or exogenous) scenario. The basic structure of capital accumulation that the Solow model translates characterizes a steady state where economic aggregates do not grow as a result of endogenous features, and therefore capital, output, consumption and investment time paths may be represented as straight lines reflecting the fixed point result.

The main argument in this paper is that such theoretical structure (should it be as simple as the Solow model or more complicated as other neoclassical / endogenous growth frameworks) is aimed at explaining a process of perfect allocation of resources, where all the possible investment opportunities are completely fulfilled. This assumption lacks empirical consistency; although available resources allow for some potential investment level, firms tend to adopt a prudent behaviour and therefore they do not develop all the possible investment projects that the available economic resources allow to. Instead of considering that effective investment levels coincide in all moments of time with the potential level of investment, we rather assume that they adopt simple piecewise linear rules that relate demand with expected demand growth.

Evidently, not all demand is understood in the same way. Expected demand concerning non durable goods or the expected demand associated with investment projects that have began in previous periods is likely to be completely fulfilled; nevertheless, there are some components of demand that may vary significantly and for these a hump-shaped / tent shaped expectations rule is considered. Then, we have also assumed that investment decisions will depend on expected demand. Adding these features to the basic structure of economic growth, we have produced endogenous cycles.

Three different demand expectation rules were considered. In each case, one has studied local bifurcations and analyzed particular cases concerning global dynamics. Flip bifurcations occur in several circumstances dividing the space of parameters in regions of stability (two eigenvalues of the Jacobian matrix inside the unit circle) and local saddle-path stability (one eigenvalue inside the unit circle and the other outside it). Global dynamics reveal that the areas locally identified as saddle-path stable regions are in fact areas where endogenous cycles prevail. For different values of parameters these endogenous fluctuations may be synonymous of regular low periodicity cycles or completely irregular cycles (chaos).

Our main conclusion is that the standard growth model in its most simple structure can easily be transformed in a model of endogenous cycles if one accounts for the difficulty firms have in perceiving the full extent of investment opportunities (which are driven by expectations about demand). The generated endogenous cycles can take a great variety of properties - it will all depend on which is in fact the rule that firms adopt to forecast the evolution of aggregate demand.

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