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Ambiguity, Risk and Portfolio Choice under Incomplete Information

Jianjun Miao*

Department of Economics, Boston University, 270 Bay State Road, Boston, MA 02215, and Zhongnan University of Economics and Law
E-mail: miaoj@bu.edu

This paper studies optimal consumption and portfolio choice in a Merton-style model with incomplete information when there is a distinction between ambiguity and risk. The latter distinction is afforded by adoption of recursive multiple-priors utility. The fundamental issues are: (i) How does the agent optimally estimate the unobservable processes as new information arrives over time? (ii) What are the effects of ambiguity and incomplete information on behavior? This paper shows that it is optimal to first use any prior to perform Bayesian estimation and then to maximize expected utility with that prior based on the resulting estimates. Finally, the paper shows that a hedging demand arises that is affected by both ambiguity and estimation risk.

Key Words: Ambiguity; Recursive multiple-priors utility; Incomplete information; Portfolio choice; Hedging; Estimation risk.
JEL Classification Numbers: D81, G11.

1. INTRODUCTION

In economic analysis, it is typically assumed that a decision maker's beliefs are represented by a single probability measure. Frank Knight (1921) emphasizes the distinction between *risk* where there are probabilities to guide choice, and *ambiguity* where likelihoods of events are too imprecise to be adequately summarized by probabilities. The Ellsberg Paradox (1961) tells us that this distinction is behaviorally significant. This suggests that there are two dimensions of the decision maker's beliefs about the likeli-

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hoods of events: risk and ambiguity. In standard models, ambiguity is neglected or it is assumed that the decision maker is indifferent to it.

Added motivation for the analysis to follow comes from the finance literature on incomplete information. In the real world, investors make consumption and investment decisions based on information available from sources such as newspapers, financial reports, and market data. It is unrealistic to assume that they observe the driving uncertainty processes underlying prices and returns. These unobservable processes (or parameters) must be learned as new information arrives over time.

In the standard Bayesian analysis, the decision maker has a unique prior over the unobservable processes. The prior then is updated by Bayes' Rule as new information arrives. Moreover, estimates of these processes are obtained by Bayesian estimation. This Bayesian approach emphasizes the effect of estimation risk on optimal behavior.

To incorporate ambiguity, this paper asks the following question: How does a decision maker choose when he is averse to ambiguity and when his information is incomplete?

The first step in addressing this question is to formulate a utility function that permits the distinction between risk and ambiguity under incomplete information. Chen and Epstein (2002) provide such a distinction under complete information by generalizing Gilboa and Schmeidler's (1989) static model to a dynamic setting; they call their model *recursive multiple-priors utility.*[1] This paper adapts their formulation to an environment with incomplete information. The resulting model of utility is applied to study a single agent's consumption and investment decisions in a continuous-time Merton-style model with incomplete information.

The issues then become: (i) How does the agent optimally estimate the unobservable uncertainty processes underlying asset prices as information arrives over time. (ii) What are the effects of ambiguity and incompleteness of information on behavior?

If one views the agent's planning problem as a control problem, for the standard expected utility model, there is a well-known separation principle in the control literature (e.g., Flemming and Rishel (1975)). This principle states that control under incomplete information can be solved separately by the two independent problems of filtering (or estimation) and control under complete information. It is natural to conjecture that this principle is also true for recursive multiple-priors utility.

Because there is a unique prior under expected utility or risk-based utility such as stochastic differential utility proposed by Duffie and Epstein (1992), estimation is not a problem because standard Bayesian analysis applies.

[1]Epstein and Schneider (2003) develop an aximomatic foundation for recursive muptiple-priors in a discrete-time framework with complete information.

However, when the agent has multiple priors, it is not clear *a priori* how to perform estimation: Can one perform Bayesian estimation using one of the priors in the set? If so, which ones are suitable?

This paper shows that the separation principle still holds for recursive multiple-priors utility. In particular, optimality is consistent with the use of *any* measure in the set of priors to perform Bayesian estimation.

In an incomplete information environment, the key to the above results is that (i) the set of priors is updated by applying Bayes' Rule to each prior in the set, and this leads to dynamic consistency; (ii) all measures in the set of priors and their restrictions on the observation filtration are mutually absolutely continuous. Thus given that the 'true' probability measure is one of the priors, one can obtain Bayesian estimates of unobservable processes using any measure in the set of priors and equivalently rewrite the agent's budget constraint in terms of these estimates under the corresponding measure. Accordingly, the agent's optimization problem is transformed into an environment with complete information and the preceding estimation procedure is optimal.

With regard to the characterization of optimal consumption and portfolio choice, I find that consistent with the separation principle, a two-step procedure consisting of ordinary filtering and ordinary martingale methods can be used to solve the agent's problem.

Finally, I provide examples with logarithmic and power felicity functions that deliver closed form solutions. I show that under complete information there is no hedging demand even when ambiguity is present. The effect of ambiguity is that the agent myopically holds a mean-variance efficient portfolio but with distorted mean values of asset returns. In contrast, there is a hedging demand under incomplete information. This demand is affected by both ambiguity and estimation risk.

1.1. Related Literature

Chen and Epstein (2002) formulate recursive multiple-priors utility in continuous time. They also apply this utility to a Lucas-style representative agent model to study asset pricing implications. Epstein and Miao (2000) apply recursive multiple-priors utility to study a heterogeneous agent model to address the consumption home bias and equity home bias puzzles. Both of these papers assume complete information.

There is a large literature studying consumption and portfolio choice with incomplete information in the expected utility framework (see Bawa, Brown and Klein (1979), Detemple (1986), Gennotte (1986), Karatzas and Xue (1991), Feldman (1992), Lakner (1995), Brennan (1998), Lakner (1998), Barberis (2000), Karatzas and Zhao (1998), and Xia (2000) and the references cited therein). Cvitanic *et al.* (2000) study the corresponding problem for stochastic differential utility. This paper adds to this literature using recursive multiple-priors utility.

My model is related to a series of papers by Hansen and Sargent and their coauthors (see Anderson *et al.* (2003), Cagetti *et al.* (2002), Hansen *et al.* (2006) and Hansen and Sargent (2001)). These papers study models of robust control where the decision maker fears model uncertainty and seeks robust decision-making, which are also motivated in part by the Ellsberg Paradox.[2]

1.2. Outline

The paper proceeds as follows. Section 2 defines recursive multiple-priors utility under incomplete information. Section 3 applies this utility to study optimal consumption and portfolio choice in a Merton-style model with incomplete information. Section 4 provides examples that deliver explicit solutions. Proofs are relegated to an appendix.

2. RECURSIVE MULTIPLE-PRIORS UTILITY

This section adapts Chen and Epstein (2001) and defines recursive multiple-priors utility under incomplete information that also conforms with the axiomatization in Epstein and Schneider (2003).

2.1. Information Structure

Time is continuous in the finite horizon $[0,T]$. There is a complete filtered probability space $(\Omega, \mathcal{F}_T, \{\mathcal{F}_t\}_{t=0}^T, P)$ on which a d'-dimensional standard Brownian motion on $\mathbb{R}^{d'}$ $W = (W^1, \ldots, W^{d'})^\intercal$ is defined.[3] The filtration $\{\mathcal{F}_t\}_{t=0}^T$ or simply $\{\mathcal{F}_t\}$ represents complete information. The probability measure P is a reference measure.

The decision maker's available information is represented by a sub-filtration $\{\mathcal{G}_t\}$ where each $\mathcal{G}_t \subset \mathcal{F}_t$. Assume that $\{\mathcal{G}_t\}$ is generated by some $\mathbb{R}^d$-valued observable diffusion process (y_t). The following assumption is crucial and common in the literature on incomplete information.

Assumption 1. There is a d-dimensional standard Brownian motion $\widehat{W}$ defined on the filtered probability space $(\Omega, \mathcal{G}_T, \{\mathcal{G}_t\}, P)$ such that the augmented natural filtration generated by $\widehat{W}$ is identical to $\{\mathcal{G}_t\}$.

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[2]See Hansen *et al.* (2006) and Hansen and Sargent (2001) for surveys of the robust control model and Epstein and Schneider (2001) for detailed comparison with the recursive multiple-priors model.

[3]All processes to appear in the sequel are progressively measurable and all equalities and inequalities involving random variables (processes) are understood to hold dP *a.s.* ($dt \otimes dP$ *a.s.*). Denote by $E_Q[\cdot]$ and $E_Q[\cdot|\cdot]$ the expectation and conditional expectation taken with respect to the measure Q. When Q is suppressed it is understood that $Q = P$. Finally, denote by $|\cdot|$ the Euclidean norm.

Because the Brownian motion W is unobservable, I use the observable Brownian motion $\widehat{W}$ to define utility under incomplete information in the sequel.

The Brownian motion $\widehat{W}$ is often referred to as an *innovation process.* It can be extracted from the decision maker's observation process (y_t) by filtering theory.[4] For example, suppose that $d' = 2$, $d = 1$ and that the decision maker observes (y_t) but not $(W_t^1, W_t^2)^\intercal$ and (x_t) where

$$dx_t = x_t dt + dW_t^1 \text{ and } dy_t = x_t dt + \sigma^y dW_t^2.$$

Assume σ^Y is a nonzero constant. Then $\widehat{W}$ is delivered by

$$d\widehat{W}_t = (\sigma^y)^{-1}(dy_t - E\left[x_t \mid \mathcal{G}_t\right] dt).$$

Note that d' might not be equal to d because the decision maker may observe an arbitrary dimensional process (y_t). However, I assume $d = d'$ in the later applications.

2.2. Consumption Space

There is a single perishable consumption good. A *consumption process* c is nonnegative, real-valued, progressively measurable with respect to the filtration $\{\mathcal{G}_t\}$ and square integrable (i.e. $E\left[\int_0^T c_t^2 dt\right] < \infty$). Denote by $\mathcal{C}$ the set of all consumption processes.

2.3. Utility

A *recursive multiple-priors utility process* $(V_t(c))$ for each $c \in \mathcal{C}$ is defined by five primitives: information structure $((\Omega, \mathcal{G}_T, \{\mathcal{G}_t\}, P))$, the Brownian motion $\widehat{W}$, the set of priors (probability measures) $\mathcal{P}$ on $(\Omega, \mathcal{G}_T)$, the discount rate $\beta > 0$, and the felicity function $u : \mathbb{R}_+ \to \mathbb{R}$.

The construction of the set of priors $\mathcal{P}$ is key.[5] Take all measures in $\mathcal{P}$ to be equivalent to P. They can be defined via their densities by use of density generators and Girsanov's Theorem. Specifically, define a *density generator* $\theta = (\theta_t)$ as an $\mathbb{R}^d$-valued $\{\mathcal{G}_t\}$-adapted process satisfying $\sup_t |\theta_i(t)| \leq \kappa_i$, $i = 1, ..., d$, where $\kappa = (\kappa_1, ..., \kappa_d)^\intercal \geq 0$. Denote by Θ the set of all such density generators. This specification of Θ is referred to as *κ-ignorance* in Chen and Epstein (2002).[6]

[4] See Liptser and Shiryayev (1977) for an introduction to filtering theory.

[5] Note that the set of priors is delivered as part of the utility representation from behavior (see Epstein and Schneider (2003)). In applications, one must specify this set so that it is consistent with behavior, e.g., some axiomatic foundation.

[6] See Chen and Epstein (2002) for more general specifications of Θ.

Then each density generator θ generates a $(P, \{\mathcal{G}_t\})$-martingale (z_t^θ) :

$$z_t^\theta = \exp\left\{-\frac{1}{2}\int_0^t |\theta_s|^2 ds - \int_0^t \theta_s \cdot d\widehat{W}_s\right\}, \quad 0 \le t \le T, \tag{1}$$

which determines a probability measure Q^θ on $(\Omega, \mathcal{G}_T)$ via

$$\frac{dQ^\theta}{dP} = z_T^\theta, \ , \text{and} \ \left.\frac{dQ^\theta}{dP}\right|_{\mathcal{G}_t} = z_t^\theta. \tag{2}$$

The set of priors is defined by

$$\mathcal{P} = \{Q^\theta : \theta \in \Theta \text{ and } Q^\theta \text{ is given by (2)}\}. \tag{3}$$

Because $\mathcal{P}$ expands as κ increases, one can interpret κ as an ambiguity aversion parameter.

Finally, define the *recursive multiple-priors utility process* $(V_t(c))$ for each $c \in \mathcal{C}$ as:

$$V_t(c) = \min_{Q\in\mathcal{P}} E_Q\left[\int_t^T e^{-\beta(s-t)}\, u(c_s)\, ds \middle| \mathcal{G}_t\right], \ 0 \le t \le T. \tag{4}$$

Abbreviate $V_0(\cdot)$ by $V(\cdot)$ and refer to it as *recursive multiple-priors utility.* The recursive multiple-priors utility model under complete information studied in Chen and Epstein (2002) corresponds to the case where $\{\mathcal{G}_t\} = \{\mathcal{F}_t\}$ and $\widehat{W} = W$. Finally, the standard expected utility model is obtained when $\kappa = 0$ in which case $\mathcal{P} = \{P\}$.

With regard to the properties of utility, first the utility process $(V_t(c))$ is dynamically consistent because the following recursive relation holds:

$$V_t = \min_{Q\in\mathcal{P}} E_Q\left[\int_t^\tau e^{-\beta(s-t)}\, u(c_s)\, ds + e^{-\beta(\tau-t)} V_\tau \middle| \mathcal{G}_t\right], \ 0 \le t < \tau \le T.$$

This property follows from the fact that the utility process $(V_t(c))$ is the unique solution to the following backward stochastic differential equation (BSDE),[7]

$$dV_t = [-u(c_t) + \beta V_t + \max_{\theta\in\Theta} \theta_t \cdot \sigma_t^V\,]\, dt + \sigma_t^V \cdot d\widehat{W}_t, \ V_T = 0. \tag{5}$$

[7]Sufficient conditions are that u be Borel measurable and that it satisfy a growth condition ensuring $E\left[\int_0^T u^2(c_t)\, dt\right] < \infty$ for all c in $\mathcal{C}$. See El Karoui *et al.* (1997) for an excellent survey of the theory and applications of the backward stochastic differential equations.

Note that the volatility (σ_t^V) at c, denoted more fully by $(\sigma_t^V(c))$, is determined as part of the solution to the BSDE; it plays a key role in the sequel.[8]

Because[9]

$$\max_{\theta \in \Theta} \theta_t \cdot \sigma_t^V = \theta_t^* \cdot \sigma_t^V, \text{ for } \theta_t^* = \kappa \otimes \operatorname{sgn}(\sigma_t^V(c)), \tag{6}$$

BSDE (5) can be written as

$$dV_t = [-u(c_t) + \beta V_t + \theta_t^* \cdot \sigma_t^V]\, dt + \sigma_t^V \cdot d\widehat{W}_t, \; V_T = 0. \tag{7}$$

Note that the measure delivered by the density generator θ^* achieves the minimum in (4).

Finally, assume that $u' > 0$, $u'' < 0$. Then by Chen and Epstein (2002), each $V_t(\cdot)$ is continuous, increasing and strictly concave. Also assume that the following Inada condition holds: $\lim_{x \to 0^+} u'(x) = \infty$ and $\lim_{x \to \infty} u'(x) = 0$.

3. OPTIMAL CONSUMPTION AND PORTFOLIO CHOICE

This section applies recursive multiple-priors utility to study the optimal consumption and portfolio choice problem with incomplete information.

3.1. The Environment

Financial markets. Uncertainty is represented by a complete filtered probability space $(\Omega, \mathcal{F}_T, \{\mathcal{F}_t\}_{t=0}^T, P)$ on which is defined a d-dimensional standard Brownian motion $W = (W^1, ..., W^d)^\intercal$. There are $d+1$ securities consisting of one riskless bond and d non-dividend-paying stocks. The price of the riskless bond is given by

$$S_t^0 = e^{rt}, \; t \in [0, T],$$

where the riskless rate r is a positive constant. Denote by S_t^i the price of the i^{th} stock and by $R_t^i = dS_t^i / S_t^i$ its return, $i = 1, ..., d$. Assume that the initial price S_0^i is a given positive constant and that the vector of returns $R_t = (R_t^1, ..., R_t^d)^\intercal$ satisfies

$$dR_t = \mu^R\, dt + \sigma^R\, dW_t, \tag{8}$$

[8] Both (V_t) and (σ_t^V) are progressively measurable with respect to $\{\mathcal{G}_t\}$ and sequare integrable.

[9] For any d-dimensional vector x, $\operatorname{sgn}(x)$ is the d-dimensional vector with i^{th} component equal to $\operatorname{sgn}(x_i) = | x_i | / x_i$ if $x_i \neq 0$ and $= 0$ if $x_i = 0$. For any $y \in \mathbb{R}^d$, $y \otimes \operatorname{sgn}(x)$ denotes the vector in $\mathbb{R}^d$ with i^{th} component $y_i \operatorname{sgn}(x_i)$.

where the volatility σ^R is a $d \times d$ matrix of real-valued constants. On the other hand, the vector of mean returns $\mu^R = (\mu_1^R, ..., \mu_d^R)^\intercal : \Omega \to \mathbb{R}^d$ is an $\mathcal{F}_0$-measurable random variable with distribution $\nu(A) = P(\mu^R \in A)$ for any Borel set A in $\mathbb{R}^d$ that satisfies:

$$\int_{\mathbb{R}^d} |b| \, \nu(db) < \infty.$$

Thus μ^R is independent of W. In the standard Bayesian analysis, ν is the prior distribution of μ^R.

Assume that the volatility matrix σ^R satisfies the following assumption which ensures that financial markets are complete (e.g., Duffie (1996)).

Assumption 2. σ^R is invertible.

Define the *market price of uncertainty* process (η_t) by[10]

$$\eta_t = (\sigma^R)^{-1}(\mu^R - r\mathbf{1}), \, 0 \le t \le T, \tag{9}$$

where $\mathbf{1}$ is the vector in $\mathbb{R}^d$ with each component equal to 1. Then the following Lemma holds (see Lakner (1995)).

Lemma 1. *The process Z defined by*

$$Z_t = \exp\left\{ -\int_0^t \eta_s \cdot dW_s - \frac{1}{2}\int_0^t |\eta_s|^2 \, ds \right\}$$

is a $(P, \{\mathcal{F}_t\})$-martingale.

Information structure. Assume that the bond price S_t^0 and stock prices S_t are given exogenously. Denote by $\{\mathcal{F}_t^S\}$ the augmented filtration generated by the price processes. Complete information is represented by $\{\mathcal{F}_t\}$, the augmented filtration generated by μ^R and W. However, the agent does not observe the Brownian motion W and the mean returns μ^R. Rather, his information is represented by the filtration $\{\mathcal{F}_t^S\}$ where each $\mathcal{F}_t^S \subset \mathcal{F}_t$. Thus we are in the set-up of section 2.1 with $\{\mathcal{G}_t\} = \{\mathcal{F}_t^S\}$.

Budget constraint. There is a single consumption good taken as the numeraire. Consumption processes lie in the consumption space $\mathcal{C}$ defined in section 2.2. Denote the *wealth process* by (X_t). A *portfolio (share)* ψ is an $\mathbb{R}^d$-valued $\{\mathcal{F}_t^S\}$-adapted progressively measurable process such that

[10] Following Chen and Epstein (2002) and Epstein and Miao (2003), the deviation from the usual terminology of *market price of risk* is to emphasize that uncertainty includes both risk and ambiguity in the model.

$\int_0^T |\psi_s|^2 ds < \infty$. The component $\psi_i(t)$ represents the proportion of wealth invested in the i th stocks at time t. Thus $1 - \psi_t \cdot \mathbf{1}$ is the proportion invested in the bond. Denote the set of all portfolios by Ψ. Endowed with initial wealth $X_0 > 0$, the agent makes consumption and investment decisions based on information represented by $\{\mathcal{F}_t^S\}$. His budget constraint is given by

$$dX_t = \left\{ \left[r + (\psi_t)^\intercal (\mu^R - r\mathbf{1}) \right] X_t \, - \, c_t \right\} \, dt + X_t (\psi_t)^\intercal \, \sigma^R \, dW_t. \qquad (10)$$

Preferences. The above environment is standard. The departure from the standard model is that preferences are represented by the recursive multiple-priors utility function V corresponding to the set of priors defined in (3).

In order to ensure that V is well defined, introduce the process $(\widehat{W}_t)$:

$$\widehat{W}_t = \int_0^t (\sigma^R)^{-1} [dR_\tau - \widehat{\mu}_\tau^R d\tau], \qquad (11)$$

where $\widehat{\mu}^R(t) \equiv E\left[\mu^R \mid \mathcal{F}_t^S\right]$ is a measurable version of the conditional expectation of μ^R with respect to the price filtration $\{\mathcal{F}_t^S\}$.

The following lemma implies that $\widehat{W}$ defined in (11) satisfies Assumption 1. Thus recursive multiple-priors utility V is well defined. The proof of this lemma is standard (see, e.g., Liptser and Shiryayev (1977)).

Lemma 2. *$\widehat{W}$ is a $(P, \{\mathcal{F}_t^S\})$-Brownian motion. Moreover, the augmented filtration generated by the Brownian motion $\widehat{W}$ coincides with $\{\mathcal{F}_t^S\}$.*

3.2. The Decision Problem and Separation Principle

Decision problem. The agent makes consumption and investment plans for the entire horizon at time zero by solving:

$$\sup_{(c,\psi) \in \mathcal{C} \times \Psi} V(c) \qquad (12)$$

subject to (10) and

$$X_t \geq 0, \ \ t \in [0, T], \ X_0 > 0 \text{ given.} \qquad (13)$$

The credit constraint (13) rules out doubling strategies (e.g., Dybvig and Huang (1988)). Note that the consumption and portfolio processes are required to be adapted to the price filtration $\{\mathcal{F}_t^S\}$. Finally, because the utility process is dynamically consistent, the optimal plan will be carried out as time proceeds.

Separation principle. I solve this problem by the separation principle. In order to understand how this principle works for recursive multiple-priors utility, consider first the standard setting where $\kappa = 0$ and V is an expected utility function. In this case, the agent's unique prior is represented by P.

By (8) and (9), $(\widehat{W}_t)$ defined in (11) satisfies

$$\widehat{W}_t = W_t + \int_0^t (\eta_s - \widehat{\eta}_s) ds, \tag{14}$$

where $\widehat{\eta}_t \equiv E\left[\eta_t \mid \mathcal{F}_t^S\right]$ is a measurable version of the conditional expectation of η_t with respect to $\{\mathcal{F}_t^S\}$. Denote $\widehat{\mu}^R(t) = (\widehat{\mu}_1^R(t), ..., \widehat{\mu}_d^R(t))^\intercal$. Then

$$\widehat{\eta}_t = (\sigma^R)^{-1}(\widehat{\mu}_t^R - r\mathbf{1}). \tag{15}$$

By (8), (11) and (15), under prior P the agent's perceived returns dynamics is

$$dR_t = \widehat{\mu}_t^R dt + \sigma^R d\widehat{W}_t \tag{16}$$

and the budget constraint (10) becomes

$$dX_t = (rX_t - c_t)dt + X_t(\psi_t)^\intercal \sigma^R [d\widehat{W}_t + \widehat{\eta}_t dt]. \tag{17}$$

Because $\widehat{W}$ and $\widehat{\eta}$ are adapted to $\{\mathcal{F}_t^S\}$, all processes in (17) are adapted to $\{\mathcal{F}_t^S\}$. Thus the agent's problem has been transformed into one with complete information and filtration $\{\mathcal{F}_t^S\}$ where the Bayesian estimate $\widehat{\eta}_t$ ($\widehat{\mu}_t^R$) is treated as the 'true' market price of uncertainty (mean returns). After using standard filtering theory (see Liptser and Shiryayev (1977)) to determine the conditional distribution of η (or μ^R), the usual optimization tools under complete information can be applied.

What happens when the agent has a set of priors? Note that the above transformation (16) is performed using the single prior P. When the agent has a set of priors $\mathcal{P}$, this transformation can take many forms depending on which prior in the set $\mathcal{P}$ is used. Formally, consider any $Q \in \mathcal{P}$ and denote by θ the corresponding density generator. By Girsanov's Theorem, the process $\widehat{W}^Q$ defined by

$$d\widehat{W}_t^Q = d\widehat{W}_t + \theta_t dt$$

is a Q-Brownian motion and the natural filtration generated by $\widehat{W}^Q$ coincides with $\{\mathcal{F}_t^S\}$. Then the budget constraint can be written as

$$dX_t = (rX_t - c_t)dt + X_t(\psi_t)^\intercal \sigma^R \left[d\widehat{W}_t^Q + (\widehat{\eta}_t - \theta_t)dt\right]. \tag{18}$$

Because $(\widehat{\eta}-\theta)$ and $\widehat{W}^Q$ are $\{\mathcal{F}_t^S\}$-adapted, when the agent treats $(\widehat{\eta}_t-\theta_t)$ as the observable estimate of the market price of uncertainty using measure Q, the problem is transformed into the complete information world. Consequently, the usual optimization tools under complete information can be applied.

In sum, because all measures in the set of priors are equivalent all corresponding transformed budget constraints are equivalent to the original one (10). Hence using any measure in the set of priors to perform estimation leads to the same optimum.

3.3. Two-step Procedure

Consistent with the separation principle, I first use the reference measure P to perform estimation and transform the budget constraint (10) into (17) as in the preceding subsection. Then the problem is reformulated as a static Arrow-Debreu problem. Finally, from this problem, I derive optimal consumption and portfolio choice (e.g., Duffie and Skiadas (1994)).

Filtering. By Lemma 1 and Girsanov's Theorem, one can define a probability measure $\widetilde{P}$ equivalent to P on $(\Omega, \mathcal{F}_T)$ via $d\widetilde{P}/dP = Z_T$ such that the d-dimensional process $\widetilde{W}$ defined by

$$\widetilde{W}_t = W_t + \int_0^t \eta_s \, ds \tag{19}$$

is a $(\widetilde{P}, \{\mathcal{F}_t\})$-Brownian motion. Then, by (8) and (9),

$$dR_t = r dt + \sigma_t^R \, d\widetilde{W}_t.$$

Thus, $\widetilde{P}$ is an *equivalent martingale measure* because the vector of 'discounted' prices, $(e^{-rt}S_t)$, is a $\widetilde{P}$-martingale.

The following facts are important for the characterization of optima. By Lakner (1998), the $(P, \{\mathcal{F}_t^S\})$-martingale $(\widehat{Z}_t)$ defined by

$$\widehat{Z}_t \equiv E\left[Z_t \mid \mathcal{F}_t^S\right], \, 0 \le t \le T,$$

is an indistinguishable version of the process

$$\exp\left\{-\int_0^t \widehat{\eta}_s \cdot d\widehat{W}_s - \frac{1}{2}\int_0^t |\widehat{\eta}_s|^2 ds\right\}, \, 0 \le t \le T. \tag{20}$$

Therefore, by (14) and Girsanov's Theorem, the process $\widetilde{W}$ defined by (19) satisfies

$$\widetilde{W}_t = W_t + \int_0^t \eta_s ds = \widehat{W}_t + \int_0^t \widehat{\eta}_s ds \tag{21}$$

and it is a $(\widetilde{P}, \{\mathcal{F}_t^S\})$-Brownian motion. Moreover, the augmented natural filtration of $\widetilde{W}$ coincides with the price filtration $\{\mathcal{F}_t^S\}$ (see Lakner (1995) Proposition 4.1). Note that $\widetilde{P}$ is also a probability measure on $(\Omega, \mathcal{F}_T^S)$ defined by $d\widetilde{P}/dP = \widehat{Z}_T$.

Static Arrow-Debreu problem. The existence of an equivalent martingale measure $\widetilde{P}$ and the credit constraint (13) rule out arbitrage opportunities (see Duffie (1996)). Because there is no arbitrage and markets are complete, a unique *state price density process* (p_t) relative to measure P is delivered by

$$p_t = e^{-rt}\widehat{Z}_t.$$

The following theorem is standard (see Karatzas and Xue (1991) or Lakner (1995)).

THEOREM 1. *(i) For any consumption process $c \in \mathcal{C}$, there exist a portfolio process ψ and a wealth process X such that (c, ψ, X) satisfies the dynamic budget constraint (17) and the credit constraint (13) if and only if*

$$E\left[\int_0^T p_t c_t dt\right] \leq X_0. \tag{22}$$

(ii) If the above inequality holds with equality, the portfolio process ψ is unique up to equivalence and given by

$$\psi_t = e^{rt}((\sigma^R)^\intercal)^{-1}\phi_t/X_t, \tag{23}$$

where

$$e^{-rt}X_t = E_{\widetilde{P}}\left[\int_0^T e^{-rt}c_t\, dt \middle| \mathcal{F}_t^S\right] = X_0 + \int_0^t \phi_s \cdot d\widetilde{W}_s.$$

The corresponding wealth process is given by

$$X_t = \frac{1}{p_t}E\left[\int_t^T p_s c_s ds \middle| \mathcal{F}_t^S\right]. \tag{24}$$

Thus the consumption process c^* can be found by solving the static Arrow-Debreu problem:

$$\sup_{c\in\mathcal{C}} V(c) \quad \text{subject to } E\left[\int_0^T p_t c_t dt\right] \leq X_0. \tag{25}$$

The optimal portfolio process ψ^* is then delivered by (23).

Utility supergradient. In order to solve problem (25), it is useful to find the supergradients for V. A *supergradient* for V at the consumption process $c \in \mathcal{C}$ is a process (π_t) satisfying

$$V(c') - V(c) \le E\left[\int_0^T \pi_t\,(c'_t - c_t)\,dt\right],$$

for all c' in $\mathcal{C}$. By Chen and Epstein (2002), for each θ^* satisfying (6), the process

$$\pi_t(c) = e^{-\beta t}\,u'(c_t)\,z_t^{\theta^*},\ 0 \le t \le T, \tag{26}$$

is a supergradient for V at c.

Optimal plan. Denote by J the value function of problem (25). Assume that $J(X_0) < \infty$. Then it is easy to show that the value function for problem (12) is also finite and equal to $J(X_0)$.

The following theorem characterizes an optimum for problem (12).

THEOREM 2. *(i) The optimal consumption process c^* is given by*

$$e^{-\beta t} z_t^{\theta^*} u'(c_t^*) = \lambda p_t, \tag{27}$$

where $\lambda > 0$ is such that

$$E\left[\int_0^T p_t c_t^* dt\right] = X_0, \tag{28}$$

and (θ_t^*) satisfies

$$\theta_t^* = \kappa \otimes sgn(\sigma_t^V(c^*)). \tag{29}$$

Here $(V_t(c^*), \sigma_t^V(c^*))$ is the unique solution to BSDE (7) for $c = c^*$.

(ii) The optimal wealth process X^* is given by (24) where $c = c^*$.

(iii) The optimal portfolio ψ^* is given by

$$\psi_t^* = e^{rt}((\sigma^R)^\intercal)^{-1}\phi_t X_t^*,$$

where (ϕ_t) satisfies

$$e^{-rt}X_t^* + \int_0^t e^{-rt}c_t^* ds = E_{\widetilde{P}}\left[\int_0^T e^{-rt}c_t^*\,dt \,\middle|\, \mathcal{F}_t^S\right] = X_0 + \int_0^t \phi_s \cdot d\widetilde{W}_s. \tag{30}$$

As is well known, the optimal portfolio is related to the integrand of the martingale representation in (30). Section 4 will give two examples to clarify the nature of the optimal portfolio.

If an optimum exists it must be unique. This is because strict concavity of $V(\cdot)$ implies the optimal consumption process is unique. Then by (27) the optimal density generator is also unique.

In sum, the following two-step procedure can be used to solve optimal consumption and portfolio choice described above.

• Step 1. (Ordinary filtering) First, use the standard filtering technique (e.g., Karatzas and Zhao (1998)) to solve for the conditional distribution of μ^R and the conditional expectation $\widehat{\mu}_t^R = E[\mu^R|\mathcal{F}_t^S]$ for each t. Next use (15) and (20) to solve for $\widehat{Z}$. Finally, let $p_t = e^{-rt}\widehat{Z}_t$.

• Step 2. (Ordinary martingale method) Given any $\theta \in \Theta$, solve the following system of two equations:

$$e^{-\beta t} z_t^\theta u'(c_t) = \lambda p_t,$$

$$E\left[\int_0^T p_t c_t dt\right] = X_0,$$

for c and λ to yield $c = g(\theta)$ where g maps Θ into $\mathcal{C}$. Second, solve BSDE (7) for the volatility of $(V_t(c))$ when $c = g(\theta)$ to obtain $\sigma(g(\theta))$. The optimal density generator θ^* is given by the following fixed point problem:

$$\theta_t = \kappa \otimes \operatorname{sgn}\left(\sigma_t^V(g(\theta))\right), \ 0 \le t \le T. \tag{31}$$

Finally, if there exists a solution θ^* to (31), the optimal consumption process c^* and portfolio ψ^* are given by Theorem 2.

3.4. Hedging Motives

In order to understand the effects of ambiguity on optimal choice, it is useful to consider first a limited observational equivalence pointed out in Chen and Epstein (2002) and Epstein and Miao (2003). Notice that equation (27) is identical to that for an expected utility maximizer who uses the single prior Q^* corresponding to the density generator θ^* :

$$dQ^*/dP = \exp\left\{-\frac{1}{2}\int_0^T |\theta_s^*|^2\, ds - \int_0^T \theta_s^* \cdot d\widehat{W}_s\right\}. \tag{32}$$

Thus the optimum characterized in Theorem 2 can be generated in a standard model without ambiguity where the agent uses a distorted belief Q^*.

Because Q^* is endogenously delivered by ambiguity, following Epstein and Miao (2003), it is natural to refer to Q^* as *ambiguity adjusted probability beliefs.*

By (16) and Girsanov's Theorem, under Q^* the agent's perceived returns dynamics is

$$dR_t = (\widehat{\mu}_t^R - \sigma^R\, \theta_t^*)\, dt + \sigma^R\, d\widehat{W}_t^*, \tag{33}$$

where the $(Q^*, \{\mathcal{F}_t^S\})$-Brownian motion $(\widehat{W}_t^*)$ is defined by

$$d\widehat{W}_t^* = d\widehat{W}_t + \theta_t^* dt. \tag{34}$$

From (33), there are two factors influencing the deviations of the agent's perceived mean returns from their true values:

$$\mu^R - (\widehat{\mu}_t^R - \sigma^R\, \theta_t^*) = (\mu^R - E[\mu^R \mid \mathcal{F}_t^S]) + \sigma^R \theta_t^*.$$

The first term represents *estimation risk* and the second term reflects ambiguity. Because these terms are time-varying, investment opportunities change over time and two separate hedging motives arise.

4. EXAMPLES

Consider the power felicity function:

$$u(x) = x^\gamma/\gamma,\ x \in \mathbb{R}_+,\ 0 \neq \gamma < 1,$$

where $1-\gamma$ is the coefficient of relative risk aversion.

The following theorem characterizes an optimum.

THEOREM 3. *(i) The optimal consumption process is given by*

$$c_t^* = \left(\frac{e^{-\beta t} z_t^{\theta^*}}{\lambda p_t} \right)^{\frac{1}{1-\gamma}}, \tag{35}$$

where

$$\theta_t^* = \kappa \otimes \left(\sigma_t^H/\gamma + \frac{1}{1-\gamma} (\widehat{\eta}_t - \theta_t^*) \right), \tag{36}$$

$$\lambda = \left(E\left[\int_0^T (p_t)^{\frac{-\gamma}{1-\gamma}} (e^{-\beta t} z_t^{\theta^*})^{\frac{1}{1-\gamma}}\ dt \right] / X_0 \right)^{1-\gamma},$$

and (H_t, σ_t^H) *is given below. The dynamics of* c^* *is given by*

$$dc_t^*/c_t^* = \mu_t^c dt + \sigma_t^c \cdot d\widehat{W}_t,$$

where (μ_t^c) *and* (σ_t^c) *satisfy*

$$\sigma_t^c = \frac{1}{1-\gamma}(\widehat{\eta}_t - \theta_t^*) \text{ and } \tag{37}$$

$$\mu_t^c = \frac{1}{1-\gamma}(r-\beta) + \frac{1}{2}(2-\gamma)\sigma_t^c \cdot \sigma_t^c + \sigma_t^c \cdot \theta_t^*. \tag{38}$$

(ii) The utility process at c^* *is given by*

$$V_t = \frac{(c_t^*)^\gamma}{\gamma} H_t, \tag{39}$$

where (H_t, σ_t^H) *is the unique solution to the BSDE:*

$$dH_t/H_t = \mu_t^H dt + \sigma_t^H \cdot d\widehat{W}_t,\ H_T = 0, \tag{40}$$

where

$$\mu_t^H = \frac{\gamma}{1-\gamma}\left[\beta/\gamma - r - \frac{(\widehat{\eta}_t - \theta_t^*)\cdot(\widehat{\eta}_t - \theta_t^*)}{2(1-\gamma)}\right] - H_t^{-1} + (\theta_t^* - \alpha\sigma_t^c)\cdot\sigma_t^H. \tag{41}$$

(iii) The optimal wealth process is given by

$$X_t^* = \frac{1}{p_t} E\left[\int_t^T p_s c_s^* ds \middle| \mathcal{F}_t^S\right] = c_t^* H_t. \tag{42}$$

(iv) The optimal portfolio is given by

$$\psi_t^* = \frac{1}{1-\gamma}(\sigma^R(\sigma^R)^\intercal)^{-1}(\widehat{\mu}_t^R - r\mathbf{1}) - \frac{1}{1-\gamma}((\sigma^R)^\intercal)^{-1}\theta_t^* + ((\sigma^R)^\intercal)^{-1}\sigma_t^H. \tag{43}$$

I focus discussions on the optimal portfolio as the behavior of optimal consumption can be deduced from (35), (37) and (38).

First, it is useful to rewrite (linear) BSDE (40) in integral form:

$$H_t = E_{\overline{Q}}\left[\int_t^T \exp\left\{\frac{\gamma}{1-\gamma}\int_t^s [r - \beta/\gamma + (1-\gamma)\sigma_\tau^c \cdot \sigma_\tau^c/2]\, d\tau\right\} ds \middle| \mathcal{F}_t^S\right], \tag{44}$$

where $d\overline{Q}/P = z_T^{\overline{\theta}}$ and $(z_t^{\overline{\theta}})$ is determined by the density generator $\overline{\theta}_t = \theta_t^* - \alpha\sigma_t^c$. Thus, $H_t > 0$. From (44) and Ito's Lemma, $H_t\sigma_t^H$ is the integrand of the martingale representation of the martingale:

$$E_{\overline{Q}}\left[\int_0^T \exp\left\{\frac{\gamma}{1-\gamma}\int_0^s [r-\beta/\gamma+(1-\gamma)\sigma^c_\tau\cdot\sigma^c_\tau/2]\,d\tau\right\}ds\,\middle|\,\mathcal{F}^S_t\right],\ 0\le t\le T.$$

Thus, substituting (37) into the above reveals that both ambiguity (represented by θ^*_t) and estimation risk (represented by $\widehat{\mu}^R_t$) affect σ^H_t which determines hedging demands represented by the third term in (43).

As shown in section 3.4, ambiguity distorts mean returns at time t by an amount of $\sigma^R\theta^*_t$ under the ambiguity adjusted belief Q^*. The second term in (43) represents this static effect due to ambiguity.

As $\gamma\to 0$, the first-order conditions converge to those for the logarithmic case. Accordingly, the optimal consumption and portfolio processes converge to the plans that are optimal in the logarithmic case. In particular, when $\gamma=0$,

$$H_t=\beta^{-1}\left[1-e^{-\beta(T-t)}\right]\ \text{and}\ \sigma^H_t=0.$$

This reflects the well known fact that with logarithmic felicity the agent behaves myopically so that there is no hedging demand against future changes of investment opportunities. As a result, the optimal portfolio rule is identical to that in a model with complete information and mean returns $(\widehat{\mu}^R_t)$.

Next, the above theorem subsumes solutions for the standard model with expected utility, obtained by setting $\kappa=0$ (e.g., Brennan (1998)).[11] In the absence of ambiguity, estimation risk is the only source of hedging demand.[12] Brennan (1998) interprets this demand as being induced by the agent's learning about the true mean returns.

Theorem 3 can also deliver solutions for the case of complete information where the agent observes $\{\mathcal{F}_t\}$ so that $\{\mathcal{F}^S_t\}=\{\mathcal{F}_t\}$. For example, under expected utility, it is easy to show that $\sigma^H_t=0$. Thus the optimal portfolio is given by the mean-variance efficient demand:

$$\psi^*_t=\frac{1}{1-\gamma}(\sigma^R(\sigma^R)^\intercal)^{-1}(\mu^R-r\mathbf{1}).$$

Under ambiguity, the optimal portfolio is characterized by the following corollary:

Corollary 1. *In the case of complete information, if* $0\le\kappa<\eta_t$, *then* $\theta^*_t=\kappa$ *and the optimal portfolio is given by*

$$\psi^*_t=\frac{1}{1-\gamma}(\sigma^R(\sigma^R)^\intercal)^{-1}(\mu^R_t-r\mathbf{1})-\frac{1}{1-\gamma}((\sigma^R)^\intercal)^{-1}\kappa.$$

[11]Brennan (1998) assumes that the distribution of μ^R is normal and that the agent maximizes expected utility from terminal weath.

[12]Explicit expression for the hedging demand can be derived from Corollary 2.

Thus under complete information, ambiguity as modeled using κ-ignorance, does not induce any hedging demand even when mean returns are random.

In contrast, under incomplete information hedging demands arise as revealed by the third component of optimal portfolio given in (43). Hedging demands naturally arise in standard models with incomplete information due to estimation risk. In my model ambiguity affects these hedging demands even when (θ_t^*) is constant as will be shown later. Therefore, ambiguity has an intertemporal hedging effect.

Finally, in general it is difficult to solve for (θ_t^*) and (ψ_t^*) explicitly because (θ_t^*) is endogenously determined by a fixed-point problem (36) and (σ_t^H) can not be characterized explicitly. However, the following corollary provides a condition to ensure that $\theta_t^* = \kappa$ and characterizes the optimal portfolio (ψ_t^*) explicitly in terms of Malliavin derivatives and stochastic integrals.[13]

COROLLARY 2. *If the following condition hold:*

$$(\widehat{\eta}_t - \kappa) + \frac{e^{rt}}{X_t^*}((\sigma^R)^\intercal)^{-1} E_{\widetilde{P}}\left[\int_t^T e^{-rs} c_s^* \left(\int_t^s \mathbb{D}_t \widehat{\eta}_\tau d\widetilde{W}_\tau\right) ds \middle| \mathcal{F}_t^S\right] > 0, \tag{45}$$

where for $\theta_t^* = \kappa$, (c_t^*) is given by (35), (X_t^*) is given by (42) and Q^* is determined by (32), then $\theta_t^* = \kappa$ is optimal and the optimal portfolio is given by

$$\psi_t^* = \frac{1}{1-\gamma}(\sigma^R(\sigma^R)^\intercal)^{-1}(\widehat{\mu}_t^S - r1) - \frac{1}{1-\gamma}((\sigma^R)^\intercal)^{-1}\kappa \tag{46}$$

$$+\frac{\gamma}{1-\gamma}\frac{e^{rt}}{X_t^*}((\sigma^R)^\intercal)^{-1} E_{\widetilde{P}}\left[\int_t^T e^{-rs} c_s^* \left(\int_t^s \mathbb{D}_t \widehat{\eta}_\tau d\widetilde{W}_\tau\right) ds \middle| \mathcal{F}_t^S\right].$$

Even though $\theta_t^* = \kappa$ is constant, the optimal wealth and consumption processes (X_t^*) and (c_t^*) depend on κ. Thus the third term in (46) is affected by κ so that ambiguity still affects the hedging demand.

[13]The Malliavin derivative operator $\mathbb{D}$ is defined on $\mathcal{D}_{1,1}$, the space of smooth functionals of $\{\widehat{W}_t; 0 \leq t \leq T\}$. For the exact definition of $\mathcal{D}_{1,1}$ and an introduction to Malliavin calculus, the reader is referred to Ocone and Karatzas (1991) and Nualart (1995).

APPENDIX

Proof of Theorem 2:

By (26), the utility supergradient at c^* is given by

$$\pi_t = e^{-\beta t} z_t^{\theta^*} u'(c_t^*),$$

where $\theta_t^* = \kappa \otimes \mathrm{sgn}(\sigma_t^V(c^*))$ and $(V_t(c^*), \sigma_t^V(c^*))$ is the unique solution to BSDE (7) for $c = c^*$. From (27), the first-order condition for the problem (25) is satisfied since λ is the Lagrange multiplier associated with the constraint (22). Since V is concave, this condition is also sufficient for c^* to be an optimum for problem (25) and hence to problem (12) subject to (17) and (13).

The remaining step is to find the optimal portfolio ψ^*. This follows immediately from Theorem 1. ▌

Proof of Theorem 3:

Equations (35), (37) and (38) follow from the first-order condition

$$e^{-\beta t} z_t^{\theta^*} (c_t^*)^{\gamma - 1} = \lambda p_t, \tag{A.1}$$

and Ito's Lemma. The Lagrange multiplier λ is determined by (28). Defer the proof of (36) for the moment.

For part (ii), it suffices to show that for V_t in (39) the process $V_t + \int_0^t ((c_s^*)^\gamma/\gamma - \beta V_s - \theta_t^* \cdot \sigma_t^V) ds$, $0 \le t \le T$, is a $(P, \mathcal{F}_t^S)$-martingale so that (V_t) solves BSDE (7).

Apply Ito's Lemma to (39) to derive

$$\frac{dV_t + ((c_t^*)^\gamma/\gamma - \beta V_t - \theta_t^* \cdot \sigma_t^V) dt}{V_t} = B_t dt + \left(\sigma_t^H + \alpha \sigma_t^c\right) \cdot d\widehat{W}_t, \tag{A.2}$$

where

$$B_t = \mu_t^H + \gamma(\mu_t^c - \sigma_t^c \cdot \theta_t^*) - \frac{1}{2}\gamma(1-\gamma)\sigma_t^c \cdot \sigma_t^c - \beta + H_t^{-1} - (\theta_t^* - \alpha \sigma_t^c) \cdot \sigma_t^H.$$

By (37), (38) and (41), one obtains that $B_t = 0$ as desired.

By (A.2), the volatility of utility process is given by

$$\sigma_t^V(c^*) = V_t(\sigma_t^H + \alpha \sigma_t^c).$$

Then, equation (36) follows from (29).

Turn to the proof of part (iii). By Ito's Lemma and eliminating the resulting martingale term after taking expectations,

$$V_t e^{-\beta t} z_t^{\theta^*} - E\left[V_T e^{-\beta T} z_T^{\theta^*} | \mathcal{F}_t^S\right] = E\left[\int_t^T e^{-\beta s} z_s^{\theta^*} (c_s^*)^\gamma / \alpha ds \mid \mathcal{F}_t^S\right].$$

Use $V_T = 0$, (A.1) and (24) to derive

$$X_t^* = \alpha V_t e^{-\beta t} z_t^{\theta^*} (\lambda p_t)^{-1} = \gamma (c_t^*)^{1-\gamma} V_t.$$

Equation (42) follows from the above identity and (39).

Finally, apply Ito's Lemma to (42) and match the resulting volatility with that in (10) to obtain

$$\psi_t = (\sigma^R)^{-1}(\sigma_t^c + \sigma_t^H).$$

Inserting (37) yields the optimal portfolio (43). ▮

Proof of Corollary 1:[1]

Guess $\theta_t^* = \kappa$. Then using the same computation as above one can show that

$$\sigma_t^c = \frac{1}{1-\gamma}(\eta_t - \kappa) \text{ and}$$

$$H_t = E_{\overline{Q}}\left[\int_t^T \exp\left\{\frac{\gamma}{1-\gamma}\left[r - \beta/\gamma + \frac{1}{1-\gamma}(\eta_t - \kappa)\cdot(\eta_t - \kappa)/2\right](s-t)\right\} ds \middle| \mathcal{F}_t\right],$$

where $d\overline{Q}/P = z_T^{\overline{\theta}}$ and $(z_t^{\overline{\theta}})$ is determined by the density generator $\overline{\theta}_t = \kappa - \alpha\sigma_t^c$. Thus $H_t > 0$. Because $\eta_t = (\sigma^R)^{-1}(\mu^R - r\mathbf{1})$ and μ^R is a random variable independent of the Brownian motion W, one can show that $\sigma_t^H = 0$.

Apply Ito's Lemma to (39) to derive

$$\sigma_t^V(c^*) = (c_t^*)^\gamma H_t \sigma_t^c.$$

Because $(c_t^*)^\gamma H_t > 0$,

$$\text{sgn}(\sigma_t^V(c^*)) = \text{sgn}(\sigma_t^c).$$

Thus if

$$0 \le \kappa_i < \eta_t^i \text{ for all } i,$$

then $\sigma_t^c > 0$. By (39), $\sigma_t^V(c^*) > 0$. Thus $\theta_t^* = \kappa$ satisfies (36) and the expression in the corollary gives the optimal portfolio. ▮

Proof of Corollary 2:

First I guess $\theta_t^* = \kappa$. The key step is to compute σ_t^H. Then one verifies that the guess is consistent with (36) so that $\theta_t^* = \kappa$ is indeed optimal.

[1] It can also be proved from Corollary 2.

Let

$$F \equiv \int_0^T e^{-rs} c_s^* ds.$$

Then by (30) and Ocone and Karatzas (1991) Theorem 2.5,

$$\phi_t = E_{\widetilde{P}}\left[\mathbb{D}_t F \mid \mathcal{F}_t^S\right] - E_{\widetilde{P}}\left[F \int_t^T \mathbb{D}_t \widehat{\eta}_\tau d\widetilde{W}_\tau \mid \mathcal{F}_t^S\right]. \tag{A.3}$$

Apply the following steps to compute this expression.

Step 1. Compute the first term in (A.3).

Use (35) and the definitions of F and p to derive

$$\mathbb{D}F = \int_0^T e^{-rs}\mathbb{D}\left(\frac{e^{-\beta s} z_s^{\theta^*}}{\lambda p_s}\right)^{\frac{1}{1-\gamma}} ds = \int_0^T e^{-rs}\left(\frac{e^{-\beta s}}{\lambda e^{-rs}}\right)^{\frac{1}{1-\gamma}} \mathbb{D}\left(z_s^{\theta^*}/\widehat{Z}_s\right)^{\frac{1}{1-\gamma}} ds. \tag{A.4}$$

By (1), (20), (21) and the chain rule of Malliavin derivative,

$$\begin{aligned}
&\mathbb{D}\left(z_s^{\theta^*}/\widehat{Z}_s\right)^{\frac{1}{1-\gamma}} \\
= \; & \mathbb{D}\exp\left\{\frac{1}{1-\gamma}\left[-\frac{1}{2}\int_0^s |\theta_\tau^*|^2 d\tau - \int_0^s \theta_\tau^* \cdot d\widehat{W}_\tau + \frac{1}{2}\int_0^s |\widehat{\eta}_\tau|^2 d\tau + \int_0^s \eta_\tau \cdot d\widehat{W}_\tau\right]\right\} \\
= \; & \frac{1}{1-\gamma}\left(z_s^{\theta^*}/\widehat{Z}_s\right)^{\frac{1}{1-\gamma}}\left\{-\int_0^s (\mathbb{D}\theta_\tau^*)\theta_\tau^* d\tau - \int_0^s \mathbb{D}\theta_\tau^* d\widehat{W}_\tau\right. \\
& \left. -\theta^*(\cdot)1_{[0,s]}(\cdot) + \int_0^s (\mathbb{D}\widehat{\eta}_\tau)\widehat{\eta}_\tau d\tau + \int_0^s \mathbb{D}\widehat{\eta}_\tau d\widehat{W}_\tau + \widehat{\eta}(\cdot)1_{[0,s]}(\cdot)\right\} \\
= \; & \frac{1}{1-\gamma}\left(z_s^{\theta^*}/\widehat{Z}_s\right)^{\frac{1}{1-\gamma}}\left\{\widehat{\eta}(\cdot)1_{[0,s]}(\cdot) - \kappa 1_{[0,s]}(\cdot) + \int_0^s \mathbb{D}\widehat{\eta}_\tau d\widetilde{W}_\tau\right\},
\end{aligned}$$

where $1_{[0,s]}(\cdot)$ is an indicator function.

Substituting this expression into (A.4) yields

$$E_{\widetilde{P}}\left[\mathbb{D}_t F | \mathcal{F}_t^S\right] = E_{\widetilde{P}}\left[\int_0^T e^{-rs}\mathbb{D}_t\left(e^{-\beta s} z_s^{\theta^*}/(\lambda p_s)\right)^{\frac{1}{1-\gamma}} ds \middle| \mathcal{F}_t^S\right]$$

$$= \frac{1}{1-\gamma}(\widehat{\eta}_t - \theta_t^*) E_{\widetilde{P}}\left[\int_t^T e^{-rs} c_s^* ds \middle| \mathcal{F}_t^S\right] + \frac{1}{1-\gamma} E_{\widetilde{P}}\left[\int_t^T e^{-rs} c_s^* \int_t^s \mathbb{D}_t \widehat{\eta}_\tau d\widetilde{W}_\tau ds \middle| \mathcal{F}_t^S\right].$$

Step 2. Compute the second term in (A.3).

By the definitions of F and Malliavin derivative,

$$E_{\widetilde{P}}\left[F\int_t^T \mathbb{D}_t \widehat{\eta}_\tau d\widetilde{W}_\tau \middle| \mathcal{F}_t^S\right] = E_{\widetilde{P}}\left[\int_t^T e^{-rs} c_s^* \left(\int_t^s \mathbb{D}_t \widehat{\eta}_\tau d\widetilde{W}_\tau\right) ds \middle| \mathcal{F}_t^S\right].$$

Step 3. Compute σ_t^H.
By (43),

$$\sigma_t^H = \frac{\gamma}{1-\gamma}\frac{e^{rt}}{X_t^*}E_{\widetilde{P}}\left[\int_t^T e^{-rs}c_s^*\left(\int_t^s \mathbb{D}_t\widehat{\eta}_\tau d\widetilde{W}_\tau\right)ds\middle|\mathcal{F}_t^S\right].$$

Thus if condition (45) holds, then $\theta_t^* = \kappa$ satisfies (36) so that it is indeed optimal.

Step 4. Compute the optimal portfolio.
By Theorem 2,

$$\psi_t^* = e^{rt}((\sigma^S)^\intercal)^{-1}\phi_t/X_t^*$$

$$= \frac{1}{1-\gamma}(\sigma^R(\sigma^R)^\intercal)^{-1}(\widehat{\mu}_t^S - r1) - \frac{1}{1-\gamma}((\sigma^R)^\intercal)^{-1}\kappa$$

$$+\frac{\gamma}{1-\gamma}\frac{e^{rt}}{X_t^*}((\sigma^R)^\intercal)^{-1}E_{\widetilde{P}}\left[\int_t^T e^{-rs}c_s^*\left(\int_t^s \mathbb{D}_t\widehat{\eta}_\tau d\widetilde{W}_\tau\right)ds\middle|\mathcal{F}_t^S\right].$$

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