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The Closed-form Solution for Pricing American Put Options

Wang Xiaodong

Room B1201, Hangnan Building, Zhichun Road, Haidian District, Beijing, China (100083)
E-mail: brice231@sohu.com

This paper proposes a closed-form solution for pricing an American put option on a non-dividend paying stock based on an optimally early-exercise strategy. An American put option should be early-exercised when the maximum option premium of early exercise is not less than the value of its European counterpart; otherwise, it should not be early-exercised. This paper also shows that Merton (1973)'s formula for pricing a perpetual American put option on a non-dividend paying stock is not perfect and shows such an option's value is equal to its strike price.

Key Words: American put option; Closed-form solution; Assets pricing.
JEL Classification Numbers: G12, G13.

1. INTRODUCTION

In 1973, the publication of the celebrated work of Black and Scholes made great contribution to the world's financial markets. Since then, pricing of European options became easier. Comparing with American options, European options are simpler financial derivatives that give their holder rights, but not obligations, to buy or sell a unit of asset at a fixed time, for a fixed price.

Black and Scholes gave the following famous closed-form solution for pricing European options on non-dividend paying stocks:

$$C_E(S_0, K, r, T, \sigma) = S_0 N(d_1) - Ke^{-rT} N(d_2) \quad (1)$$

$$P_E(S_0, K, r, T, \sigma) = Ke^{-rT} N(-d_2) - S_0 N(-d_1) \quad (2)$$

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Where

$$d_1 = \frac{\ln \frac{S_0}{K} + \left(r + \frac{1}{2}\sigma^2\right) T}{\sigma\sqrt{T}} \tag{3}$$

$$d_2 = \frac{\ln \frac{S_0}{K} + \left(r - \frac{1}{2}\sigma^2\right) T}{\sigma\sqrt{T}} \tag{4}$$

American options differ from European options by virtue of the fact that they can be exercised at any time during the lifetime of options. This makes for a more complicated pricing problem. Up to now, there is no closed-form solution for pricing American options although many people made great efforts.

An American Call Option on a non-dividend-paying stock should never be exercised prior to expiration, so an American call option on a non-dividend paying stock has the same value as its European counterpart.

$$C_A(S_0, K, r, T, \sigma) = C_E(S_0, K, r, T, \sigma) \tag{5}$$

However, an American put option may be rationally early-exercised, no matter there is dividend paying or not. This is because the holder can get interests by early exercising and put money in a risk-free banking account. In this paper, we only focus on American put options on non-dividend paying stocks.

In order to continue our analysis, we take the following assumptions, which are similar to that of Black-Scholes Model.

(a) There are no transaction costs or taxes.

(b) We are in a risk-neutral economy, which means all the market expected return is equal to the risk -free interest rate. The risk-free interest rate is known and constant; borrowing and lending are possible at the risk-free interest rate.

(c) There are non-dividends for the underlying stock, and short-sell such an underlying stock is allowed and possible.

A short sale of such an underlying stock entails market participants borrowing a share of underlying stock from the lender in the market and then selling it, receiving cash. Later, buy back an underlying stock, paying cash for it, and return it to the lender (Since there are non-dividends for the underlying stock, only an underlying stock is needed to return to the lender). A short sale can be viewed, then, just as a way of borrowing money.

(d) The stock price is Log normal distributed.

$\ln(S_t/S_0)$ is normally distributed with mean $\left(r - \frac{1}{2}\sigma^2\right) t$ and variance $\sigma^2 t$:

$$\ln(S_t/S_0) \sim N\left[\left(r - \frac{1}{2}\sigma^2\right) t, \sigma^2 t\right] \tag{6}$$

This gives us two equivalent ways to write an expression for the stock prices:

$$\ln(S_t/S_0) = \left(r - \frac{1}{2}\sigma^2\right)t + \sigma\sqrt{t}z \quad (7)$$

$$S_t = S_0 e^{(r-\frac{1}{2}\sigma^2)t+\sigma\sqrt{t}z} \quad (8)$$

Here z is a standard normal random variable. $z \sim N(0, 1)$

(e) The variation of the stock's continuous combine return is known and constant.

(f) The market participants take advantage of any arbitrage opportunities once they occur. In other words, there are no risk-free arbitrage opportunities in the market.

Since the holder of an American put option may rationally early-exercise and put the money into a risk-free banking account, he/she may get more gains than the holder of an European counterpart. This is why the value of the former is greater than that of the later. Obviously, if and only if early-exercise can bring the holder more gains than the value of an European counterpart, the holder will early-exercise this American put option.

The pricing of an American put option on a non-dividend paying stock could be described as an optimization problem. When the maximum option premium of early exercise is no less than the value of its European counterpart, the holder of an American put option would prefer early exercise; otherwise, the holder should not early exercise. So the price of an American put option on a non-dividend paying stock should be equal to the expected value of the maximum option premium.

What is the economic meaning of an American put option contract? When the buyer goes into the contract with the writer of an American put option on a non-dividend paying stock, the writer gives the buyer rights, but not obligation to short-sell an underlying stock to the writer in the strike price at any time before the expiration date. We can also say the writer gives the buyer rights, but not obligation to borrow money, the same amount as the strike price from the writer at any time before the expiration date. Of course, if the buyer uses the rights, he/she will have obligation to buy back an underlying stock in the market price from the writer before the expiration date, or we say the buyer will have obligation to return money (be equal to the market price of an underlying stock at that point) to the writer before the expiration date.

Since the buyer has rights whether exercise the option or not, and when exercise the option, he/she will take an optimal exercise strategy to gain maximum profits, this will bring same amount of loss to the writer of option. In fact, the value of the option actually is the compensation for the option writer.

The rest of the paper proceeds as follows. Section 2 derives a closed form solution to price an American put option on a non-dividend paying stock by setting up a series of Propositions. Section 3 shows the price of a perpetual American put option on a non-dividend paying stock is equal to its strike price. It shows that Merton (1973)'s formula is not perfect. The final section makes concluding remarks.

All the mathematics notations are given in the **Appendix 2**.

2. THE MODEL

First of all, some preparation should be made before beginning our tour. The following two Theorems are given. Related proof can be found in the reference books.

THEOREM 1. *An European put option with a higher strike price is at least as valuable as an otherwise identical one with a lower strike price. That is, if $K_1 \geq K_2$, then*

$$P_E(S_0, K_1, r, T, \sigma) \geq P_E(S_0, K_2, r, T, \sigma) \tag{9}$$

THEOREM 2. *Upper and lower bounds for an American put option on a non-dividend paying stock is:*

$$K - S_0 \leq P_A(S_0, K, r, T, \sigma) \leq C_A(S_0, K, r, T, \sigma) + (K - S_0) \leq K \tag{10}$$

From the formulation in (5), we know above formula can be written as

$$K - S_0 \leq P_A(S_0, K, r, T, \sigma) \leq C_E(S_0, K, r, T, \sigma) + (K - S_0) \leq K \tag{10'}$$

In addition, the following formula should also be satisfied.

$$P_E(S_0, K, r, T, \sigma) \leq P_A(S_0, K, r, T, \sigma) \tag{11}$$

Next, let's begin to use probabilistic approach to price an American put option on a non-dividend paying stock.

PROPOSITION 1. *The value of an American put option $P_A(S_0, K, r, T, \sigma)$ on a non-dividend paying stock is equal to the expected value of the maximum option premium.*

$$\begin{aligned} & P_A(S_0, K, r, T, \sigma) \\ = \ & E^Q\{\max[P_E(S_0, K, r, T, \sigma), Max\Pr emium(early exercise)]\} \end{aligned} \tag{12}$$

Here $Max\Pr emium(early exercise)$ *is the maximum option premium when an American put option is optimally early-exercised.*

Proof. According to the definition of an American put option, the holder has rights, but not obligation to exercise it at any time during its lifetime. As we know, when an American put option is not early-exercised, the premium will be equal to its European counterpart.

$$P_A(S_0, K, r, T, \sigma) = P_E(S_0, K, r, T, \sigma)$$

The holder of an American put option should take an optimal exercise strategy to get the maximum option premium. So the pricing of an American put option is such an optimization problem:

(1) When the maximum option premium of optimally early exercise is no less than $P_E(S_0, K, r, T, \sigma)$, the American put option should be optimally early-exercised and get the max premium:

$$P_A(S_0, K, r, T, \sigma) = Max\Pr emium(early exercise) \tag{13}$$

(2) Otherwise, the American put option should not be early-exercised and get the same premium as its European counterpart:

$$P_A(S_0, K, r, T, \sigma) = P_E(S_0, K, r, T, \sigma) \tag{14}$$

$$\begin{aligned} \therefore\ & P_A(S_0, K, r, T, \sigma) \\ =\ & E^Q\{\max[P_E(S_0, K, r, T, \sigma), Max\Pr emium(early exercise)]\} \end{aligned}$$

∎

PROPOSITION 2. *As long as an American put option* $P_A(S_0, K, r, T, \sigma)$ *on a non-dividend paying stock is optimally early-exercised, an underlying stock should be shorted in the strike price* K *at time 0.*

Proof. The process of early-exercise an American put option on a non-dividend paying stock can be divided into two steps: short-sell one underlying stock in the strike price K and put the money into a risk-free banking account; buy back one underlying stock in the market price immediately or later before the expiration date.

First, suppose $P_A(S_0, K, r, T, \sigma)$ is optimally early-exercised, and an underlying stock is shorted in the strike price K not at time 0, but at time t $(0 < t < T)$. The premium will be equal to

(1) If buy back an underlying stock at time t immediately

$\Pr emium(earlyexercise) = Payoff(S_t < K)$
$= [K - E(S_t/S_t < K)]e^{-rt} Prob(S_t < K)^1 = P_E(S_0, K, r, t, \sigma)$

(2) If buy back an underlying stock at time Ψ $(t < \Psi \le T)$

At time Ψ $(t < \Psi \le T)$, the holder will have $Ke^{r(\Psi - t)}$, since the holder can put the money K into a risk-free banking account during the time period $t \sim \Psi$.

$$\begin{aligned} & \Pr emium(earlyexecise) = Payoff(S_\Psi < Ke^{r(\Psi-t)}) \\ = \; & [Ke^{r(\Psi-t)} - E(S_\Psi/S_\Psi < Ke^{r(\Psi-t)})]e^{-r\Psi} Prob(S_\Psi < Ke^{r(\Psi-t)}) \\ = \; & P_E(S_0, Ke^{r(\Psi-t)}, r, \Psi, \sigma) \end{aligned}$$

However, if an underlying stock is shorted in the strike price K at time 0, the premium will be equal to

(1) If buy back an underlying stock at time t $(0 < t < T)$

At time t, the holder will have Ke^{rt}, since the holder can put the money K into a risk-free banking account during the time period $0 \sim t$.

$$\begin{aligned} & \Pr emium(earlyexercise) = Payoff(S_t < Ke^{rt}) \\ = \; & [Ke^{rt} - E(S_t/S_t < Ke^{rt})]e^{-rt} Prob(S_t < Ke^{rt}) \\ = \; & P_E(S_0, Ke^{rt}, r, t, \sigma) \end{aligned}$$

(2) If buy back an underlying stock at time Ψ $(t < \Psi \le T)$

At time Ψ, the holder will have $Ke^{r\Psi}$, since the holder can put the money K into a risk-free banking account during the time period $0 \sim \Psi$.

$$\begin{aligned} & \Pr emium(earlyexecise) = Payoff(S_\Psi < Ke^{r\Psi}) \\ = \; & [Ke^{r\Psi} - E(S_\Psi/S_\Psi < Ke^{r\Psi})]e^{-r\Psi} Prob(S_\Psi < Ke^{r\Psi}) \\ = \; & P_E(S_0, Ke^{r\Psi}, r, Psi, \sigma) \end{aligned}$$

According to **Theorem 1**,

$$P_E(S_0, Ke^{rt}, r, t, \sigma) > P_E(S_0, K, r, t, \sigma)$$

$$P_E(S_0, Ke^{r\Psi}, r, \Psi, \sigma) > P_E(S_0, Ke^{r(\Psi-t)}, r, \Psi, \sigma)$$

So, as long as an American put option on a non-dividend paying stock is optimally early-exercised, an underlying stock should be shorted in the strike price K at time 0. ∎

[1] $Prob(S_t < K)$: Means probability of $S_t < K$

Proposition 3. *As long as an American put option $P_A(S_0, K, r, T, \sigma)$ on a non-dividend paying stock is optimally early-exercised, the expected value of maximum option premium will be:*

$$\begin{aligned} & E[Max\,\Pr emium(earlyexercise)] \\ = \; & P_E(S_0, Ke^{rT}, r, T, \sigma)N(-d_4) + (K - S_0)N(d_4) \end{aligned} \tag{15}$$

In other words, the maximum option premium under the optimally early-exercised strategy is either $P_E(S_0, Ke^{rT}, r, T, \sigma)$ or $(K - S_0)$, and the probability to take them is $N(-d_4)$ or $N(d_4)$, respectively.

Where

$$P_E(S_0, Ke^{rT}, r, T, \sigma) = KN(-d_4) - S_0N(-d_3) \tag{16}$$

$$d_3 = \frac{\ln \frac{S_0}{K} + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} \tag{17}$$

$$d_4 = \frac{\ln \frac{S_0}{K} - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} \tag{18}$$

Proof. According to **Proposition 2**, as long as an American put option $P_A(S_0, K, r, T, \sigma)$ on a non-dividend paying stock is optimally early-exercised, an underlying stock should be shorted in the strike price K at time 0. Then at time ζ $(0 < \zeta \leq T)$, the option holder will have $Ke^{r\zeta}$, since the holder can put the money K into a risk-free banking account during the time period $0 \sim \zeta$.

(1) When $S_\zeta < Ke^{r\zeta}$ $(0 < \zeta \leq T)$

$$\begin{aligned} & \Pr emium(earlyexercise) \\ = \; & [Ke^{r\zeta} - E(S_\zeta / S_\zeta < Ke^{r\zeta})]e^{-r\zeta} Prob(S_\zeta < Ke^{r\zeta}) \\ = \; & P_E(S_0, Ke^{r\zeta}, r, \zeta, \sigma) \end{aligned}$$

(2) When $S_\zeta > Ke^{r\zeta}$ $(0 < \zeta \leq T)$

$$\begin{aligned} & \Pr emium(earlyexercise) \\ = \; & [Ke^{r\zeta} - E(S_\zeta / S_\zeta > Ke^{r\zeta})]e^{-r\zeta} Prob(S_\zeta > Ke^{r\zeta}) \\ = \; & -C_E(S_0, Ke^{r\zeta}, r, \zeta, \sigma) < 0 \end{aligned}$$

It can be proved that $\frac{\partial P_E(S_0,Ke^{r\zeta},r,\zeta,\sigma)}{\partial \zeta} > 0$ (**Appendix 1**), which means $P_E(S_0, Ke^{r\zeta}, r, \zeta, \sigma)$ increases while ζ $(0 < \zeta \leq T)$ increases.

Since $C_E(S_0, Ke^{r\zeta}, r, \zeta, \sigma) = P_E(S_0, Ke^{r\zeta}, r, \zeta, \sigma) - (K - S_0)$

So $\frac{\partial[-C_E(S_0,Ke^{r\zeta},r,\zeta,\sigma)]}{\partial \zeta} < 0$, which means $-C_E(S_0, Ke^{r\zeta}, r, \zeta, \sigma)$ decreases while ζ $(0 < \zeta \leq T)$ increases.

Now we suppose T' is a point at $[0, T]$, $0 < T' < T$, let $T = n\Delta, T' = m\Delta$, Δ is a very short time period$m$ and n are nature numbers, $m < n$.

As long as $P_A(S_0, K, r, T, \sigma)$ is early-exercised under the optimally early-exercise strategy, the option holder will take the following optimal strategy to get the maximum gains:

When $S_T < Ke^{rT}$, $Max\,\Pr emium(earlyexercise) = P_E(S_0, Ke^{rT}, r, T, \sigma)$

When $S_T > Ke^{rT}$ and $S_{T-\Delta} < Ke^{r(T-\Delta)}$,

When $S_T > Ke^{rT}$, $S_{T-\Delta} > Ke^{r(T-\Delta)}$ and $S_{T-2\Delta} < Ke^{r(T-2\Delta)}$,

$$\begin{aligned} & Max\,\Pr emium(earlyexercise) \\ = \; & P_E(S_0, Ke^{r(T-2\Delta)}, r, T - 2\Delta, \sigma) - C_E(S_0, Ke^{r(T-\Delta)}, r, T - \Delta, \sigma) \\ \approx \; & (K - S_0) \end{aligned}$$

......

When $S_T > Ke^{rT}$, $S_{T-\Delta} > Ke^{r(T-\Delta)}$, $S_{T-2\Delta} > Ke^{r(T-2\Delta)}$, ... and $S_{T-m\Delta} < Ke^{r(T-m\Delta)}$,

$$\begin{aligned} & Max\,\Pr emium(earlyexercise) \\ = \; & P_E(S_0, Ke^{r(T-m\Delta)}, r, T - m\Delta, \sigma) - C_E(S_0, Ke^{r[T-(m-1)\Delta]}, r, T - (m-1)\Delta, \sigma) \\ \approx \; & (K - S_0) \end{aligned}$$

......

When $S_T > Ke^{rT}$, $S_{T-\Delta} > Ke^{r(T-\Delta)}$, $S_{T-2\Delta} > Ke^{r(T-2\Delta)}$, ..., $S_{T-m\Delta} > Ke^{r(T-m\Delta)}$, ... and $S_\Delta > Ke^{r\Delta}$,

$$Max\,\Pr emium(earlyexercise) = (K - S_0)$$

So, as long as $P_A(S_0, K, r, T, \sigma)$ is early-exercised under the optimally early-exercise strategy, the expected value of the maximum gains will be:

$$\begin{aligned} & E[Max\,\Pr emium(earlyexercise)] = P_E(S_0, Ke^{rT}, r, T, \sigma)Prob(S_T < Ke^{rT}) \\ & +(K - S_0)Prob(S_T > Ke^{rT})Prob(S_{T-\Delta} < Ke^{r(T-Delta)}) \\ & +(K - S_0)Prob(S_T > Ke^{rT})Prob(S_{T-\Delta} > Ke^{r(T-\Delta)})Prob(S_{T-2\Delta} < Ke^{r(T-2\Delta)}) \\ & \quad \ldots\ldots \\ & +(K - S_0)Prob(S_T > Ke^{rT})Prob(S_{T-\Delta} > Ke^{r(T-\Delta)}) \cdots Prob(S_{T-m\Delta} < Ke^{r(T-m\Delta)}) \\ & \quad \ldots\ldots \\ & +(K - S_0)Prob(S_T > Ke^{rT})Prob(S_{T-\Delta} > Ke^{r(T-\Delta)}) \cdots Prob(S_\Delta > Ke^{r\Delta}) \end{aligned}$$

Now let Δ tends to be zero, and T' tends to T, we will have:

$$\begin{aligned}
Prob(S_T < Ke^{rT}) &\approx Prob(S_{T-\Delta} < Ke^{r(T-\Delta)}) \approx Prob(S_{T-2\Delta} < Ke^{r(T-2\Delta)}) \\
\ldots\ldots &\approx Prob(S_{T-m\Delta} < Ke^{r(T-m\Delta)}) = N(-d_4) \qquad (19) \\
Prob(S_T > Ke^{rT}) &\approx Prob(S_{T-\Delta} > Ke^{r(T-\Delta)}) \approx Prob(S_{T-2\Delta} > Ke^{r(T-2\Delta)}) \\
\ldots\ldots &\approx Prob(S_{T-m\Delta} > Ke^{r(T-m\Delta)}) = N(d_4) \qquad (20)
\end{aligned}$$

Here we take $d_4 = \frac{\ln\frac{S_0}{K} - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}$

Please note $N(d_4) < 1$, and $N(-d_4) + N(d_4) = 1$. Omitting the high-level items, it is easy to have:

$$\begin{aligned}
& E[Max\,\Pr emium(earlyexercise)] \\
= \;& P_E(S_0, Ke^{rT}, r, T, \sigma)N(-d_4) \\
+ \;& (K - S_0)N(-d_4)[N(d_4) + N(d_4)^2 + \cdots + N(d_4)^m] \\
= \;& P_E(S_0, Ke^{rT}, r, T, \sigma)N(-d_4) + (K - S_0)N(-d_4)\frac{N(d_4)[1 - N(d_4)^{m-1}]}{1 - N(d_4)} \\
= \;& P_E(S_0, Ke^{rT}, r, T, \sigma)N(-d_4) + (K - S_0)N(d_4)
\end{aligned}$$

This tells us that the maximum option premium under the optimally early-exercised strategy is either $P_E(S_0, Ke^{rT}, r, T, \sigma)$ or $(K - S_0)$, and the probability to take them is $N(-d_4)$ or $N(d_4)$, respectively. ▮

Now, it is the time to propose the closed form solution for pricing an American put option on a non-dividend paying stock.

PROPOSITION 4. *The price of an American put option $P_A(S_0, K, r, T, \sigma)$ on a non-dividend paying stock is:*

$$\begin{aligned}
P_A(S_0, K, r, T, \sigma) &= P_E(S_0, Ke^{rT}, r, T, \sigma)N(-d_4) \\
&+ \max[(K - S_0), P_E(S_0, K, r, T, \sigma)]N(d_4) \qquad (21)
\end{aligned}$$

Where,

$$\begin{aligned}
P_E(S_0, K, r, T, \sigma) &= Ke^{-rT}N(-d_2) - S_0N(-d_1) \qquad (22) \\
P_E(S_0, Ke^{rT}, r, T, \sigma) &= KN(-d_4) - S_0N(-d_3) \qquad (23)
\end{aligned}$$

$$d_1 = \frac{\ln \frac{S_0}{K} + \left(r + \frac{1}{2}\sigma^2\right) T}{\sigma\sqrt{T}}, \qquad d_2 = \frac{\ln \frac{S_0}{K} + \left(r - \frac{1}{2}\sigma^2\right) T}{\sigma\sqrt{T}} \tag{24}$$

$$d_3 = \frac{\ln \frac{S_0}{K} + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}, \qquad d_4 = \frac{\ln \frac{S_0}{K} - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} \tag{25}$$

Proof. According to Proposition 1 and Proposition 3, we have:

$$P_A(S_0, K, r, T, \sigma) = E^Q\{\max[P_E(S_0, K, r, T, \sigma), Max\,\Pr emium(earlyexercise)]\}$$

Here $Max\,\Pr emium(earlyexercise)$ is either $P_E(S_0, Ke^{rT}, r, T, \sigma)$ or $(K-S_0)$, with the probability $N(-d_4)$ or $N(d_4)$, respectively.

Since $P_E(S_0, Ke^{rT}, r, T, \sigma) > P_E(S_0, K, r, T, \sigma)$

$$\begin{aligned} \therefore\ & P_A(S_0, K, r, T, \sigma) \\ =\ & E^Q\{\max[P_E(S_0, K, r, T, \sigma), Max\,\Pr emium(earlyexercise)]\} \\ =\ & P_E(S_0, Ke^{rT}, r, T, \sigma)N(-d_4) + \max[(K - S_0), P_E(S_0, K, r, T, \sigma)]N(d_4) \end{aligned}$$

∎

3. PERPETUAL AMERICAN PUT OPTION

A perpetual American put option is a special kind of American put option. It grants its holder rights, but not obligation to sell an underlying stock in a fixed price at any time up until infinite future. Since the maturity time of such an option is infinite future, it is also called an expiration-less option.

Obviously, a perpetual American put option on a non-dividend paying stock should at least satisfy:

$$K - S_0 \le P_{PA}(S_0, K, r, \infty, \sigma) \le K \tag{26}$$

And

$$P_A(S_0, K, r, t, \sigma) \le P_{PA}(S_0, K, r, \infty, \sigma) \tag{27}$$

Where $0 < t \le \infty$

3.1. Merton's formula

Merton (1973) proposed a closed-form solution for pricing a perpetual American put option. McDonald and Siegel (1986) discussed the link between the perpetual American put and perpetual American call.

The model is, for one dividend-paying stock, with current stock price S_0, strike price K, risk-free interest rate r, expiration time ∞, volatility σ and continuous dividend-paying rate δ, the formula[2] for pricing a perpetual American put option is:

$$P_{PA}(S_0, K, r, \infty, \sigma, \delta) = \frac{K}{1-h_2}\left(\frac{h_2-1}{h_2}\frac{S_0}{K}\right)^{h_2} \tag{28}$$

Where,

$$h_2 = \frac{1}{2} - \frac{r-\delta}{\sigma^2} - \sqrt{\left(\frac{r-\delta}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}\partial} \tag{29}$$

For non-dividend paying stock, $\delta = 0$, the formula becomes:

$$P_{PA}(S_0, K, r, \infty, \sigma) = \frac{K}{1-h_2}\left(\frac{h_2-1}{h_2}\frac{S_0}{K}\right)^{h_2} \tag{30}$$

Where,

$$h_2 = -\frac{2r}{\sigma^2} \tag{31}$$

Formula (30) tells us that the price of a perpetual American put option on a non-dividend paying stock is related to S_0, K, r and σ.

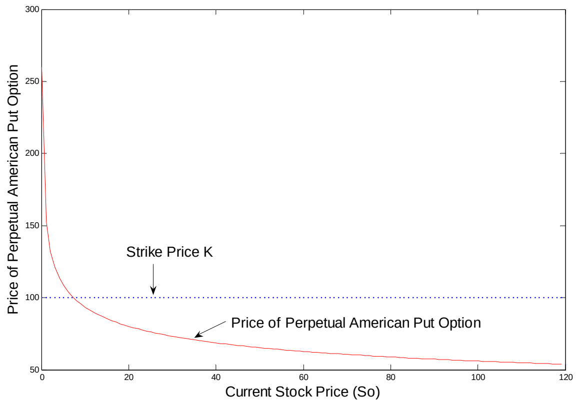

FIG. 1. Price of PAPO at different current stock prices based on Formula (30) ($K = 100, r = 0.04, \sigma = 0.6$)

However, some examples can easily show formula (30) does not do a good job. Figure 1-3[3] shows the results based on formula (30) for the price of

[2]Please see Robert L. McDONALD, 2003, Derivatives Markets, Pearson Education, Inc. 392-393

[3]All the figures in this paper are drawn with MATLAB

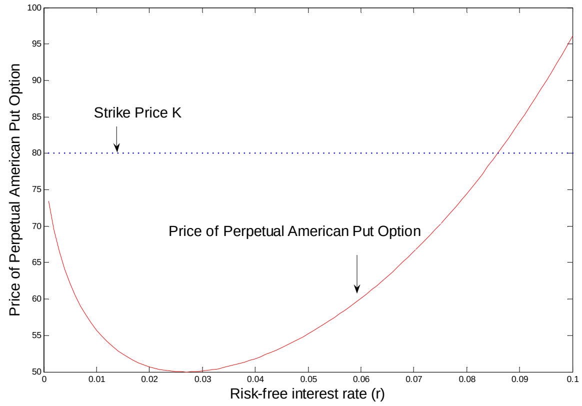

FIG. 2. Price of PAPO at different risk-free interest rate based on Formula (30) $(S_0 = 30, K = 80, \sigma = 0.3)$

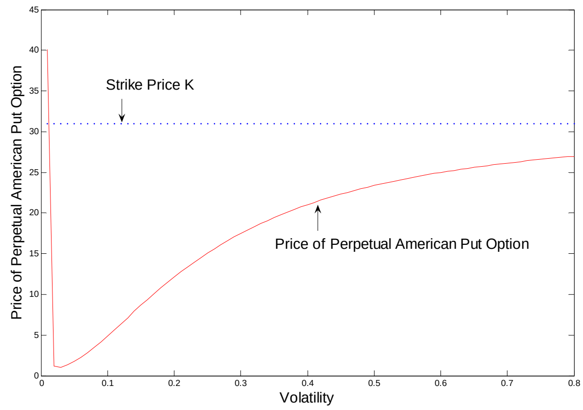

FIG. 3. Price of PAPO at different volatility based on Formula (30) $(S_0 = 30, K = 31, r = 0.01)$

perpetual American put options in different current stock price, different risk-free interest rate and different volatility, respectively. It is very clear that in some cases, the result is greater than the strike price, which is contrary to the upper limit in (26). This is apparently not reasonable.

3.2. Formula for pricing PAPO

In fact, the value of a perpetual American put option on a non-dividend paying stock could be drawn from **Proposition 4** easily.

PROPOSITION 5. *The price of a perpetual American put option* $P_{PA}(S_0, K, r, \infty, \sigma)$ *on a non-dividend paying stock is equal to its strike* *price:*

$$P_{PA}(S_0, K, r, \infty, \sigma) = K \tag{32}$$

Proof. A perpetual American put option on a non-dividend paying stock is a special kind of American put option on a non-dividend paying stock while the maturity time is infinite future.

$$P_{PA}(S_0, K, r, \infty, \sigma) = P_A(S_0, K, r, T, \sigma) \;\; (when \;\; T \to \infty) \tag{33}$$

From **Proposition 4**, we know that

$$\begin{aligned} P_A(S_0, K, r, T, \sigma) &= P_E(S_0, Ke^{rT}, r, T, \sigma)N(-d_4) \\ &+ \max[(K - S_0), P_E(S_0, K, r, T, \sigma)]N(d_4) \end{aligned}$$

$$d_4 = \frac{\ln \frac{S_0}{K} - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} = \xrightarrow{T\to\infty} -\infty \tag{34}$$

$$d_3 = \frac{\ln \frac{S_0}{K} + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} = \xrightarrow{T\to\infty} \infty \tag{35}$$

$$\therefore N(d_4) \xrightarrow{T\to\infty} 0; N(-d_4) \xrightarrow{T\to\infty} 1; N(-d_3) \xrightarrow{T\to\infty} 0 \tag{36}$$

$$P_E(S_0, Ke^{rT}, r, T, \sigma) = KN(-d_4) - S_0 N(-d_3) = \xrightarrow{T\to\infty} K \tag{37}$$

$$\begin{aligned} &P_A(S_0, K, r, T, \sigma) = P_E(S_0, Ke^{rT}, r, T, \sigma)N(-d_4) \\ &+ \max[(K - S_0), P_E(S_0, K, r, T, \sigma)]N(d_4) = \xrightarrow{T\to\infty} K \end{aligned} \tag{38}$$

$$\therefore P_{PA}(S_0, K, r, \infty, \sigma) = K$$

∎

This result can be seen clearly from Figure 4-6. Based on formula in (21), **Proposition 4**, the relationship between the price of an American put option (at the money, in the money, out of money, respectively) and the maturity time T is show in Figure 4-6, from which we can notice that the price of an American put option rises while maturity time T increases and tends to K while the maturity time T is big enough.

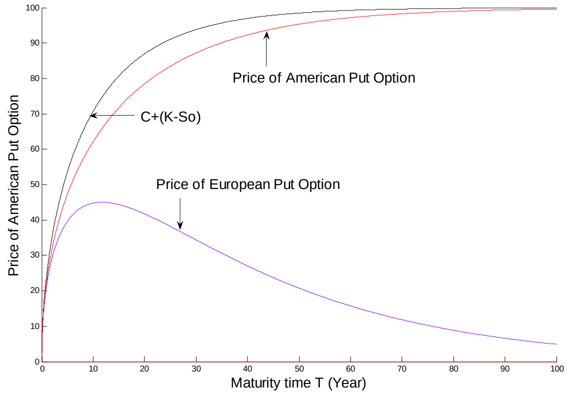

FIG. 4. Price of APO at different T based on formula (21) $(S_0 = 100, K = 100, r = 0.03, \sigma = 0.6)$

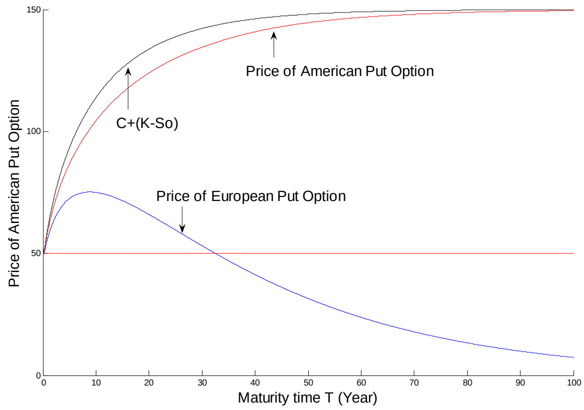

FIG. 5. Price of APO at different T based on formula (21) $(S_0 = 100, K = 150, r = 0.03, \sigma = 0.6)$

It will be easier to understand this result from the economic meaning of an American put option described in Section 1:

When $T \to \infty$, since the present value of $E(S_T/S_T < Ke^{rT})$ is

$$\begin{aligned} PV[E(S_T/S_T < Ke^{rT})] &= E(S_T/S_T < Ke^{rT})e^{-rT} = S_0 e^{rT} \frac{N(-d_3)}{N(-d_4)} e^{-rT} \\ &= S_0 \frac{N(-d_3)}{N(-d_4)} \xrightarrow{T \to \infty} 0 \end{aligned} \tag{39}$$

$$Prob(S_T/S_T < Ke^{rT}) = N(-d_4) \xrightarrow{T \to \infty} 1 \tag{40}$$

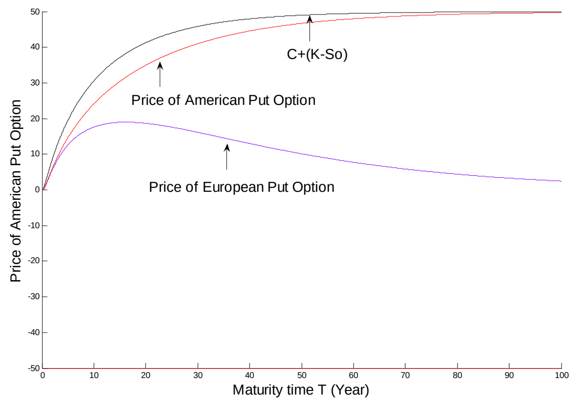

FIG. 6. Price of APO at different T based on formula (21) ($S_0 = 100, K = 50, r = 0.03, \sigma = 0.6$)

If the writer of an American put option gives the buyer rights, but not obligation to borrow money K from the writer at any time during option's lifetime, and return money $E(S_T/S_T < Ke^{rT})$ in the infinite future. How much of compensation should the writer ask for? Of course, the compensation should be equal to K.

3.3. Arbitrage Opportunity

In fact, there will be arbitrage opportunity if the price of a perpetual American put option on a non-dividend paying stock is not equal to its strike price.

PROPOSITION 6. *The price of a perpetual American put option $P_{PA}(S_0, K, r, \infty, \sigma)$ on a non-dividend paying stock is equal to its strike price K. Otherwise, there will be arbitrage opportunity.*

Proof. (1) If the price of a perpetual American put option

$$P_{PA}(S_0, K, r, \infty, \sigma) > K \tag{41}$$

Writing a cash-secured put would earn arbitrage profits.[4]

(2) If the price of a perpetual American put option

$$P_{PA}(S_0, K, r, \infty, \sigma) < K \tag{42}$$

[4]Please see YUH-DAUH LYUU, 2002, Financial Engineering and Computation, Principles, Mathematics, Algorithms, Cambridge University Press, 86

Let

$$P_{PA}(S_0, K, r, \infty, \sigma) = K' < K \tag{43}$$

An arbitrager can make risk-free profits as follows: buy one perpetual American put option $P_{PA}(S_0, K, r, \infty, \sigma)$ in K' (His cost is K'), exercise this option immediately (short-sell a stock in K), and keep the money K in a banking account. The only thing the arbitrager needs to do next is just waiting to a time point t_x until the stock price S_{t_x} at time t_x is less than $(K - K')e^{rt_x}$, which means

$$S_{t_x} < (K - K')e^{rt_x} \tag{44}$$

Then buy back a stock in S_{t_x} at time t_x. Since before infinite future, the arbitrager surely has such an opportunity, so the present value of the arbitrager's risk-free profit will be

$$PV(Profit) = K - K' - S_{t_x}e^{-rt_x} > 0 \tag{45}$$

In a word, the price of a perpetual American put option $P_{PA}(S_0, K, r, \infty, \sigma)$ on a non-dividend paying stock could not be greater or less than its strike price K, so

$$P_{PA}(S_0, K, r, \infty, \sigma) = K$$

∎

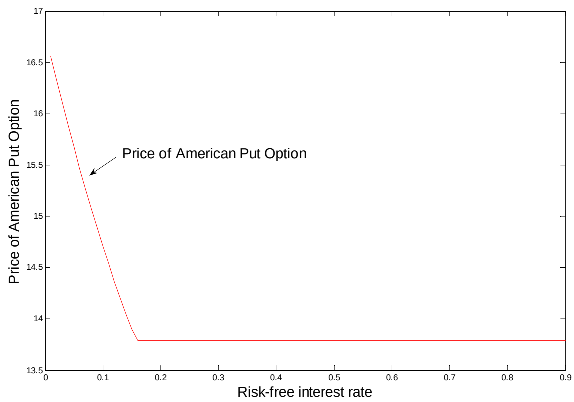

FIG. 7. Price of APO at different risk-free interest rate r based on formula (21) ($S_0 = 100, K = 108, T = 1, \sigma = 0.3$)

This conclusion tells us that time could 'erase' all differences. Since the maturity time of a perpetual American put option is the infinite future, as

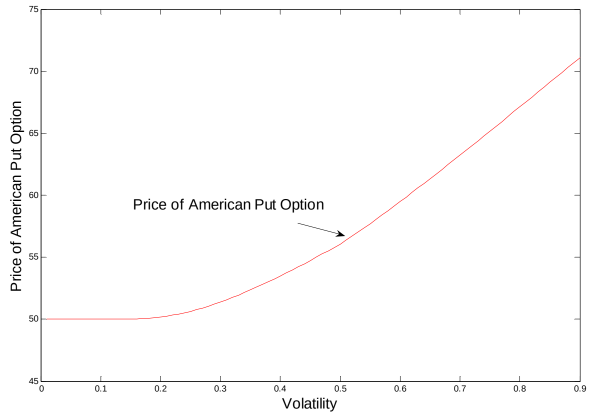

FIG. 8. Price of APO at different volatility σ based on formula (21) $(S_0 = 100, K = 150, T = 1, r = 0.06)$

long as the strike prices are same, the value of these perpetual American put options will be same. They will be equal to the strike price.

4. CONCLUSIONS

An American put option grants its holder rights, but not obligation to sell an underlying stock in the strike price at any time up until maturity, so the holder of an American put option has more rights than that of an otherwise equivalent European put option. The pricing of an American put option on a non-dividend paying stock could be described as an optimization problem: when the maximum option premium of early exercise is no less than the value of its European counterpart, the option should be early-exercised; otherwise, it should not be early-exercised. The price of an American put option on a non-dividend paying stock is equal to the expected value of maximum option premium.

The value of a perpetual American put option on a non-dividend paying stock is equal to its strike price. This can be shown by the no-arbitrage theory and by the formula this paper provides. On the other hand, this also proves that Merton (1973)'s formula is not perfect.

Based on formula in (21), **Proposition 4**, we could also find that:

(1) Keep others constant, as the risk-free interest rate r increases, the value of $P_A(S_0, K, r, T, \sigma)$ decreases (See Figure 7); Keep others constant, as the volatility σ increases, the value of $P_A(S_0, K, r, T, \sigma)$ also increases (See Figure 8)

(2) The upper and lower bounds for an American put option on a non-dividend paying stock in **Theorem 2** are satisfied.

APPENDIX

Here we prove $\frac{\partial P_E(S_0,Ke^{r\zeta},r,\zeta,\sigma)}{\partial\zeta} > 0$

Proof.

$$\begin{aligned}
&\frac{\partial P_E(S_0,Ke^{r\zeta},r,\zeta,\sigma)}{\partial\zeta} \\
= &\frac{\partial\left[KN\left(-\frac{\ln\frac{S_0}{K}-\frac{1}{2}\sigma^2\zeta}{\sigma\sqrt{\zeta}}\right)-S_0N\left(-\frac{\ln\frac{S_0}{K}+\frac{1}{2}\sigma^2\zeta}{\sigma\sqrt{\zeta}}\right)\right]}{\partial\zeta} \\
= &K\frac{\partial N\left(-\frac{\ln\frac{S_0}{K}-\frac{1}{2}\sigma^2\zeta}{\sigma\sqrt{\zeta}}\right)}{\partial\zeta}-S_0\frac{\partial N\left(-\frac{\ln\frac{S_0}{K}+\frac{1}{2}\sigma^2\zeta}{\sigma\sqrt{\zeta}}\right)}{\partial\zeta} \\
= &S_0N'(d_3)\frac{\partial\frac{\ln\frac{S_0}{K}+\frac{1}{2}\sigma^2\zeta}{\sigma\sqrt{\zeta}}}{\partial\zeta}-KN'(d_4)\frac{\partial\frac{\ln\frac{S_0}{K}-\frac{1}{2}\sigma^2\zeta}{\sigma\sqrt{\zeta}}}{\partial\zeta}
\end{aligned}$$

Where $N'(d_3)=\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(d_3)^2}$, $N'(d_4)=\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(d_4)^2}$ Since

$$\begin{aligned}
N'(d_4) &= \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(d_4)^2}=\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(d_3-\sigma\sqrt{\zeta})^2} \\
&= \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(d_3)^2+d_3\sigma\sqrt{\zeta}-\frac{1}{2}\sigma^2\zeta}=N'(d_3)\frac{S_0}{K}
\end{aligned}$$

So, we have

$$\begin{aligned}
\frac{\partial P_E(S_0,Ke^{r\zeta},r,\zeta,\sigma}{\partial\zeta} &= S_0N'(d_3)\left[\frac{\partial\frac{\ln\frac{S_0}{K}+\frac{1}{2}\sigma^2\zeta}{\sigma\sqrt{\zeta}}}{\partial\zeta}-\frac{\partial\frac{\ln\frac{S_0}{K}-\frac{1}{2}\sigma^2\zeta}{\sigma\sqrt{\zeta}}}{\partial\zeta}\right] \\
&= S_0N'(d_3)\left[\frac{\frac{1}{2}\sigma^2\zeta-\ln\frac{S_0}{K}}{2\sigma^{\frac{3}{2}}}-\frac{-\frac{1}{2}\sigma^2\zeta-\ln\frac{S_0}{K}}{2\sigma\zeta^{\frac{3}{2}}}\right] \\
&= S_0N'(d_3)\frac{\sigma}{2\sqrt{\zeta}}>0
\end{aligned}$$

∎

APPENDIX: MATHEMATICS NOTATIONS

APO: American put option
$PAPO$: Perpetual American put option
S_0: Current stock price (Stock price at time 0)

K: Strike price of option, or exercise price of option
T: Time to expiration of option
t: A future point in time
S_T: Stock price at maturity (Stock price at time T)
r: Continuously compounded risk-free interest rate for an investment maturing in time T
σ: Volatility of the stock price
$N(x)$: Cumulative probability a variable with a standardized normal distribution is less than x. A standardized normal distribution is a normal distribution has a mean of zero and standard deviation of 1.0.
$P_E(S_0, K, r, T, \sigma)$: Value of an European put option, with current stock price S_0, strike price K, risk-free interest rate r, expiration time T, volatility σ
$C_E(S_0, K, r, T, \sigma)$: Value of an European call option, with current stock price S_0, strike price K, risk-free interest rate r, expiration time T, volatility σ
$P_A(S_0, K, r, T, \sigma)$: Value of an American put option, with current stock price S_0, strike price K, risk-free interest rate r, expiration time T, volatility σ
$C_A(S_0, K, r, T, \sigma)$: Value of an American call option, with current stock price S_0, strike price K, risk-free interest rate r, expiration time T, volatility σ
$P_{PA}(S_0, K, r, \infty, \sigma)$: Value of a perpetual American put option, with current stock price S_0, strike price K, risk-free interest rate r, expiration time ∞, volatility σ

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