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Endogenous Default Penalties in Nominal Incomplete Markets *

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This paper endogeneizes the utility penalties, initiated by Dubey, Geanakoplos and Shubik (2005), by introducing a benevolent central planner choosing the value of marginal default utility penalties for each consumer. We also prove equilibrium existence with endogenous default penalties for incomplete markets with nominal assets.

Key Words: Equilibrium; Endogenous default penalties; Incomplete markets.
JEL Classification Numbers: D52, D91.

1. INTRODUCTION.

Classical models of General Equilibrium Theory, both with complete markets and incomplete markets, assume that all borrowers fully keep their promises. This assumption was heavily criticized as it does not reflect actual financial markets practices. In the Nineties, general equilibrium models have allowed for default. More precisely, borrowers may actually default on future asset returns.

When default is allowed, one must impose some mechanisms that urge agents to pay back a part of their debts. In the literature, there are at least

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two mechanisms that encourage agents to partially honor their promises and, therefore, protect lenders against full default. The first mechanism that urges the agents to honor a part of their promises is to require the borrowers to constitute collateral in terms of durable goods protecting the lenders. This collateral will be seized and given to the lenders in case of default (see, for instance, Dubey, Geanakoplos and Zame (1995) and Araujo, Pascoa and Torres–Martinez (2002)). The second mechanism that protects the lenders against full default is to assume that defaulters suffer utility penalties. These default penalties can be interpreted as social or/and economic sanctions or loose of reputation. For such a model, Dubey–Geanakoplos–Shubik(2005) prove equilibrium existence for a finite–horizon real incomplete market model with a stochastic structure characterized by a finite number of states of nature and bounded short sales. Zame (1993) proves that default improves risk sharing for a finite–horizon models with utility penalties. Some models combine these two default punishments, that is borrowers are required to constitute collateral in terms of durable goods and suffer utility penalties in case of default (see for instance Zame (1993), Dubey, Geanakolpos and Zame (1995) for a finite–horizon model and Pascoa and Seghir (2009) for an infinite–horizon model).

The utility penalties used in Dubey–Geanakoplos–Shubik (2005) and Pascoa and Seghir (2008) are exogenously given and represent a characterization of the consumers via their utility functions. More precisely, these penalties, which are assumed to be proportional to the value of total default committed, are assumed to be chosen exogenously ex-ante by a benevolent central planer, or by society as an all. However, the choice of these penalties was not modeled in Dubey–Geanakoplos–Shubik(2005).
Default penalties vary from one society to another , ranging from flexible (a payment penalty frees for example) to harsh (imprisonment). The aim of this paper is to make an explicit modeling of society's choice of the values for the default penalties. To do so, we introduce, in the context of a standard two period incomplete market model with nominal assets, a benevolent central planner choosing the value of marginal default utility penalties for each consumer, for the default that he may commit on each asset and in each state of nature. The objective of this central planner is to maximize a social utility function, which is assumed to be utilitarian.

It is well known that when assets are nominal and default is not allowed, one needs to assume that the return matrix has full rank. Indeed, when default is not allowed, the rank of the (promised) returns may drop and equilibrium fails to exist (see Geanakoloplos (1990) for an excellent discussion of this problem). In this paper, we prove that this assumption is not needed anymore when default is possible. In fact, *default can make assets, that are linearly dependent in terms of promised returns, independent in* *terms of effective returns.* We prove equilibrium existence without making any assumption on the rank of the returns matrix.

The contents of the paper are as follows. First, we present the model and the concepts of equilibrium. Next, we present the assumptions that guarantee equilibrium existence. To prove equilibrium existence, we first define a compactified economy in which consumers' decision variables are bounded from above. Section 4 proves equilibrium existence in such compactified economies. In Section 4, We also prove equilibrium existence for the original economy using asymptotic results.

2. THE MODEL

Let an economy extend over two time periods $t = 0, 1$ with a finite set S of possible states of nature $s \in S$ in period 1. For convenience, we assume that $0 \notin S$, and $s = 0$ denotes the state of the world (known with certainty) at period 0. We denote $\overline{S} = \{0\} \cup S$. There is a finite number G of commodities for consumption, indexed by $g = 1, \ldots, G$, at date 0 and in each state at date 1. There is a finite number J of nominal assets, indexed by $j = 1, \ldots, J$, for trade at date 0. Each asset $j \in J$ yields $r_j(s)$ units of account in state s at date 1. For every state of nature $s \in S$, let $r(s) = (r_j(s))_{j \in J}$ and let $R = (r_j(s))_{(s,j) \in S \times J}$ denote the $(S \times J)$-matrix of returns.

There is a finite set I of consumers in the economy. At period 0, each agent $i \in I$ chooses a portfolio $z^i := (z^i_j,\ j \in J)$, with $z^i = \theta^i - \varphi^i$ where:

- $\theta^i := (\theta^i_j,\ j \in J) \in \mathbb{R}^J_+$ are the quantities of assets bought by the agent i at period 0,
- $\varphi^i := (\varphi^i_j,\ j \in J) \in \mathbb{R}^J_+$ are the quantities of assets sold by the agent i at period 0.

At each state of nature $s \in \overline{S}$, each agent $i \in I$ chooses a consumption bundle $x^i(s) \in X^i(s) \subset \mathbb{R}^G_+$.

Each consumer has a consumption set $X^i := \prod_{s \in \overline{S}} X^i(s) \subset \mathbb{R}^{G\overline{S}}$, an initial endowment vector $\omega^i \in \mathbb{R}^{G\overline{S}}$, and an utility function U^i. The collection $\mathcal{E} = \big((X^i, U^i, \omega^i)_{i \in I}, R\big)$ describes our economy. Vectors of commodity and asset prices will be denoted by $p \in \mathbb{R}^{G\overline{S}}$and $q \in \mathbb{R}^J$, respectively.

At each state of nature $s \in S$, the debt of an agent $i \in I$ (induced by the sale of φ^i_j units of the asset $j \in J$ in period 0) is $r_j(s)\varphi^i_j$. Since default is allowed, an agent $i \in I$ chooses to pay, at each state of nature $s \in S$, $\Delta^i_j(s)$ such that $\forall s \in S,\ \forall j \in J,\ 0 \leqslant \Delta^i_j(s) \leqslant r_j(s)\varphi^i_j$.

On the other hand, since the markets are anonymous, each agent expects to receive a part of his credits. Let $K := (K_j(s),\ s \in S,\ j \in J) \in [0,1]^{S \times J}$ be the $(S \times J)$-matrix of delivery rates on assets.

To stimulate the agents to pay back at least a part of their debts, we assume that they feel a disutility, which is represented by utility penalties, from defaulting. More precisely, given the marginal penalties $\lambda_j^i(s)$, the utilities associated with an individual choice variable plan $(x^i, \theta^i, \varphi^i, \Delta^i)$ of an agent $i \in I$ is:

$$U^i_{\lambda^i}(x^i, \theta^i, \varphi^i, \Delta^i) = V_0^i(x_0^i) + \sum_{s=1}^{S} \left[V_s^i(x_s^i) - \sum_{j \in J} \lambda_j^i(s)[r_j(s)\varphi_j^i - \Delta_j^i(s)] \right].$$

Note that $[r_j(s)\varphi_j^i - \Delta_j^i(s)]$ is exactly the default of the agent $i \in I$ at node s, on his promise from the sale of the asset j at period 0. We will define two equilibria concepts. In the first definition, utility penalties are exogenously given. In the second one these penalties will be endogenously determined by a benevolent central planner (from now on referred as the *Lawmaker)* in order to maximize society's overall well being (measured by a certain social utility function), and will reflect society's degree of acceptance towards default, inducing agents to default in a higher or lower scale.

The following definitions summarize the economy.

Definition 2.1. [Budget sets]
Given (p, q, K), the budget set $B^i(p, q, K)$ of an agent $i \in I$ is the set of $(x^i, \theta^i, \varphi^i, \Delta^i)$ in $X^i \times \mathbb{R}_+^J \times \mathbb{R}_+^J \times \mathbb{R}_+^{JS}$ which verify:

$$p(0) \cdot x^i(0) + q \cdot \left(\theta^i - \varphi^i\right) \leqslant p(0) \cdot \omega^i(0), \tag{1}$$

and

$$p(s) \cdot x^i(s) + \sum_{j \in J} \Delta_j^i(s) \leqslant p(s) \cdot \omega^i(s) + \sum_{j \in J} K^j(s) r_j(s) \theta_j^i, \ \forall s \in S. \tag{2}$$

Our first equilibrium definition is the following:

Definition 2.2. [Equilibrium with exogenous default penalties]
An equilibrium of $\mathcal{E}$ with exogenous default penalties is a collection $(\overline{p}, \overline{q}, \overline{K}, (\overline{x}^i, \overline{\theta}^i, \overline{\varphi}^i, \overline{\Delta}^i)_{i \in I})$ such that, for a given vector λ:

(i) For each agent $i \in I$, $(\overline{x}^i, \overline{\theta}^i, \overline{\varphi}^i, \overline{\Delta}^i) \in \operatorname{argmax} U^i_{\lambda^i}(x, \theta, \varphi, \Delta)$ over $B^i(\overline{p}, \overline{q}, \overline{K})$,

(ii) $\sum_{i\in I}(\overline{x}^i - \omega^i) = 0,$

(iii) $\sum_{i\in I}\overline{\theta}^i = \sum_{i\in I}\overline{\varphi}^i,$

(iv) $\forall s \in S,\ \forall j \in J,\ \overline{K}^j(s)\sum_{i\in I} r_j(s)\overline{\varphi}^i_j = \sum_{i\in I}\overline{\Delta}^i_j(s).$

Condition (i) is the optimality of agents' choices over their budget sets. Conditions (ii) and (iii) require commodity and asset markets to clear. Condition (iv) says that, at each node and for each asset, the total effective delivery made by the sellers (borrowers) is equal to the total expected delivery made by the buyers (lenders).

Remark 2.1. The Lawmaker may perfectly choose to set a very flexible default penalty in some state of nature, that is $\lambda^i_j(s) = 0$, or close to 0, for some agent $i \in I$, some asset $j \in J$, and some state of nature $s \in S$, in order to allow agents to default for example in a state of nature with low probability and where they have a very low endowment. However, at equilibrium, one must have:

$$\forall i \in I,\ \forall j \in J \text{ such that } \overline{q}_j \neq 0,\ \exists s \in S:\ \overline{\lambda}^i_j(s) \neq 0. \tag{3}$$

That is, for each agent i and each asset j with nonzero price, there must exist at least one state of nature for which the penalty is not equal to zero. Indeed, otherwise, $\exists i \in I,\ \exists j \in J:\ \overline{q}^j > 0$ and $\forall s \in S,\ \overline{\lambda}^i_j(s) = 0$. Let us consider the following change on the short sales of this agent i with respect the asset j :

$$\widehat{\varphi}^i_j = \overline{\varphi}^i_j + \epsilon,\ \epsilon > 0.$$

Moreover, by paying the same thing, that is $\overline{\Delta}^i_j(s)$, at period 1, the agent will not loose more utility (because $\forall s \in S,\ \overline{\lambda}^i_j(s) = 0$). In addition, by borrowing $\widehat{\varphi}^i_j$ he can improve his consumption at period 0, which contradicts equilibrium optimality. In other words, zero penalties in all states of nature is not consistent with equilibrium existence.

Let $W(x,\theta,\varphi,\Delta,p,q,K) = \sum_{i\in I}\beta_i U^i_{\lambda_i}\left(x^i,\theta^i,\varphi^i,\Delta^i\right)$, with $\sum_{i\in I}\beta_i = 1$, be the social utility function that the Lawmaker will seek to maximize. Then, our second equilibrium concept is the following:

Definition 2.3. [Equilibrium with endogenous default penalties]

An equilibrium of $\mathcal{E}$ with endogenous default penalties is a collection $(\overline{p}, \overline{q}, \overline{K}, \overline{\lambda}, (\overline{x}^i, \overline{\theta}^i, \overline{\varphi}^i, \overline{\Delta}^i)_{i\in I}, \lambda)$ such that:

(i) For each agent $i \in I$, $(\overline{x}^i, \overline{\theta}^i, \overline{\varphi}^i, \overline{\Delta}^i) \in \operatorname{argmax} U^i_{\lambda^i}(x, \theta, \varphi, \Delta)$ over $B^i(\overline{p}, \overline{q}, \overline{K})$,

(ii) $\sum_{i\in I} (\overline{x}^i - \omega^i) = 0$,

(iii) $\sum_{i\in I} \overline{\theta}^i = \sum_{i\in I} \overline{\varphi}^i$,

(iv) $\forall s \in S,\ \forall j \in J,\ \overline{K}^j(s) \sum_{i\in I} r_j(s)\overline{\varphi}^i_j = \sum_{i\in I} \overline{\Delta}^i_j(s)$.

(v) The Lawmaker chooses utility penalties $\overline{\lambda} \in \mathbb{R}^{IJS}$ such that

$$W(\overline{x}, \overline{\theta}, \overline{\varphi}, \overline{\Delta}, \overline{p}, \overline{q}, \overline{K}, \overline{\lambda}) \geqslant W\,(x, \theta, \varphi, \Delta, p, q, K, \lambda)$$

for each $(x, \theta, \varphi, \Delta, p, q, K)$ that constitutes an equilibrium with exogenous default penalties for λ.

Condition (v) defines the mechanism of endogenous choice for the marginal default penalties vector.

3. THE ASSUMPTIONS AND THE EXISTENCE RESULT

We make on $\mathcal{E}$ the following assumptions:

- **Assumption [A1].** For each asset $j \in J$, for each state of nature $s \in S,\ r^j(s) \geqslant 0$.
- **Assumption [A2].** For each agent $i \in I,\ X^i = \mathbb{R}^{G\overline{S}}_+$, and for each state of nature $s \in \overline{S},\ V^i_s : X^i(s) \longrightarrow \mathbb{R}_+$ is continuous, strictly monotone, concave, differentiable at each point $x \in \mathbb{R}^{G\overline{S}}_{++}$ and satisfies the following Inada conditions: $\|\nabla V^i_s(x^i(s))\| \longrightarrow \infty$ as $x^i(g,s) \longrightarrow 0$, for some $g \in \{1,\ldots,G\}$ and $\frac{\partial V^i_s(x^i(s))}{\partial x^i(g,s)} \longrightarrow 0$ as $x^i(s,g) \longrightarrow \infty$, $\forall g = 1,\ldots,G$.
- **Assumption [A3].** For each agent $i \in I$, for every state of nature $s \in \overline{S},\ \omega^i(s) \gg 0$.

Assumption [A1] states that all assets are supposed to have nonnegative returns in all states of nature. Assumption [A2] is classical to guarantee equilibrium existence. Assumption [A3] requires all agents to have a positive initial endowment at each state of nature.

Our main existence result is:

THEOREM 1. *Under the assumptions [A1]–[A3] stated above, the Economy $\mathcal{E}$ has an equilibrium $(\overline{p}, \overline{q}, \overline{K}, \overline{\lambda}, (\overline{x}^i, \overline{\theta}^i, \overline{\varphi}^i, \overline{\Delta}^i)_{i\in I})$ with endogenous default penalties.*

4. THE TRUNCATED ECONOMIES

This section is devoted to prove the equilibrium existence in a compactified economy. Indeed, following an idea due to Florenzano–Gourdel–Marakulin (1998), we define the economy $\mathcal{E}^M$ which has the same characteristics than the economy $\mathcal{E}$ but we suppose that for $\mathcal{E}^M$ the positive and negative parts of the admissible portfolios for the agents are bounded from above. Moreover, in the economy $\mathcal{E}^M$, we suppose that the marginal penalties are also bounded from above. Formally, for each $M > 0$ and for each agent $i \in I$, we define the following sets:

$$\Phi^{iM} := \{\varphi^i \in \mathbb{R}^J_+ : \varphi^i_j \leqslant M,\ \forall j \in J\},$$

$$\Theta^{iM} := \{\theta^i \in \mathbb{R}^J_+ : \theta^i_j \leqslant M,\ \forall j \in J\}.$$

Remark 4.1. Note that the definitions of the sets Φ^{iM} and Θ^{iM} immediately imply that one has $-M \leqslant \theta^i_j - \varphi^i_j \leqslant M,\ \forall j \in J, \forall \theta^i_j \in \Phi^{iM}$, $\forall \varphi^i_j \in \Theta^{iM}$.

Since $0 \leqslant \Delta^i_j(s) \leqslant r_j(s)\varphi^i_j$, consumer i's deliveries are immediately restricted to be in the set

$$D^{iM} := \{\Delta^i \in \mathbb{R}^{J\times S}_+ : \Delta^i_j(s) \leqslant r_j(s)M,\ \forall (s,j) \in S \times J\}.$$

Notice that, without assumption [A3], D^{iM} could be empty for every $i \in I$. Moreover, for each $M > 0$, we define:

$$\Lambda^M := \{(\lambda^i_j(s),\ i \in I,\ j \in J,\ s \in S) \in \mathbb{R}^{I\times J\times S}_+ : \lambda^i_j(s) \leqslant M,\ \forall (i,s,j) \in I{\times}S{\times}J\}.$$

Let $\Pi := \left\{(p,q) \in \mathbb{R}^{L\overline{S}}_+ \times \mathbb{R}^J_+ : \|p(0)\|_1 + \|q\|_1 = 1 \wedge \|p(s)\|_1 = 1,\ \forall s \in S\right\}$ be the set of admissible prices.

Let us define the set of feasible allocations:

$$\hat{X} := \left\{x = (x^i)_{i\in I} \in \mathbb{R}^{G\overline{SI}}_+ : \sum_{i\in I} x^i \leqslant \sum_{i\in I} \omega^i\right\}.$$

For each agent $i \in I$, let $\hat{X}^i$ be the projection of $\hat{X}$ on X^i. Notice that $\hat{X}^i$ is compact.

Let us denote by $B_0(0,r)$ the open ball in $\mathbb{R}^{G\overline{S}}$ with center 0 and radius r. We can choose a real number r such that $\forall i \in I,\ \hat{X}^i \subset B_0(0,r)$. For each agent $i \in I$, let us define $\widetilde{X}^i = \mathbb{R}_+^{G\overline{S}} \cap \overline{B_0(0,r)}$, where $\overline{B_0(0,r)}$ denotes the closure of $B_0(0,r)$. Once the optimality in $\widetilde{X}^i \times \Phi^{iM} \times \Theta^{iM} \times D^{iM}$ will be shown, we can deduce the optimality in $X^i \times \Phi^{iM} \times \Theta^{iM} \times D^{iM}$, given exogenous utility penalties $\lambda \in \Lambda^M$, by a classical argument which uses the concavity of the utility functions.

Proposition 1. *Under assumptions [A1]-[A3] stated above, each truncated economy $\mathcal{E}^M$ has an equilibrium with exogenous default penalties $(\overline{p}^M, \overline{q}^M, \overline{K}^M, (\overline{x}^{iM}, \overline{\theta}^{iM}, \overline{\varphi}^{iM}, \overline{\Delta}^{iM})_{i\in I})$, for $\lambda \in \Lambda^M$ given.*

Proof of Proposition 1.
The proof of existence of equilibrium in each compactfied economy, $\mathcal{E}^M$, is based on the classical argument of the lower semicontinuity of the budget correspondences (assured by the non-vacuity of their interiors). The interiority of the initial endowments of each agent at each node and for each commodity, assumed in Hypothesis [A3], guarantees this non-emptiness. Formally, let us define for each asset $j \in J$ and for each state of nature $s \in S$:

$$K_j(s)((\varphi)_{i\in I}, (\Delta)_{i\in I}) = \begin{cases} \min\left(1, \frac{\sum_{i\in I} \Delta_j^i(s)}{r_j(s) \sum_{i\in I} \varphi_j^i}\right) & \text{if } r_j(s) \sum_{i\in I} \varphi_j^i \neq 0 \\ [0,1] & \text{if not} \end{cases}. \quad (4)$$

For each agent $i \in I$, let us define the budget correspondence $\widetilde{B}^{iM} : \Pi \times [0,1]^{S\times J} \longrightarrow \widetilde{X}^i \times \Theta^{iM} \times \Phi^{iM} \times D^{iM}$ as follows: $\forall (p,q,K) \in \Pi \times [0,1]^{S\times J}$, $\widetilde{B}^{iM}(p,q,K) := \{(x^i, \theta^i, \varphi^i, \Delta^i) \in \widetilde{X}^i \times \Theta^{iM} \times \Phi^{iM} \times D^{iM}$ satisfying the budget constraints of Definition 1$\}$. For each agent $i \in I$, we define also the correspondence $\widetilde{B}'^{iM} : \Pi \times [0,1]^{S\times J} \longrightarrow \widetilde{X}^i \times \Theta^{iM} \times \Phi^{iM} \times D^{iM}$ as follows: $\forall (p,q,K) \in \Pi \times [0,1]^{S\times J}$, $\widetilde{B}'^{iM}(p,q,K) := \{(x^i, \theta^i, \varphi^i, \Delta^i) \in \widetilde{X}^i \times \Theta^{iM} \times \Phi^{iM} \times D^{iM}$ satisfying strictly the budget constraints $\}$

To simplify the notation, we will denote $v = (p,q,K)$ and $w = (x, \theta, \varphi, \Delta)$.

We define also the best response correspondence for agent i :

$$\Psi_\lambda^{iM}(v,w) = \left\{ w^i \in \widetilde{B}^{iM}(v) : U_\lambda^i(w^i) \geqslant U_\lambda^i(w'^i), \forall w'^i \in \widetilde{B}^{iM}(v) \right\}.$$

Notice that $\widetilde{B}^{iM}(v)$ is compact for each $v \in (v,w) \in \Pi \times [0,1]^{S\times J}$, and that $\Psi_{\lambda}^{iM}(v,w) \neq \emptyset$ for each $(v,w) \in \Pi \times [0,1]^{S\times J} \times \widetilde{X}^i \times \Theta^{iM} \times \Phi^{iM} \times D^{iM}$, and each $i \in I$.

Claim 4.1. For each agent $i \in I$, $\widetilde{B}^{iM}$ is a closed correspondence (then upper semicontinuous).

Proof. First, notice that for each given $v \in \Pi \times [0,1]^{S\times J}$, B^{iM} is closed-valued, thus it has closed graph. Moreover, since it is a correspondence with convex values and it takes values on the compact set $\hat{X}^i \times \Theta^{iM} \times \Phi^{iM} \times D^{iM}$, one gets that it is upper semicontinuous. □

Claim 4.2. $\forall i \in I,\ \forall (p,q,K) \in \Pi \times [0,1]^{S\times J},\ \widetilde{B}'^{iM}(p,q,K) \neq \emptyset.$

Proof. Let $i \in I$ and $(p,q,K) \in \Pi \times [0,1]^{S\times J}$. For each $s \in S$, in view of assumption [A3] and since $p(s) \neq 0$, one can choose $x^i(s)$ such that $p(s) \cdot (x^i(s) - \omega^i(s)) < 0$ and let us choose $\Delta^i(s) = 0$.
Now, if $p(0) \neq 0$, in view of assumption [A3], one can choose $x^i(0)$ such that $p(0) \cdot (x^i(0) - \omega^i(0)) < 0$ and let us set $\theta^i = \varphi^i = 0$. If $p(0) = 0$, then there exists $j_0 \in J$ such that $q_{j_0} \neq 0$. In this case, we choose $\theta^i = 0$ and $\varphi^i_{j_0} > 0$. □

Claim 4.3. $\forall i \in I,\ \widetilde{B}^{iM}$ is lower semicontinuous.

Proof. It follows from the convexity and the non-emptiness of $\widetilde{B}'^{iM}(p,q,K)$ for each $(p,q,K) \in \Pi \times [0,1]^{S\times J}$ that $\widetilde{B}^{iM}(p,q,K) = \overline{\widetilde{B}'^{iM}(p,q,K)}$. The Claim follows from the fact that $\widetilde{B}'^{iM}$ has an open graph. □

As in Gale–Mas-Colell (1975, 1979), we define the following reaction correspondence for a fictitious agent who fixes prices:

$$\Psi^{0M}(v,w) = \left\{ (p',q') \in \Pi : \sum_{s\in\overline{S}} \left[(p'(s) - p(s)) \cdot \sum_{i\in I} (x^i(s) - \omega^i(s)) \right] + \right.$$
$$\left. + (q' - q) \cdot \sum_{i\in I} \left(\theta^i - \varphi^i\right) > 0 \right\}.$$

We also define the next correspondence for another hypothetical, who will set the repayment rates $K_j(s)$:

$$\Psi^{KM}(v,w)=\left\{K\in[0,1]^{SJ}:K_j(s)(v,w)=\left\{\begin{array}{cc}\min\left(1,\frac{\sum_{i\in I}\Delta_j^i(s)}{r_j(s)\sum_{i\in I}\varphi_j^i}\right) & \text{if } r_j(s)\sum_{i\in I}\varphi_j^i\neq 0\\ [0,1] & \text{if not}\end{array}\right.\right\}$$

Claim 4.4. Ψ^{0M} is lower semicontinuous.

Proof. Ψ^{0M} has open graph, thus is lower semicontinuous. □

Claim 4.5. Ψ^{KM} is upper semicontinuous.

Proof. Consider a convergent sequence $(v_n,w_n)=(x_n,\theta_n,\varphi_n,\Delta_n,p_n,q_n,K_n)$, whose limit is $(\overline{v},\overline{w})=\left(\overline{x},\overline{\theta},\overline{\varphi},\overline{\Delta},\overline{p},\overline{q},\overline{K}\right)$. If $\sum_{i\in I}\overline{\varphi}_j^i=0$ for some $j\in J$, then $\lim K_j(s)(v_n,w_n)\in[0,1]$ since $K_j(s)(v_n,w_n)\in[0,1]$ for each n. For $j\in J$ such that $\sum_{i\in I}\overline{\varphi}_j^i\neq 0$, then $\overline{K}_j(s)=\lim K_j(s)(v_n,w_n)$ by continuity of $\min\left(1,\sum_{i\in I}\Delta_j^i(s)/r_j(s)\sum_{i\in I}\varphi_j^i\right)$. □

Claim 4.6. For each $i\in I$, Ψ_λ^{iM} is upper semicontinuous and convexed

valued.

Proof. The first part results immediately from the continuity of the $\widetilde{B}^{iM}$ correspondences and Berge's (1963) Maximum Theorem. The second part then results from the concavity of U_λ^i. □

Now we apply the following theorem:

Theorem. [Gourdel Fixed Point Theorem]

Let $X=\prod_{i=1}^{m+n}X_i$, where, for each i, X_i is a non-empty, convex and compact subset of a normal finite dimensional vector space E^i. Let F_i $(i=1,...,m)$ be m lower semicontinuous correspondences from X to X_i with convex and eventually empty values, and let F_i $(i=m+1,...,m+n)$ be n upper semicontinuous correspondences from X to X_i with convex and eventually empty values. Then there is $x^*=(x_i^*)$ in X such that, for each i, either $x_i^*\in F_i(x_i^*)$ or $F_i(x_i^*)$ is empty.

Since the correspondences Ψ^{0M}, Ψ^{KM} and Ψ_λ^{iM}, $i \in I$, satisfy the conditions of the above theorem, there exists

$$(\overline{v}^M, \overline{w}^M) = \left(\overline{p}^M, \overline{q}^M, \overline{K}^M, \left(\overline{x}^{iM}, \overline{\theta}^{iM}, \overline{\varphi}^{iM}, \overline{\Delta}^{iM}\right)_{i\in I}\right)$$

such that:

a) $\Psi^{0M}(\overline{v}^M, \overline{w}^M) = \emptyset$, that is,

$$\sum_{s\in \overline{S}} \left[(p(s) - \overline{p}^M(s)) \cdot \sum_{i\in I}(\overline{x}^{iM}(s) - \omega^i(s))\right] + (q - \overline{q}^M) \cdot \sum_{i\in I} \left(\overline{\theta}^{iM} - \overline{\varphi}^{iM}\right) \leqslant 0, \ \forall (p, q) \in \Pi \tag{5}$$

b) $\overline{K}^M \in \Psi^{KM}(\overline{w}^M, \overline{v}^M)$, that is, $\overline{K}_j^M(s)$

$$= \begin{cases} \min\left(1, \frac{\sum_{i\in I} \overline{\Delta}_j^{iM}(s)}{r_j(s) \sum_{i\in I} \overline{\varphi}_j^{iM}}\right) & \text{if } r_j(s) \sum_{i\in I} \overline{\varphi}_j^{iM} \neq 0 \\ [0,1] & \text{if not} \end{cases}$$

c) For each $i \in I$, $\overline{w}^{iM} = \left(\overline{x}^{iM}, \overline{\theta}^{iM}, \overline{\varphi}^{iM}, \overline{\Delta}^{iM}\right) \in \Psi_\lambda^{iM}(\overline{w}^{iM}, \overline{v}^{iM})$, that is, $\overline{w}^{iM} \in \underset{w^i \in \widetilde{B}^{iM}(\overline{v})}{\arg\max}\, U_\lambda^i(w^i)$.

To show that this fixed point constitutes an equilibrium for the truncated economy, we must show that it satisfies conditions (i)-(iv) on the definition of equilibrium.

Claim 4.7. $(\overline{w}^M, \overline{v}^M)$ constitutes an equilibrium with exogenous default penalties for the truncated economy $\mathcal{E}^M$.

Proof. Conditions (i) and (iv) in definition 2 are immediately satisfied due to the definition of the correspondences Ψ_λ^{iM} and Ψ^{KM}.

To prove conditions (ii) and (iii), setting in (5), $p(s) = \overline{p}^M(s)$, $\forall s \in S$, one gets: $\forall (p(0), q) \in \mathbb{R}_+^{GJ} : \|p(0)\|_1 + \|q\|_1 = 1$,

$$p(0) \cdot \sum_{i\in I}[\overline{x}^{iM}(0) - \omega^i(0)] + q \cdot \sum_{i\in I} \left(\overline{\theta}^{iM} - \overline{\varphi}^{iM}\right) \leqslant \overline{p}^M(0) \cdot \sum_{i\in I}[\overline{x}^{iM}(0) - \omega^i(0)] + \overline{q}^M \cdot \sum_{i\in I} \left(\overline{\theta}^{iM} - \overline{\varphi}^{iM}\right) = 0.$$

where the last equality results from the fact that at $(\overline{w}^{iM}, \overline{v}^{iM})$ the agents' budget constrains must be saturated due to preferences monotonicity.

Now, setting $p(0) = 0$ in the last equality, one gets:

$$q \cdot \sum_{i \in I} \left(\overline{\theta}^{iM} - \overline{\varphi}^{iM} \right) \leqslant 0, \ \forall q \in \mathbb{R}^J_+ : \|q\|_1 = 1,$$

implying that $\sum_{i \in I} \left(\overline{\theta}^{iM} - \overline{\varphi}^{iM} \right) \leqslant 0$. Similarly, by setting $q = 0$ we obtain

$$p(0) \cdot \sum_{i \in I} [\overline{x}^{iM}(0) - \omega^i(0)] \leqslant 0, \forall p(0) \in \mathbb{R}^G_+ : \|p(0)\|_1 = 1,$$

implying $\sum_{i \in I} [\overline{x}^{iM}(0) - \omega^i(0)] \leqslant 0$.

Now, suppose that $\overline{p}_g(0) = 0$ for some $g \in G$. By strict monotonicity of the consumers utility functions, each one of them would like to buy r units of good g, implying an aggregate demand equal to $rI > \sum_{i \in I} \omega^i(0)$[1], contradicting $\sum_{i \in I} [\overline{x}^{iM}(0) - \omega^i(0)] \leqslant 0$. Thus $\overline{p}(0) \in \mathbb{R}^G_{++}$, and so if for some $g \in G$ we had $\sum_{i \in I} [\overline{x}^{iM}_g(0) - \omega^i_g(0)] < 0$ the price auctioneer could improve his payoff by changing to some other price vector in Π such that $p_g(0) = 0$.

Similarly, if for some asset $j \in J$ we had simultaneously $\overline{q}_j > 0$ and $\sum_{i \in I} \left(\overline{\theta}^{iM}_j - \overline{\varphi}^{iM}_j \right) < 0$, the auctioneer would not be maximizing his objective function. Thus we can only have $\sum_{i \in I} \left(\overline{\theta}^{iM}_j - \overline{\varphi}^{iM}_j \right) < 0$ if $\overline{q}_j = 0$. A consumer i will only be willing to sell asset j at zero price if $\lambda^i_j(s) r_j(s) = 0$ for all $s \in S$. This in turn would imply $\sum_{i \in I} \overline{\Delta}^{iM}_j(s) = 0$, hence $\overline{K}^M_j(s) = 0$ for all $s \in S$. Thus we can take in this case, without loss of generality, $\theta^i_j = \varphi^i_j = 0$ for all $i \in I$, and hence condition (iii) is satisfied.

Setting in (5), $p(0) = \overline{p}^M(0)$ and $q = \overline{q}^M$, one gets:
$\forall p(s) \in \mathbb{R}^G_+ : \|p(s)\|_1 = 1$,

$$\sum_{s \in S} \left[(p(s) - \overline{p}^M(s)) \cdot \sum_{i \in I} (\overline{x}^{iM}(s) - \omega^i(s)) \right] \leqslant 0.$$

[1]Remember that $\hat{X}^i$ is the projection of $\hat{X}$ on $X^i = \left\{ x = (x^i)_{i \in I} \in \mathbb{R}^{G\overline{S}I}_+ : \sum_{i \in I} x^i \leqslant \sum_{i \in I} \omega^i \right\}$, and r is chosen such that $\forall i \in I$, $\hat{X}^i \subset B_0(0, r)$. Thus $r > \max_{s \in \overline{S}} \left\{ \max_{g \in G} \left\{ \max_{i \in I} \left\{ w^i_g \right\} \right\} \right\}$.

□

By saturation of state s budget we have

$$\sum_{s\in S}\overline{p}^M(s)\cdot\sum_{i\in I}(\overline{x}^{iM}(s)-\omega^i(s))+\sum_{j\in J}\sum_{i\in I}\overline{\Delta}_j^{iM}(s)=\sum_{j\in J}\sum_{i\in I}\overline{K}_j^M(s)\overline{\theta}_j^{iM},$$

and since equilibrium condition (ii) and (iv) are satisfied, it follows that $\sum_{j\in J}\sum_{i\in I}\overline{\Delta}_j^{iM}(s)=\sum_{j\in J}\sum_{i\in I}\overline{K}_j^M(s)\overline{\theta}_j^{iM}$.

Then

$$\sum_{s\in S}p(s)\cdot\sum_{i\in I}(\overline{x}^{iM}(s)-\omega^i(s))\leqslant 0,\ \forall p(s)\in\mathbb{R}_+^G:\|p(s)\|_1=1.$$

Now, suppose that $\overline{p}_g(s)=0$ for some $g\in G$. By strict monotonicity of the consumers utility functions, each one of them would like to buy r units of good g, implying an aggregate demand equal to $rI>\sum_{i\in I}\omega^i(s)^2$, contradicting $\sum_{i\in I}[\overline{x}^{iM}(s)-\omega^i(s)]\leqslant 0$. Thus $\overline{p}(s)\in\mathbb{R}_{++}^G$, and so if for some $g\in G$ we had $\sum_{i\in I}[\overline{x}_g^{iM}(s)-\omega_g^i(s)]<0$ the price auctioneer could improve his payoff by changing to some other price vector in Π such that $p_g(s)=0$. So condition (ii) is satisfied. □

This concludes the proof of proposition 1. □

Proposition 2. *Under assumptions [A1]–[A3] stated above, each truncated economy $\mathcal{E}^M$ has an equilibrium with endogenous default penalties $(\overline{p}^M,\overline{q}^M,\overline{K}^M,(\overline{x}^{iM},\overline{\theta}^{iM},\overline{\varphi}^{iM},\overline{\Delta}^{iM})_{i\in I},\overline{\lambda})$.*

Proof of Proposition 2. We start by showing that the set of vectors which are candidates to be an equilibrium with endogenous default penalties for an economy $\mathcal{E}^M$ closed, where $M=\mathbb{N}\cup\{\infty\}$.

Claim 4.8. For each $\lambda\in\Lambda^M$, let $T(\mathcal{E}^M,\lambda)\subset\widetilde{X}^i\times\Theta^{iM}\times\Phi^{iM}\times D^{iM}\times\Pi\times[0,1]^{S\times J}$ be the set of equilibriums for the truncated economy $\mathcal{E}^M$ associated with λ. Then the set

$$T^M=\left\{\left(p,q,K,\left(x^i,\theta^i,\varphi^i,\Delta^i\right)_{i\in I},\lambda\right):\left(p,q,K,\left(x^i,\theta^i,\varphi^i,\Delta^i\right)_{i\in I}\right)\in T(\mathcal{E}^M,\lambda)\right\}$$

[2]see note 1 above.

is closed.

Proof. Consider a sequence of equilibrium vectors $\left(p_n, q_n, K_n, (x_n^i, \theta_n^i, \varphi_n^i, \Delta_n^i)_{i\in I}, \overline{\lambda}\right)$ such that $\left(p_n, q_n, K_n, (x_n^i, \theta_n^i, \varphi_n^i, \Delta_n^i)_{i\in I}, \overline{\lambda}\right) \in T(\mathcal{E}^M, \lambda)$ for each $n \in \mathbb{N}$. Since this is a sequence over a compact set, it converges, maybe passing to a subsequence, to a point $(\overline{v}, \overline{w}, \overline{\lambda})$. Suppose there exists $(x^i, \theta^i, \varphi^i, \Delta^i)$ in $\tilde{B}^{iM}(\overline{p}^M, \overline{q}^M, \overline{K}^M)$ such that $U^i_{\overline{\lambda}}(x^i, \theta^i, \varphi^i, \Delta^i) > U^i_{\overline{\lambda}}(\overline{x}^i, \overline{\theta}^i, \overline{\varphi}^i, \overline{\Delta}^i)$. Then, $(\tilde{x}^i, \tilde{\theta}^i, \tilde{\varphi}^i, \tilde{\Delta}^i) := \tau(x^i, \theta^i, \varphi^i, \Delta^i) + (1-\tau)(\overline{x}^i, \overline{\theta}^i, \overline{\varphi}^i, \overline{\Delta}^i)$ is budgetary feasible for agent i at prices $(\overline{p}^M, \overline{q}^M, \overline{K}^M)$, for τ small enough. By quasiconcavity of $U^i_{\overline{\lambda}}$, one gets $U^i_{\overline{\lambda}}(\tilde{x}^i, \tilde{\theta}^i, \tilde{\varphi}^i, \tilde{\Delta}^i) > U^i_{\overline{\lambda}}(\overline{x}^i, \overline{\theta}^i, \overline{\varphi}^i, \overline{\Delta}^i)$. Now, there exists $N_0 \in \mathbb{N}$ such that:

$$\begin{aligned} & \mid U^i_{\overline{\lambda}}(\overline{x}^i, \overline{\theta}^i, \overline{\varphi}^i, \overline{\Delta}^i) - U^i_{\overline{\lambda}}(x_n^i, \theta_n^i, \varphi_n^i, \Delta_n^i) \mid \\ < \; & \frac{1}{2}\Big(U^i_{\overline{\lambda}}(\tilde{x}^i, \tilde{\theta}^i, \tilde{\varphi}^i, \tilde{\Delta}^i) - U^i_{\overline{\lambda}}(\overline{x}^i, \overline{\theta}^i, \overline{\varphi}^i, \overline{\Delta}^i)\Big), \text{ for } n > N_0. \end{aligned}$$

There also exists $N_1 \in \mathbb{N}$ such that:

$$\begin{aligned} & \mid U^i_{\overline{\lambda}}(x_n^i, \mid \theta_n^i, \varphi_n^i, \Delta_n^i) - U^i_{\lambda_n}(x_n^i, \theta_n^i, \varphi_n^i, \Delta_n^i) \mid \\ < \; & \frac{1}{2}\Big(U^i_{\overline{\lambda}}(\tilde{x}^i, \tilde{\theta}^i, \tilde{\varphi}^i, \tilde{\Delta}^i) - U^i_{\overline{\lambda}}(\overline{x}^i, \overline{\theta}^i, \overline{\varphi}^i, \overline{\Delta}^i)\Big), \text{ for } n > N_1. \end{aligned}$$

Hence, $U^i_{\overline{\lambda}}(\tilde{x}^i, \tilde{\theta}^i, \tilde{\varphi}^i, \tilde{\Delta}^i) > U^i_{\lambda_n}(x_n^i, \theta_n^i, \varphi_n^i, \Delta_n^i)$, for $n > \max\{N_0, N_1\}$. There is also $N_2 \in \mathbb{N}$ such that $\mid (U^i_{\lambda_n}(\tilde{x}^i, \tilde{\theta}^i, \tilde{\varphi}^i, \tilde{\Delta}^i) - (U^i_{\overline{\lambda}}(\tilde{x}^i, \tilde{\theta}^i, \tilde{\varphi}^i, \tilde{\Delta}^i) \mid < (U^i_{\overline{\lambda}}(\tilde{x}^i, \tilde{\theta}^i, \tilde{\varphi}^i, \tilde{\Delta}^i) - U^i_{\lambda_n}(x_n^i, \theta_n^i, \varphi_n^i, \Delta_n^i)$, for $n > N_2$. Thus, $(U^i_{\lambda_n}(\tilde{x}^i, \tilde{\theta}^i, \tilde{\varphi}^i, \tilde{\Delta}^i) > U^i_{\lambda_n}(x_n^i, \mid \theta_n^i, \varphi_n^i, \Delta_n^i)$, for $n > \max\{N_0, N_1, N_2\}$. Finally, there exists N_3 such that $(\tilde{x}^i, \tilde{\theta}^i, \tilde{\varphi}^i, \tilde{\Delta}^i) \in \tilde{B}^{iM}(p_n, q_n, K_n)$, for $n > N_3$. But this contradicts the fact that $\left(p_n, q_n, K_n, (x_n^i, \theta_n^i, \varphi_n^i, \Delta_n^i)_{i\in I}, \lambda_n\right) \in T(\mathcal{E}^M, \lambda_n)$, for $n > \max\{N_0, N_1, N_2, N_3\}$. □

Proposition 3. *Under assumptions [A1]–[A3] stated above, each truncated economy $\mathcal{E}^M$ has an equilibrium with endogenous default penalties* $(\overline{p}^M, \overline{q}^M, \overline{K}^M, (\overline{x}^{iM}, \overline{\theta}^{iM}, \overline{\varphi}^{iM}, \overline{\Delta}^{iM})_{i\in I}, \overline{\lambda}^M)$.

Proof of Proposition 3. The problem of the Lawmaker is equivalent to:

$$\begin{array}{ll} \max\limits_{\lambda \in \Lambda_M} & W(x, \theta, \varphi, \Delta) \\ \text{s.t.} & (x, \theta, \varphi, \Delta, \lambda) \in T^M(\mathcal{E}^M) \end{array} \tag{6}$$

A vector $\left(p^M, q^M, K^M, (\overline{x}^{iM}, \overline{\theta}^{iM}, \overline{\varphi}^{iM}, \overline{\Delta}^{iM})\right)$ is an equilibrium with endogenous default penalties for $\mathcal{E}^M$ if and only if it solves (6). Since $T^M(\mathcal{E}^M)$ is contained in the compact set $\tilde{X}^{iM} \times \tilde{\Theta}^{iM} \times \tilde{\Phi}^{iM} \times \tilde{D}^{iM} \times \Pi \times [0,1]^{S\times J}$, over which $W(x,\theta,\varphi,\Delta,\lambda)$ is continuous, $W(x,\theta,\varphi,\Delta,\lambda)$ is bounded over $T^M(\mathcal{E}^M)$. Since $T^M(\mathcal{E}^M)$ is closed, Problem (6) has a solution.

5. ASYMPTOTIC RESULTS.

In this section, we show, by letting $M \to +\infty$, that the original economy $\mathcal{E}$ has an equilibrium. Indeed, we prove that (i) there is a subsequence of $(\overline{p}^M, \overline{q}^M, \overline{K}^M, (\overline{x}^{iM}, \overline{\theta}^{iM}, \overline{\varphi}^{iM}, \overline{\Delta}^{iM})_{i\in I}, \overline{\lambda}^M)$ which converges, and (ii) the cluster point is an equilibrium of the original economy $\mathcal{E}$.

In this part, we will adopt the following convention: $0 \times \infty = 0$ which is also adopted, for example, in Aliprantis and Border (1999), (chapter 1, page 7). This will be important to define the value of a consumer's expected utility when he defaults infinitely in an asset and in a state where his marginal default penalty is zero, or if he gives zero subjective probability to that state.

It follows from our adopted normalization and the definition of the expected delivery rates that the sequence $(\overline{p}^M, \overline{q}^M, \overline{K}^M)$ is bounded, and therefore we can find a convergent subsequence, denoted by $(\overline{p}^{k_M}, \overline{q}^{k_M}, \overline{K}^{k_M})$. Let $(\overline{p}, \overline{q}, \overline{K})$ denote its limit. On the other hand, by the physical market clearing, the sequence $\overline{x}^M$ is bounded, and so we can find a convergent subsequence, denoted by $(\overline{x}^{k_M})$. Let $\overline{x}$ denote its limit.

Now, we cannot have $\overline{p}_{gk_M} \to 0$, for any commodity g and state s, because otherwise we would have, for k_M high enough, the aggregate demand of good g in state s equal to $k_M \times I$, while its aggregate supply would be equal to zero, contracting the fact that the k_M-th truncated economy is in equilibrium.

Remark 5.1.
We can perfectly have, for some asset j, $\overline{K}^{jk_M}(s) = 0$ or converging to zero for all states of nature.

In this paragraph, we state the following lemma which will be useful to guarantee that $\overline{z}^{iM}$ is bounded for each agent $i \in I$. For each asset $j \in J$, let us define the subset $S_j^+ \subset S$ as follows: $S_j^+ := \{s \in S : \ r^j(s) > 0\}$.

LEMMA 1.

(a)If $\lim_{M\to+\infty} \overline{q}_{jM} = 0$ *for some* $j \in J$ *then* $\lim_{M\to+\infty} \overline{K}_{jM} = 0$ *for all* $s \in S_j^+$.

(b)If $\lim_{M\to+\infty} \overline{K}_{jM} = 0$ *but* $\lim_{M\to+\infty} \overline{q}_{jM} \neq 0$*, then* $\lim_{M\to+\infty} \sum_{i\in I} \overline{\theta}_j^{iM} = \lim_{M\to+\infty} \sum_{i\in I} \overline{\varphi}_j^{iM} = 0$*. Moreover,* $\lim_{M\to+\infty} [\overline{q}_{jM} \sum_{i\in I} \overline{\theta}_j^{iM}] = \lim_{M\to+\infty} [\overline{q}_{jM} \sum_{i\in I} \overline{\varphi}_j^{iM}] = 0$.

(c)If If $\lim_{M\to+\infty} \overline{K}_{jM} = 0$ *for all* $s \in S_j^+$ *but* $\lim_{M\to+\infty} \sum_{i\in I} \overline{\theta}_j^{iM} \neq 0$*, then* $\lim_{M\to+\infty} \overline{q}_{jM} = 0$.

Proof. See Appendix A.

Now, to guarantee that $\overline{z}^{iM}$ does not diverge, we must rule out arbitrage opportunities in equilibrium. To this end, we start by proving some auxiliary lemmas:

LEMMA 2. *If* $\overline{\varphi}_j^{iM} \to +\infty$ *for some* $i \in I$ *and some* $j \in J$*, then for each state of nature* $s \in S$*, one has: either (i)* $\frac{\overline{\Delta}_j^{iM}(s)}{r^j(s)\,\overline{\varphi}_j^{iM}} \to 1$*, or (ii)* $\frac{\overline{\Delta}_j^{iM}(s)}{r^j(s)\,\overline{\varphi}_j^{iM}} \to 0$.
Also, if (ii) occurs, it must be true that $\overline{\lambda}_j^{iM}(s) \to 0$*, for some state* s.

Proof. See Appendix A.

Note that the sequence $\overline{z}^{iM}$ takes values on $\overline{\mathbb{R}}^J$, which is compact. Therefore, it must have a convergent subsequence in $\overline{\mathbb{R}}^J$. The next lemma says that its limits belongs to $\overline{\mathbb{R}}^J$.

LEMMA 3. *If consumers attribute subjective positive probability to every state of nature, then no arbitrage opportunity can exist in equilibrium for the truncated economies for* M *sufficiently high, and* $\lim_{M\to+\infty} \overline{z}^{iM}$ *will be finite, for every consumer* $i \in I$.

Proof. See Appendix A.

Obviously, the arguments in the proof of Lemma 3 are also valid for the truncated economies, for M high enough, since we would have in these truncated economies not an infinite excess demand and supply of different assets, but yet a very large one, implying that $\overline{q}_M$ cannot be optimal for the auctioneer, and we would also have very high levels of default, implying that $\overline{\lambda}_M$ cannot be optimal for the Lawmaker.

LEMMA 4. *If* $\lim_{M\to+\infty} \overline{q}_{jM} = 0$*, then* $\lim_{M\to+\infty} [\overline{q}_{jM} \sum_{i\in I} \overline{\theta}_j^{iM}] = \lim_{M\to+\infty} [\overline{q}_{jM} \sum_{i\in I} \overline{\varphi}_j^{iM}] < +\infty$.

Proof. See Appendix A.

LEMMA 5. *For each agent $i \in I$ and for each state $s \in S$,* $\lim_{M\to+\infty} \sum_{j\in J} \left[\overline{K}_{jM} r^j(s)\overline{\theta}_j^{iM} - \overline{\Delta}_j^{iM}(s)\right]$ *is finite.*

Proof. See Appendix A.
Finally, we state one last lemma relating the agents:

LEMMA 6. $\overline{\theta}_j^i$ *and* $\overline{\varphi}_j^M$ *are bounded from above for each pair* $(i, j) \in I \times J$.

Proof. See Appendix A.

LEMMA 7. *The sequence* $\overline{\lambda}_j^i(s)$ *can be, without loss of generality, bounded for each triple* $(i, j, s) \in I \times J \times S$.

Proof. See Appendix A.

Claim 5.1. [Optimality of the cluster point]

Proof of Claim 5.1. By contraposition, let us assume that one of the following cases holds:

(i) There exists $i \in I$, there exists $(\tilde{x}^i, \tilde{\theta}^i, \tilde{\varphi}^i, \tilde{\Delta}^i) \in \text{Int}B^i(\overline{p}, \overline{q}, \overline{K})$ such that:

$$U^i_{\overline{\lambda}}(\tilde{x}^i, \tilde{\theta}^i, \tilde{\varphi}^i, \tilde{\Delta}^i) > U^i_{\overline{\lambda}}(\overline{x}^i, \overline{\theta}^i, \overline{\varphi}^i, \overline{\Delta}^i).$$

(ii) There exists $\hat{\lambda}$ such that:

$$\sum_{i\in I} \beta^i U^i_{\hat{\lambda}}(\overline{x}^i(\hat{\lambda}^i), \overline{\theta}^i(\hat{\lambda}^i), \overline{\varphi}^i(\hat{\lambda}^i), \overline{\Delta}^i(\hat{\lambda}^i)) > \sum_{i\in I} \beta^i U^i_{\overline{\lambda}}(\overline{x}^i(\overline{\lambda}^i), \overline{\theta}^i(\overline{\lambda}^i), \overline{\varphi}^i(\overline{\lambda}^i), \overline{\Delta}^i(\overline{\lambda}^i)).$$

If (i) occurs, then $(\tilde{x}^i, \tilde{\theta}^i, \tilde{\varphi}^i, \tilde{\Delta}^i)$ is budgetary feasible for agent i at prices $(\overline{p}, \overline{q}, \overline{K})$, so is $\tau\ (\tilde{x}^i, \tilde{\theta}^i, \tilde{\varphi}^i, \tilde{\Delta}^i) + (1-\tau)\ (\overline{x}^i, \overline{\theta}^i, \overline{\varphi}^i, \overline{\Delta}^i)$, for $\tau \in [0, 1]$ small enough. By convexity of preferences, we have $\tau\ (\tilde{x}^i, \tilde{\theta}^i, \tilde{\varphi}^i, \tilde{\Delta}^i) + (1 - \tau)\ (\overline{x}^i, \overline{\theta}^i, \overline{\varphi}^i, \overline{\Delta}^i) \succ^i (\overline{x}^i, \overline{\theta}^i, \overline{\varphi}^i, \overline{\Delta}^i)$. By continuity of preferences, $\tau\ (\tilde{x}^i_{k_M}, \tilde{\theta}^i_{k_M}, \tilde{\varphi}^i_{k_M}, \tilde{\Delta}^i_{k_M}) + (1-\tau)\ (\overline{x}^i, \overline{\theta}^i, \overline{\varphi}^i, \overline{\Delta}^i)$ would have been chosen instead of $(\overline{x}^i_{k_M}, \overline{\theta}^i_{k_M}, \overline{\varphi}^i_{k_M}, \overline{\Delta}^i_{k_M})$, a contradiction.
If (ii) occurs, by the maximum theorem, the mapping $\lambda \mapsto U^i_\lambda(\overline{x}^i_{k_M}(\lambda^i), \overline{\theta}^i_{k_M}(\lambda^i), \overline{\varphi}^i_{k_M}(\lambda^i), \overline{\Delta}^i_{k_M}(\lambda^i))$ is continuous, so the mapping $\lambda \mapsto \sum_{i\in I} \beta^i U^i_\lambda(\overline{x}^i_{k_M}(\lambda^i), \overline{\theta}^i_{k_M}(\lambda^i), \overline{\varphi}^i_{k_M}(\lambda^i), \overline{\Delta}^i_{k_M}(\lambda^i))$ is also continuous, for each k_M and converges pointwise to the mapping $\lambda \mapsto \sum_{i\in I} \beta^i U^i_\lambda(\overline{x}^i(\lambda^i), \overline{\theta}^i(\lambda^i), \overline{\varphi}^i(\lambda^i), \overline{\Delta}^i(\lambda^i))$ is continuous. By the discussion on convergence above, the Lawmaker is allowed to choose for each (i, j, s)

a value $\lambda_j^i(s)$ sufficiently high such that agent i will repay fully in state s the returns he owes from the sale of asset j, and note that for k_M sufficiently large this choice is available for the Lawmaker in the truncated economy $\mathcal{E}^{k_M}$. If there is $\hat{\lambda}$ such that $\sum_{i\in I} \beta^i U^i_{\hat{\lambda}}(\overline{x}^i(\hat{\lambda}^i), \overline{\theta}^i(\hat{\lambda}^i), \overline{\varphi}^i(\hat{\lambda}^i), \overline{\Delta}^i(\hat{\lambda}^i)) > \sum_{i\in I} \beta^i U^i_{\hat{\lambda}}(\overline{x}^i(\overline{\lambda}^i), \overline{\theta}^i(\overline{\lambda}^i), \overline{\varphi}^i(\overline{\lambda}^i), \overline{\Delta}^i(\overline{\lambda}^i))$, it must be within these bounds, and since $\sum_{i\in I} \beta^i U^i_{\lambda}(\overline{x}^i_{k_M}(\lambda^i), \overline{\theta}^i_{k_M}(\lambda^i), \overline{\varphi}^i_{k_M}(\lambda^i), \overline{\Delta}^i_{k_M}(\lambda^i))$ converges $\sum_{i\in I} \beta^i U^i_{\lambda}(\overline{x}^i(\lambda^i), \overline{\theta}^i(\lambda^i), \overline{\varphi}^i(\lambda^i), \overline{\Delta}^i(\lambda^i))$ and $\overline{\lambda}_{k_M}$ converges to $\overline{\lambda}$ we would have:

$$\begin{aligned} &\sum_{i\in I} \beta^i U^i_{\lambda}(\overline{x}^i_{k_M}(\hat{\lambda}^i), \overline{\theta}^i_{k_M}(\hat{\lambda}^i), \overline{\varphi}^i_{k_M}(\hat{\lambda}^i), \overline{\Delta}^i_{k_M}(\hat{\lambda}^i)) \\ > &\sum_{i\in I} \beta^i U^i_{\lambda}(\overline{x}^i_{k_M}(\overline{\lambda}^i), \overline{\theta}^i_{k_M}(\overline{\lambda}^i), \overline{\varphi}^i_{k_M}(\overline{\lambda}^i), \overline{\Delta}^i_{k_M}(\overline{\lambda}^i)), \end{aligned}$$

a contradiction.

APPENDIX A

Proofs of lemmas 1 to 7.

Proof of Lemma 1.

- Proof of Item (a). We cannot have $\overline{q}_{jk_M} \to 0$ unless $\overline{K}_{jk_M} \to 0$ and $\overline{\lambda}^i_{jk_M} \to 0$ for all $s \in S_j^+$ and for all $i \in I$, because otherwise, for k_M high enough, we would have every agent wanting to buy k_M units of this asset, and no agent would want to sell it, since even if he does not repay he suffers a certain default penalty in return of virtually null price he receives in period 0.

- Proof of Item (b).If $\lim_{M\to+\infty} \overline{K}_{jk_M} = 0$, then $\lim_{M\to+\infty} \overline{K}_{jk_M}\, r^j(s) = 0$ for all $s \in S_j^+$. So, since $\lim_{M\to+\infty} \overline{q}_{jk_M} \neq 0$, there will be an order beyond which no agent will buy the asset, that is beyond some order $\lim_{M\to+\infty} \sum_{i\in I} \overline{\theta}^{iM}_j = 0$. Because beyond some order each truncated economy is in equilibrium, $\sum_{i\in I} \overline{\varphi}^{iM}_j = 0$.

- Proof of Item (c). By contraposition, let us assume that $\lim_{M\to+\infty} \overline{K}_{jM} = 0$ for all $s \in S_j^+$, $\lim_{M\to+\infty} \sum_{i\in I} \overline{\theta}^{iM}_j \neq 0$ and $\lim_{M\to+\infty} \overline{q}_{jM} \neq 0$. It follows immediately from item (b) that $\lim_{M\to+\infty} \sum_{i\in I} \overline{\theta}^{iM}_j = 0$, a contradiction. □

Proof of Lemma 2. Let $i \in I$ and $j \in J$ such that $\overline{\varphi}_j^{iM} \to +\infty$. First, let us assume by contraposition, that there is a state of nature $s \in S_j^+$ such that $\frac{\overline{\Delta}_j^{iM}}{r^j(s)\overline{\varphi}_j^{iM}} \to 0$ and $\overline{\lambda}_j^{iM}$ does not converge to zero. Then, $\frac{r^j(s)\overline{\varphi}_j^{iM} - \overline{\Delta}_j^{iM}}{r^j(s)\overline{\varphi}_j^{iM}} \to 1$, which implies that $[r^j(s)\overline{\varphi}_j^{iM}] \to +\infty$. Therefore, $\overline{\lambda}_j^{iM}[r^j(s)\overline{\varphi}_j^{iM}] \to +\infty$. Therefore, the expected utility of agent i is going to $-\infty$ and he must not be maximizing his utility.
Now, since for each $i \in I$, $\overline{x}^{iM}$ converges, then the marginal utility of income $\overline{\mu}_1^{iM}(s)$ also converges, and let us denote by $\overline{\mu}_1^i(s)$ its limit. Let us distinguish the following cases:

- For assets such that $\lim_{M\to+\infty} \overline{\lambda}_j^{iM} > \overline{\mu}_1^i(s)$, we will have that beyond some order, equation (C.3) (see Appendix C) is verified with equality and with $\overline{\mu}_\Delta^{iM}(s) > 0$, implying that $\overline{\Delta}_j^{iM}(s) = r^j(s)\,\overline{\varphi}_j^{iM}$, and therefore $\frac{\overline{\Delta}_j^{iM}(s)}{r^j(s)\,\overline{\varphi}_j^{iM}} = 1$, beyond some order.
- For assets such that $\lim_{M\to+\infty} \overline{\lambda}_j^{iM} < \overline{\mu}_1^i(s)$, we will have that beyond some order, equation (C.3) (see Appendix C) is verified with strict inequality implying that $\overline{\mu}_\Delta^{iM}(s) = 0$, and therefore $\frac{\overline{\Delta}_j^{iM}(s)}{r^j(s)\,\overline{\varphi}_j^{iM}} = 0$, beyond some order.
- Finally, for assets such that $\lim_{M\to+\infty} \overline{\lambda}_j^{iM} = \overline{\mu}_1^i(s)$, agent i will have in the limit $\Delta_j^i(s)$ chosen such that the budget constraint in consumption space given the vector $(\Delta_j^i(s),\ j \in J)$ passes exactly through the point $\overline{x}^i(s)$. Now, if $\lim_{M\to+\infty} \frac{\overline{\Delta}_j^{iM}(s)}{r^j(s)\,\overline{\varphi}_j^{iM}} < 1$, this would imply that $\lim_{M\to+\infty} \frac{\overline{r^j(s)\,\overline{\varphi}_j^{iM} - \Delta_j^{iM}(s)}}{r^j(s)\,\overline{\varphi}_j^{iM}} > 0$, and so $\lim_{M\to+\infty} \overline{[r^j(s)\,\overline{\varphi}_j^{iM} - \Delta_j^{iM}(s)]} = +\infty$. But, since $\lim_{M\to+\infty} \overline{\lambda}_j^{iM} > 0$, agent i's expected utility would be converging to $-\infty$. Then, we must have $\lim_{M\to+\infty} \frac{\overline{r^j(s)\,\overline{\varphi}_j^{iM} - \Delta_j^{iM}(s)}}{r^j(s)\,\overline{\varphi}_j^{iM}} = 0$, implying that $\lim_{M\to+\infty} \frac{\overline{\Delta}_j^{iM}(s)}{r^j(s)\,\overline{\varphi}_j^{iM}} = 1$.

□

Proof of Lemma 3. Suppose by contraposition that $|\ \overline{z}^{iM}\ | \to +\infty$, then it must be true that for some asset $j \in J$, $\overline{\varphi}^{iM} \to +\infty$. Then, by Lemma 2, we know that the payment rate for agent i on this asset will be at the limit either equal to 0 or equal to 1 and it will be only equal to zero when the utility penalty for agent i with respect to asset j is going to zero. If an asset‘portfolio $z = \theta - \varphi$ offers an arbitrage opportuniuy for agent i, in a truncated economy $\mathcal{E}^M$, then he will wanv to exchange the ma—imum amount of this portfolio allowed in this truncated economy. If there is no order beyond which this arbitrage opportunity disappears, then we will have all assets j for which $\varphi^j > 0$, being traded infinitely in the truncated economy. So, the returns an agent i has to pay on this portfolio z will tend to $Q^i \otimes R$, where:

$$Q(\lambda_j^i(s) := \begin{cases} 1 & \text{if } \lambda_j^i(s) > 0, \\ 0 & \text{if } \lambda_j^i(s) = 0, \end{cases} \quad \text{and } Q^i := \Big(Q(\lambda_j^i(s))\Big)_{(s,j)\in S\times J}.$$

One arbitrage possibility would be the existence of a portfolio $z' = \theta' - \varphi'$ such that $\theta' \neq \varphi'$, $qz' \leq 0$ and $K \otimes R\theta' > (Q^i \otimes R)\theta'$ for at least some agent i.[1] Then, the consumers for which this condition is verified would want to buy an infinite multiple amount of portfolio z' (or the maximum they can buy, in a truncated economy), leading to an excess demand of some assets and an excess supply in other assets, implying that (q, K, λ) cannot be an equilibrium triple of asset price vector, expected deliveries rate matrix and default penalties.

Other possibility wold be to have a portfolio $\hat{z} = \hat{\theta} - \hat{\varphi}$, two disjoint sets of consumers, I_1 and I_2, and a partition (S_1, S_2) of S such that agents in I_1 give subjective 0 probability to states in S_2, agents in I_2 give subjective 0 probability to states in S_1, $\Big(K\otimes R(\hat{\theta}+\hat{\varphi})\Big)(s) > \Big(Q^i\otimes R(\hat{\theta}+\hat{\varphi})\Big)(s) \geq 0,\ s \in S_1,\ i \in I_1$ and $\Big(K\otimes R(\hat{\theta}+\hat{\varphi})\Big)(s) > \Big(Q^i\otimes R(\hat{\theta}+\hat{\varphi})\Big)(s) \geq 0,\ s \in S_2,\ i \in I_2$. Then, agents in both I_1 and I_2 would like to buy and sell simultaneously an unlimited amount of portfolio $\hat{z}$.[2] The amounts sold by agents in I_1 would be purchased, at the current price vector q, by agents in I_2, and vice versa. We prevent this phenomenon from happening by restricting consumers' subjective expectations to give positive probability to every state of nature $s \in S$.

Yet another possibility would be to have an asset portfolio $\hat{z} = (\hat{\theta} - \hat{\varphi})$, two disjoint sets of consumers, I_1 and I_2, such that $K \otimes R(\tilde{\theta} + \tilde{\varphi}) - (Q^i \otimes R)(\tilde{\theta}+\tilde{\varphi}) > 0,\ i \in I_1$ and $K\otimes R(\tilde{\theta}+\tilde{\varphi}) - (Q^i\otimes R)(\tilde{\theta}+\tilde{\varphi}) > 0,\ i \in I_2$. Then again, agents in both I_1 and I_2 would like to buy and sell simultaneously, from each other, an unlimited amount of portfolio $\tilde{z}$, at the current price vector q, and this would not necessarily cause excess demand or excess supply of assets. But we can nonetheless rule out this possibility by noting that every agent in $I_1 \bigcup I_2$ would become infinitely wealthy in some state of nature, and by monotonicity of his preferences his demand would be

[1]Note that if such an arbitrage opportunity exists, the agent will become infinitely healthy in each state of nature of period 1, implying that his marginal utility of income will be virtually equal to 0, and he will chooses to repay fully his debts as long as the default penalty is different from 0. Check equation (C.3) in Appendix C.

[2]That is, to buy $+\infty \cdot (\hat{\theta} - \hat{\varphi})$ and to sell $+\infty \cdot (\hat{\theta} - \hat{\varphi})$, at net price of zero.

infinite, whereas goods supply is fixed at a finite level, in all truncated economies, contradicting the fact that we are in equilibrium.[3] □

Proof of Lemma 4. $\lim_{M\to+\infty} \overline{q}_{jM} = 0$ implies, by Item (a) of Lemma 1, that $\lim_{M\to+\infty} \overline{K}_{jM}(s) = 0$, for all $s \in S^+$. This can happen in three conceivable ways:

(i) $\lim_{M\to+\infty} \sum_{i\in I} \overline{\Delta}^i_{jM}(s) = 0$, for all $s \in S^+$. Because the truncated economies are in equilibrium for M high enough, we must have, beyond some order, $\lim_{M\to+\infty} \overline{K}_{jM}(s) \sum_{i\in I} r_j(s)\overline{\theta}^i_{jM} = \lim_{M\to+\infty} \sum_{i\in I} \overline{\Delta}^i_{jM}(s) = 0$. That is, asset j buyers aggregate expected returns tend to zero in every state of nature, so their aggregate expenditure in this asset purchase must be tending also to zero. Since the truncated economy is in equilibrium, the total revenue from asset j sales must also approach zero, that is: $\lim_{M\to+\infty} \overline{q}_{jM} \sum_{i\in I} \overline{\theta}^i_{jM} = \lim_{M\to+\infty} \overline{q}_{jM} \sum_{i\in I} \overline{\varphi}^i_{jM} = 0$.

(ii) $0 < \lim_{M\to+\infty} \sum_{i\in I} \overline{\Delta}^i_{jM}(s) < +\infty$, for all $s \in S^+$. Again, we must have, beyond some order, $\lim_{M\to+\infty} \overline{K}_{jM}(s) \sum_{i\in I} r_j(s)\overline{\theta}^i_{jM} = \lim_{M\to+\infty} \sum_{i\in I} \overline{\Delta}^i_{jM}(s) < +\infty$. That is, asset j buyers' aggregate expected returns do not tend to $+\infty$ in no state of nature, so their aggregate expenditure in this asset purchase should not be tending also to $+\infty$, because otherwise they would not be maximizing their utility. Since the truncated economy is in equilibrium, $\lim_{M\to+\infty} \overline{q}_{jM} \sum_{i\in I} \overline{\theta}^i_{jM} = \lim_{M\to+\infty} \overline{q}_{jM} \sum_{i\in I} \overline{\varphi}^i_{jM} < +\infty$.

(iii) $\lim_{M\to+\infty} \sum_{i\in I} \overline{\Delta}^i_{jM}(s) = +\infty$, for all $s \in S^+$ and $\lim_{M\to+\infty} r_j(s) \sum_{i\in I} \overline{\varphi}^i_{jM}(s) = +\infty$. But, this contradicts Lemma 2.

□

Proof of Lemma 5. The budget constraints of an agent $i \in I$ at the second period can be written at equilibrium as follows: for each state $s \in S$,

$$\overline{p}^M \cdot (\overline{x}^{iM}(s) - \omega^i(s)) = \sum_{j\in J} [\overline{K}_{jM} r^j(s)\overline{\theta}^{iM}_j - \overline{\Delta}^{iM}_j(s)].$$

Then, in view of our normalization, one gets easily:

$$\sum_{j\in J} [\overline{K}_{jM} r^j(s)\overline{\theta}^{iM}_j - \overline{\Delta}^{iM}_j(s)] \le \max_{g\in G} \omega^i(s,g).$$

[3] If the reader finds another type of arbitrage opportunity, we are very certain that it cab be ruled out by the same kind of arguments. The intuition is that there can be no free lunches.

Moreover,

$$\sum_{j\in J}\left[\overline{K}_{jM} r^j(s)\overline{\theta}_j^{iM} - \overline{\Delta}_j^{iM}(s)\right] \geq -\max_{g\in G}\,\omega^i(s,g).$$

Proof of Lemma 6.
For each agent $i \in I$, for each state of nature $s \in S$ and for each asset $j \in J$, let us define agent i's repayment rate as follows:

$$K_j^i(s) := \begin{cases} 1\frac{\Delta_j^i(s)}{r^j(s)\varphi_j^i} & \text{if } r^j(s)\varphi_j^i > 0, \\ 0 & \text{if } r^j(s)\varphi_j^i = 0, \end{cases}$$

$\overline{K}_j^{iM}(s)$ takes values on the interval $[0,1]$ for each triple (i,j,s), and it thus has a convergent subsequence, $\overline{K}_j^i(s)$. By Lemma 3, the vector $\overline{z}^{iM}$ has also a convergent subsequence; let a^i denote its limit. So, if consumer i chooses in the limit to buy θ_j^i (finite or infinite) units of asset j, he will sell in the limit $\varphi_j^i = \theta_j^i - a_j^i$ units of asset j. Note that the price for his asset portfolio remains fixed at $\overline{q}a^i$. So, depending on his choice of the vector θ^i, his net effective return across all states will approach:

$$\overline{K} \otimes R\theta^i - \overline{K}^i \otimes (\theta^i - a^i) = (\overline{K} - \overline{K}^i) \otimes R\theta^i + \overline{K}^i \otimes Ra^i,$$

where $\overline{K}^i = (\overline{K}_j^i(s))_{(s,j)\in S\times J}$. The term $\overline{K}^i \otimes Ra^i$ is fixed, but since $(\overline{K} - \overline{K}^i) \otimes R\theta^i$ depends on θ^i, the consumer could change his net effective returns in all states while keeping $\overline{z}^i$ and $\overline{K}^i$ fixed. If he increases θ_j^i also φ_j^i (in order to keep $\overline{z}_j^i$ unchanged), and the default penalty he suffers in state s, $\overline{\lambda}_j^i(s)\Big(1-\overline{K}_j^i(s)\Big)r^j(s)\varphi_j^i$, will increase unless either $\overline{\lambda}_j^i(s) = 0$ or $\overline{K}_j^i(s) = 1$. Therefore, unless $\overline{\lambda}_j^i(s)\Big(1 - \overline{K}_j^i(s)\Big) = 0$, by increasing unboundedly θ_j^i and φ_j^i while keeping $\overline{z}^i$ and $\overline{K}^i$ fixed, the agent would suffer an infinite default penalty in state of nature s, and thus, his expected utility would be equal to $-\infty$.

For the case where $\overline{\lambda}_j^i(s)\Big(1 - \overline{K}_j^i(s)\Big) = 0$, we begin by noting that every vector θ^i with one or more infinite coordinates must lay in some ray from the origin of $\mathbb{R}_+^J$; that is, each vector θ^i with one or more infinite coordinates is an infinite multiple of all the finite vectors in that ray. We show that each finite vector of $\mathbb{R}_+^J$ either cannot be feasibly multiplied infinitely (as in case (i) below), or strictly dominates his infinite multiple in terms of utility maximization (as in cases (ii) and (iii) below), or dominates weakly his infinite multiple in terms of utility maximization (as in cases (iv) below). Every vector $\theta \in \mathbb{R}_+^J$ belongs to one of the four possible disjoint cases:

(i) Consider an arbitrary vector $\theta \in \mathbb{R}^J_+$ such that $\Big(\overline{K}(s) - \overline{K}^i(s)) \otimes R(s)\theta \geq 0$, for all $s \in S$, with strict inequality for at least one s. Then, the consumer could increase unboundedly his expected net effective returns in at least one state, without decreasing it in any other, by multiplying θ by $+\infty$. But by doing that he would violate the upper bounds of Lemma 5.

(ii) Consider an arbitrary vector $\theta \in \mathbb{R}^J_+$ such that $\Big(\overline{K}(s) - \overline{K}^i(s)) \otimes R(s)\theta \leq 0$, for all $s \in S$, with strict inequality for at least one s. Then, by multiplying θ by some $\gamma > 1$, the agent would be simply decreasing his net effective return in at least one state of nature, without increasing it in any other state, thus decreasing his attainable level of expected utility.

(iii) Consider an arbitrary vector $\theta \in \mathbb{R}^J_+$ such that $\Big(\overline{K}(s) - \overline{K}^i(s)) \otimes R(s)\theta > 0$ for at least one state s and $\Big(\overline{K}(s') - \overline{K}^i(s')) \otimes R(s')\theta > 0$ for another state s'. By multiplying θ by some $\gamma > 1$, the agent would increase his net effective returns in state s and decrease it in state s'. For γ high enough his marginal utility of income in state s would be approaching 0, while his marginal utility of income at state s' would be approaching $+\infty$, implying that he would not be maximizing his expected utility.

(iv) Consider an arbitrary non–null vector $\theta \in \mathbb{R}^J_+$ such that $\Big(\overline{K}(s) - \overline{K}^i(s)) \otimes R(s)\theta = 0$, for all $s \in S$. Then the consumer would be indifferent between his vector and the vector $\gamma\,\theta$, for any $\theta \in \mathbb{R}^+$. Thus, he could choose, without loss of generality, the lowest γ such that $\gamma\,\theta_j - a^i \geq 0$ for all $j \in J$.

Proof of Lemma 7.
Suppose that $\overline{\lambda}^i_{jM}(s)$ diverges. Since $\overline{x}^i_M(s)$ is converging, so is the consumer's marginal utility of income $\overline{\mu}_{1M}(s)$, and thus the difference $\overline{\lambda}^i_{jM}(s) - \overline{\mu}_{1M}(s)$ will be at some point positive, and by inspecting equations (C.3) and (C.4) in appendix C, we see that when $\overline{\lambda}^i_{jM}(s) - \overline{\mu}_{1M}(s)$ becomes positive $\overline{\mu}^r_{\Delta_j M}(s)$ must also be positive, implying $r^j(s)\overline{\varphi}^i_{jM} - \overline{\Delta}^i_{jM}(s) = 0$. That is, when $\overline{\lambda}^i_{jM}(s)$ goes to infinity $r^j(s)\overline{\varphi}^i_{jM} - \overline{\Delta}^i_{jM}(s)$ does not simply go asymptotically to 0; there is a $\hat{\lambda}^i_{jM}(s)$ beyond which it becomes exactly 0, for every M such that $\overline{\lambda}^i_{jM}(s) > \hat{\lambda}^i_{jM}(s)$. So, we can, with no loss of generality, restrict the Lawmaker to choose $\lambda^i_j(s)$ between 0 and $\hat{\lambda}^i_{jM}(s)$.

APPENDIX B

Why we do not need to assume that rank R=J.

The reader has perhaps noticed that, in any step of the proof of Theorem 1, we assume that rank $R = J$, as is commonly done in general equilibrium economies with incomplete markets (GEI). It turns out that when default is allowed this assumption becomes meaningless, because the dimension of the space of possible income transfers between states of nature is endogenous, and independent of the rank of R. In this appendix we explain in detail why.

For each agent, the effective returns matrix he faces on his long position (that is, on θ^i) is given by $\overline{K} \otimes R$. The Kronecker product does not verify the property that rank $\overline{K}\otimes = J$ if and only if rank $\overline{K} =$ rank $R = J$. By the contrary, it is trivial to see to see that we may have rank $\overline{K} \otimes R = J$ even if rank $\overline{K} < J$, and on the other hand rank $\overline{K} =$ rank $R = J$ does not imply rank $\overline{K} \otimes R = J$.

For the first case, suppose that:

$$\begin{pmatrix} \overline{K}_1(1) & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & & & \vdots \\ \vdots & & \overline{K}_j(j) & \ddots & 0 \\ 0 & 0 & \cdots & 0 & \overline{K}_J(J) \\ 0 & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & & & \vdots \\ \vdots & & 0 & & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix}$$

such that

$$\overline{K} \otimes R = \begin{pmatrix} \overline{K}_1(1)r^1(1) & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & & & \vdots \\ \vdots & & \overline{K}_j(j)r^j(j) & \ddots & 0 \\ 0 & 0 & \cdots & 0 & \overline{K}_J(J)r^J(J) \\ 0 & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & & & \vdots \\ \vdots & & 0 & & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix}.$$

whose rank is equal to J, no matter what is the rank of R, as long as $\overline{K}_j(s)r^j(s) \neq 0$, for all $j = s$.

For the second case, suppose there are three states of nature and two assets, with return matrix:

$$R = \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{pmatrix}.$$

and suppose that

$$\overline{K} = \begin{pmatrix} 1 & 1 \\ 1/2 & 1 \\ 1/3 & 1 \end{pmatrix}.$$

such that

$$\text{rank } (\overline{K} \otimes R) = \text{rank } \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix} = 1$$

Another pathological case would be to have rank$R = J$, and, for some $j \in J$, $\overline{K}_j(s) = 0$, $\forall s \in S$. Then $\overline{K}$ has one column of zeros and the rank of $\overline{K} \otimes R$ will be, at most, equal to $J - 1$.

By other words, *even if the J assets available in the economy are not linearly independent, in terms of promised returns, they can become linearly independent in terms of effective payments, which is an integral part of the equilibrium.* On the other hand, it is also possible that default makes some assets that are linearly independent in terms of promised returns redundant in terms of effective returns. The set of positive (received) returns available for all agents in equilibrium is equal to $P := \{(\overline{K} \otimes R)\theta : \theta \in \mathbb{R}^j_+\}$, which is a convex cone in the positive orthant of $\mathbb{R}^J_+$, of dimension no greater than J.

On the short sales side, the effective returns that agent i will optimally choose to pay in equilibrium are given by the matrix $\overline{K}^i \otimes R$. Again, this matrix may have rank equal or smaller than J independently of the ranks of both $\overline{K}^i$ and R. The set of negative (payed) returns available for agent i is equal to $N^i := \{(K^i \otimes \varphi) : \varphi \in \mathbb{R}^J_+, \ K^i \in [0,1]\}^S$.

If we fix j, and set $K^i_j(1) = 1$ and $K^i_j(s) = 0$ for $s \neq 1$, we see that agent i is able to "transform" asset j in one in which he has to repay only in state 1. By the same reasoning, agent i could choose to repay something to asset j's buyer only in state 2, or only in state 3, and so on. But this implies that $N^i = \mathbb{R}^S_+$, $\forall i \in I$, even if rank $R = 1$. Then, the set , $T := \{\alpha - \beta : \alpha \in P, \beta \in \mathbb{R}^S_+\}$,of all possible net return profiles available for all agents coincides with $\mathbb{R}^S$, even if dim $P = 1$, if for all $s \in S$, there exists $j \in J$ such that $\overline{K}_j(s) r^j(s) > 0$.

Thus the sticking reality is that when default is allowed for, an agent can conceivably choose any vector of income transfer across the S states of nature, even if he has only one linearly independent asset available. The greater or smaller easiness an agent will have to choose his most preferred point in N^i will obviously depend on his marginal utility default penalties. But in our setting these are chosen in order to maximize social welfare, and to achieve that goal, the Lawmaker must try to make each individual as best off as possible, without hurting too much all others. Nonetheless, the most linearly independent assets are presented to an agent, the greater is the set of possible income transfers available to him without resorting to default, and the more likely it is that his most preferred point in N^i can be achieved without needing to default. In particular, if rank $R = J$, then any point in $\mathbb{R}^S_-$ can be achieved by every agent without default, and if $\overline{K}_j(s) = 1$, for all (s, j), then we are assured that rank $\overline{K} \otimes R$ =rank $R = S$, so any point in $\mathbb{R}^S$ can be attained.

APPENDIX C

Agents' optimization problem.

Agent i's problem can be written as follows:

$$\max_{x,\theta,\varphi,\Delta} \Big\{ V^i(x^i) - \sum_{s\in S}\sum_{j\in J} \lambda^i_j(s) \big[r^j(s)\varphi^i_j - \Delta^i_j(s) \big] \Big\}$$
$$\text{s.t.} \quad p(0)\cdot x^i(0) + q\cdot z^i \le p(0)\cdot \omega^i(0),$$
$$p(s)\cdot x^i(s) + \sum_{j\in j} \Delta^i_j(s)(s) \le p(s)\cdot \omega^i(s) + \sum_{j\in J} K^j(s) r^j(s)\theta^i_j,$$
$$0 \le \Delta^i_j(s) \le r^j(s)\varphi^i_j, \ \forall s, \ \forall j,$$
$$z^i_j = \theta^i_j - \varphi^i_j, \ \forall j,$$
$$\theta^i_j \ge 0, \ \forall j,$$
$$\varphi^i_j \ge 0, \ \forall j,$$

The associated Lagrangian is therefore:

$$\begin{aligned}
\mathcal{L} \ =\ & V^i(x^i) - \sum_{s\in S}\sum_{j\in J} \lambda^i_j(s) \big[r^j(s)\varphi^i_j - \Delta^i_j(s) \big] \\
+\ & \mu_0 [p(0)\cdot\omega^i(0) - p(0)\cdot x^i(0) - q\cdot(\theta^i - \varphi^i)] \\
+\ & \sum_{s=1}^{S} \mu_1(s) \Big[p(s)\cdot\omega^i(s) + \sum_{j\in J} K^j(s) r^j(s)\theta^i_j - p(s)\cdot x^i(s) - \sum_{j\in J} \Delta^i_j(s) \Big] \\
+\ & \sum_{s=1}^{S} \mu^r_{\Delta_j}(s) [r^j(s)\varphi^i_j - \Delta^i_j(s)].
\end{aligned}$$

The necessary Kuhn–Tucker conditions are:

$$\frac{\partial \mathcal{L}}{\partial x_g^i(0)} = \frac{\partial V^i}{\partial x_g^i(0)} - \mu_0 p_g(0) \leq 0,\ x_g^i(0) \geq 0,\ x_g^i(0)\frac{\partial \mathcal{L}}{\partial x_g^i(0)} = 0,\ \forall g.$$

$$\frac{\partial \mathcal{L}}{\partial x_g^i(s)} = \frac{\partial V^i}{\partial x_g^i(0)} - \mu_1(s) p_g(s) \leq 0,\ x_g^i(s) \geq 0,\ x_g^i(s)\frac{\partial \mathcal{L}}{\partial x_g^i(s)} = 0,\ \forall g,\ \forall s.$$

$$\frac{\partial \mathcal{L}}{\partial \theta_j^i} = -\mu_0 q_j + \sum_{s=1}^{S} \mu_1(s) K^j(s) r^j(s) \leq 0,\ \theta_j^i \geq 0,\ \theta_j^i \frac{\partial \mathcal{L}}{\partial \theta_j^i} = 0,\ \forall j \quad \text{(C.1)}$$

$$\frac{\partial \mathcal{L}}{\partial \varphi_j^i} = \sum_{\lambda_j^i(s) r^j(s)} + \mu_0 q_j + \sum_{s=1}^{S} \mu_{Delta_j}^r(s) r^j(s) \leq 0,\ \varphi_j^i \geq 0,\ \varphi_j^i \frac{\partial \mathcal{L}}{\partial \varphi_j^i} = 0,\ \forall j \quad \text{(C.2)}$$

$$\frac{\partial \mathcal{L}}{\partial \Delta_j^i(s)} = \lambda_j^i(s) - \mu_1(s) - \mu_{\Delta_j}^r(s) \leq 0,\ \Delta_j^i(s) \geq 0,\ \Delta_j^i(s)[\lambda_j^i(s) - \mu_1(s) - \mu_{\Delta_j}^r(s)] = 0,\ \forall j,\ \forall s. \quad \text{(C.3)}$$

$$\frac{\partial \mathcal{L}}{\partial \mu_0} = p(0)\cdot\omega^i(0) - p(0)\cdot x^i(0) - q\cdot(\theta^i - \varphi^i) \geq 0,\ \mu_0 \geq 0,\ \mu_0 \frac{\partial \mathcal{L}}{\partial \mu_0} = 0.$$

$$\frac{\partial \mathcal{L}}{\partial \mu_1(s)} = p(s)\cdot\omega^i(s) - p(s)\cdot x^i(s) + \sum_{j\in J}[K^j(s) r^j(s)\theta_j^i - \Delta_j^i(s)],\ \mu_1(s) \geq 0,\ \mu_1(s)\frac{\partial \mathcal{L}}{\partial \mu_1(s)} = 0,\ \forall s.$$

$$\frac{\partial \mathcal{L}}{\partial \mu_{\Delta_j}^r(s)} = r^j(s)\varphi_j^i - \Delta_j^i(s) \geq 0,\ \mu_{\Delta_j}^r(s) \geq 0,\ \mu_{\Delta_j}^r(s)\frac{\partial \mathcal{L}}{\partial \mu_{\Delta_j}^r(s)} = 0,\ \forall s. \quad \text{(C.4)}$$

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