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The Role of Market-Implied Severity Modeling for Credit VaR*

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In this paper a beta-component mixture is proposed to model the market-implied severity. Recovery rates are extracted and identified from credit default swaps instead of using defaulted bonds instead using defaulted bonds because it allows us to identify recovery rates of low probability of default companies. An empirical analysis is carried out and the results show that a single beta distribution is rejected as a correct specification for implied severity while a beta-component mixture is accepted. Furthermore, the importance of this modeling approach is highlighted by focusing on its role for credit VaR.

Key Words: Implied severity; Credit default swaps; Beta-component mixture; Credit VaR.
JEL Classification Numbers: G13, G20.

1. INTRODUCTION

At present there is a growing interest in modeling severity, which is defined as one minus the recovery rate. It is essential to approximate the severity distribution because risk quantities, such as expected credit loss, loss given default and credit VaR, rely on it. Usually in credit risk practice, it has been approximated by the analysis of recovery rates on defaulted bond issues (Altman, Brady and Sinori, 2005 and Acharya, Bharath and Srinivasan, 2007), eventhough there exists a lack of data on recoveries. The main weakness of this approach is that it does not allow to estimate the

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severity distribution of low probability of default companies, which have not enough defaulted bonds to estimate recovery rates accurately. Most of the industry-sponsored models, such as Portfolio Manager, CreditMetrics and Moody's KMV model, treat recovery rates as stochastic variables modeled through a beta distribution. Although there is no theoretical reason that this is the right shape, it has been widely used in practice to describe the observed behavior of recovery rates since beta distribution is one of the few common "named" distributions that give probability 1 to a finite interval. Nevertheless, in the related literature, there is strong evidence that the recovery rate distribution may exhibit several local modes (Asarnow and Edwards, 1995, Gourieroux and Monfort, 2006, Hagmann, Renault and Scaillet, 2005, Renault and Scaillet, 2004, and Schuermann, 2005, among others). The several modes can arise from different periods (recession and expansion), different types of collateral securing the instruments or from various seniority levels in the same data set (senior secured, senior unsecured, subordinated and junior subordinated). Using data from Moody's Default Risk Service Database, Schuermann (2005) illustrates that recovery rate distributions conditioned to the stage of the business are clearly multimodal. Renault and Scaillet (2004) shows that nonparametric plots of the recovery function frequently exhibit more than two local modes using data from Standard & Poor's/PMD classified by seniority and by industry. The presence of multimodality can be suggestive of more than one underlying unimodal distribution, each referring to a certain group of recovery rates. These groups can be estimated by means of beta-component mixtures. They have simple tractability for modeling and flexibility enough to describe unknown and multimodal distributional shapes which apparently can not be modeled by a beta distribution.

The rapid growth of credit derivative market enables to make use of credit default swaps (CDS) as market indicators. As Düllmann and Sosinska (2007) pointed out credit default swaps are less limitated as market indicators than credit spreads of subordinated debt issues, since CDS represent insurance premia for default events and measure credit risk more directly. Most of studies on analyzing the usefulness of CDS as market indicators infer probability of default from CDS, imposing an exogenously constant recovery rate. The market convenction is to assume that the average recovery rate is around 50%. Under such assumption, the term structure of CDS spreads can be used to extract the term structure of risk-neutral default probabilities either using a structural model (Finger, Lardy, Pan, Ta and Tierney, 2002 and Düllmann and Sosinska, 2007) or a reduced-form framework (Jarrow, 2001, Duffie and Singleton, 1999, Jarrow, Lando and Turnbull, 1997 and Madan, Guntay and Unal, 2003). However it is unrealistic to consider that recovery rates held fixed given that the pattern of recovery rate distribution can vary significantly across seniority level, industries, stages of business cycle, etc.

At present recovery rate extraction from CDS is relatively scarce. The approaches which focus on extracting simultaneously both the probability of default and recovery rates may be classified into time-series dependent approaches, cross-sectional approaches and panel data approaches (Christensen, 2005, Pan and Singleton, 2008, Chava, Stefanescu and Turnbull, 2006 and Acharya, Bharath and Srinivasan, 2007). Das and Hanouna (2009) adopt a calibration approach for bootstrapping implied recovery rates from CDS spread curves at any single point in time. Their procedure uses information from the equity market, the credit default swap market and it also uses the forward curve of riskless rates, thereby incorporating information from the interest rate market as well. In contrast to the above approaches only information on a given trading day is used, an entire forward term structure of recovery is delivered and a dynamic model of recovery is offered through a functional relation between recovery and state variables. Das and Hanouna (2009) model is flexible and robust. It is flexible in the sense that it can be used with different state variables, alternate recovery functional forms and calibrated to multiple debt tranches of the same issuer. It is robust because it evidences parameter stability over time, is stable to changes in inputs and provides similar recovery term structures for different functional specifications. Finally, their model is easy to calibrate.

In this paper we approximate the severity distribution of a given company at any single point in time using recovery rates implicit in the term structure of CDS. In doing this, we implement the approach introduced by Das and Hanouna (2009). Our main objective is to model the market-implied severity as a mixture of beta components in order to capture the observed multimodality. Futhermore, we highlight the importance of this modeling approach by focusing on its role for credit VaR, which is a commonly used risk quantity. Specifically, simulation experiments are carried out to evaluate the implications of computing credit VaR in the case where a beta distribuion is wrongly assumed when the true underlying severity distribution is a beta-component mixture.

The paper is organized as follows. Section 2 describes briefly the methodology developed by Das and Hanouna (2009) to extract and identify the implied forward curve of recovery rates. Section 3 describes our proposal of modeling market-implied severity by finite mixtures of beta components. This section reports the representation, interpretation and estimation of mixture distributions using the Expectation-Maximization (EM) algorithm. Section 4 reports empirical results based on four companies which belong to the European stock index EUROSTOXX 50. In Section 5 an application to credit VaR estimation for two sets of portfolios is carried out. Finally, conclusions are drawn in Section 6.

2. IDENTIFICATION OF IMPLIED RECOVERY RATES FROM CDS

The most important instrument in the credit derivative market is the credit default swap (CDS), which essentially provides insurance against the default of an issuer (the reference credit) or on a specific underlying bond (the reference security). In its most basic form the buyer of the protection pays an annual or semiannual premium until either the maturity of the contract or default on the reference entity, whichever comes first. If a default occurs, the seller of the protection compensates the buyer for the loss on the reference security by either paying the face value of the bond in exchange for the defaulted bond (physical settlement) or by paying an amount of cash which compensates the buyer of the protection for the difference between the post-default market value of the bond and the par value (cash settlement). Typically, the underlying credit of a default swap is a rated firm with publicly traded debt or a sovereign entity. More details on CDS may be found in Lando (2004).

The steps of Das and Hanouna (2009) methodology to identify implied, endogenous, dynamic functions of the recovery rate and default probability from CDS can be sum up as follows:

Step 1: The standard relationship of CDS spreads to default intensities and recovery rates is presented, considering that the fair pricing of a default swap must be such that the expected present value of payments made by buyer and seller are equal:

$$C_N = \sum_{j=1}^{N} S_{j-1} D_j = \sum_{j=1}^{N} S_{j-1}(1 - e^{-\lambda_j}) D_j (1 - \phi_j) \tag{1}$$

where N is the number of periods in the model, indexed by $j = 1, \ldots, N$; C_N is the fair pricing of a default swap (it is the premium); $S_j = e^{-\sum_{k=1}^{j} \lambda_k}$ is the survival function of a firm and λ_j denotes the default intensity, $\lambda_j = \lambda(j-1, j)$, constant over forward period j (it is assumed that $S(0) = 1$, which represents that a firm is solvent); D_j is the discount function, written as function of forward rates, $D_j = e^{-\sum_{k=1}^{j} f_k}$; $\phi_j = \phi(j-1, j)$ is the recovery rate in the event of default (it is the recovery rate in the event that default occurs in period j; then, the loss payment on default is equal to $(\phi_j - 1)$); $S_j(1 - e^{-\lambda_j})$ is the probability of surviving until period $j - 1$ (i.e., the expected loss payment in period j is based on the probability of default in period j conditional on no default in a prior period).

Step 2: Default intensities are represented in terms of spreads and recovery rates. Through a process of bootstrapping, the general form of the intensity probability is (for all j):

$$\lambda_j = (-1)\ln\left\{\frac{S_{j-1}D_j(1-\phi_j+\sum_{k=1}^{j-1}G_k - C_j\sum_{k=1}^{j}H_j}{S_{j-1}D_j(1-\phi_j)}\right\}$$
$$G_j \equiv S_{j-1}(1-e^{-\lambda_j})D_j(1-\phi_j) \tag{2}$$
$$H_j \equiv S_{j-1}D_j$$

Step 3: A functional relationship of recovery rates to default intensities is choosen, which may generally be written as $\phi = g[\lambda, \theta]$, where both $\lambda, \phi \in R^n$ are term structure vectors and θ is a parameter set.

Step 4: An iterative fixed-point algorithm is begun using a starting value for $\phi(T) = 0.5$, for all T. In the iteration process, (i) finding $\lambda(T)$ from equation (2) and (ii) finding $\phi(T)$ from $\lambda(T)$ using the loglinear regresion relationship. The system stabilizes rapidly within a few iterations.

The approach which is taken in this paper is to use information from the equity market through the Merton model (Merton, 1974). The identification function between recovery rate and default intensity for the iterative process is given by the following loglinear relationship:

$$\ln(\phi(T)) = \theta_0 + \theta_1 \ln(\lambda(T)) \tag{3}$$

The term structure of interest rates is estimated using the commonly used Nelson and Siegel model (Nelson and Siege, 1987), which uses a single exponential functional form over the entire maturity range. This model suggests a parsimonious parametrization of the instantaneous forward rate curve given as follows:

$$f(t) = \alpha_1 + \alpha_2 e^{-t/\tau} + \alpha_3 \frac{t}{\tau} e^{-t/\tau} \tag{4}$$

The parameters α_1, α_2, α_3 and τ can be interpreted as: $\alpha_1 + \alpha_2$ is the instantaneous short rate; α_1 is the consol rate; that is, $\lim_{t\to\infty} f(t) = \alpha_1$; $-\alpha_2$ is the slope of the term structure of forward rates; α_3 affects the curvature of the term structure over the intermediate terms; $\tau > 0$ is the speed of convergence of the term structure toward the consol rate. These four parameters are estimated by minimizing the sum of squared errors:

$$\min \sum_{i=1}^{k} \epsilon_i^2 \tag{5}$$

where ϵ_i is the difference between the ith bond's market price and its theoretical price.

3. BETA-COMPONENT MIXTURES IN MODELING IMPLIED RECOVERY RATES

We propose to use finite mixtures of beta distributions in proportions $\pi_1, \ldots, \pi_g$ to model implied recovery rates. The mixing proportions represent the percentage of recovery rates belonging to each component of the mixture, are non-negative and sum to 1. Such distributions provide an extremely flexible method of modeling unknown and multimodal distributional shapes which apparently cannot be modelled by a single beta distribution. Each component represents a local area of support of the true distribution which may reflect the behaviour of recovery rates, for instance, belonging to a particular industry, with a specific seniority level or during a stage of the business cycle.

The probability density function of the recovery rates is given by

$$f(y; \pi, p, q) = \sum_{j=1}^{g} \pi_j f_j(y|p_j, q_j), \tag{6}$$

in which $\pi = (\pi_1, \ldots, \pi_g)$, $p = (p_1, \ldots, p_g)$, $q = (q_1, \ldots, q_g)$ and $f_j(y; p_j, q_j)$, $j = 1, \ldots, g$, denotes the values of the univariate beta probability function specified by the parameters p_j and q_j, given by

$$f_j(y; p_j, q_j) = \frac{1}{B(p_j, q_j)} y^{p_j - 1}(1 - y)^{q_j - 1}, 0 < y < 1, p_j > 0, q_j > 0 \tag{7}$$

where $B(p_j, q_j)$ denotes the beta function, p_j is the shape parameter and q_j is the scale parameter.

For a given value of g the unknown parameters in the beta mixture model are estimated by the EM (Expectation-Maximization) algorithm (Dempster, Laird and Rubin, 1977). Under the assumption that $y_1, \ldots, y_n$ are independent and identically distributed random variables following a beta mixture distribution, the log-likelihood function is given by

$$\log L(\pi, p, q|y) = \sum_{i=1}^{n} \log \sum_{j=1}^{g} \pi_j f_j(y_i|p_j, q_j) \tag{8}$$

With the maximum likelihood approach to the estimation of $\Psi = (\pi, p, q)$, an estimate is provided by an appropiate root of the likelihood equation

$$\frac{\partial \log L(\Psi|y)}{\partial \Psi} = 0 \tag{9}$$

The EM algorithm is used to find solutions of (9) corresponding to local maxima and it is guaranteed to converge to the MLE. Overall, it is based on the idea of replacing one difficult likelihood maximization with a sequence of easier maximizations whose limit is the answer to the original problem.

In the EM framework, the observed univariate data vector $Y = (Y_1, \ldots, Y_n)$ is completed with a component-label vector $Z = (Z_1, \ldots, Z_n)$. The label variable $Z_{ij} = Z_i(j)$, $i = 1, \ldots, n$, $j = 1, \ldots, g$, is 0 or 1 according to whether i corresponds to the component j. Hence, $Z = (Z_1, \ldots, Z_n)$ is an unobservable vector of component-indicator variables, and Z_i, $i = 1, \ldots, n$, are assumed to be independent random variables from a multinomial distribution consisting of one draw on g categories with respective probabilities $\pi_1, \ldots, \pi_g$. That is,

$$Z_1, \ldots, Z_n \sim Mult_g(1, \pi) \tag{10}$$

where $\pi = \pi_1, \ldots, \pi_g$. The complete-data log-likelihood is

$$\log L_c(\pi, p, q|y) = \sum_{j=1}^{g} \sum_{i=1}^{n} z_{ij} \log\{\pi_j f_j(y_i; p_j, q_j)\} \tag{11}$$

The EM algorithm allows us to maximize $L(\pi, p, q|y)$ by working with $\log L_c(\pi, p, q|y)$. The EM algorithm is an iterative procedure. Each iteration comprises of the "E-step", which calculates the expected log likelihood, and the "M-step", which finds its maximum.

Now, the algorithm starts: From an initial value $\Psi^{(0)} = (\pi^{(0)}, p^{(0)}, q^{(0)})$, a sequence is created according to

$$\begin{aligned} &\Psi^{(r+1)} = \text{the value that maximizes} \\ &E[\log L(\Psi|y, z)|\Psi^{(r)}, y] = Q(\Psi; \Psi^{(r)}) \end{aligned} \tag{12}$$

which is the conditional expectation of the complete data log-likelihood $\log L_c(\pi, p, q|y, z)$, given the observed data y, using the current fit $\Psi^{(r)}$ for Ψ.

On the (r+1) iteration, the E-step requires the calculation of $Q(\Psi; \Psi^{(r)})$. Since $Z = (Z_1, \ldots, Z_n$ is non observed data, the E-step is affected by replacing z_{ij} by its conditional expectation given y_j, using $\Psi^{(r)}$ for Ψ. That is, z_{ij} is replaced by $\tau(y_i; \Psi^{(r)}) = E_{\Psi^{(r)}}(Z_{ij}|y_i) = Pr_{\Psi^{(r)}}(Z_{ij} = 1|y_i)$. On the M-step, on the (r+1) iteration we choose the value of Ψ, say $\Psi^{(r+1)}$, that maximizes $Q(\Psi; \Psi^{(r)})$. Then, the vector $\Psi^{(r+1)}$ is obtained as an appropiate root of

$$\sum_{j=1}^{g} \sum_{i=1}^{n} \tau_j(y_i, \Psi^{(r)}) \frac{\partial \log L(\Psi_j|y)}{\partial \Psi} = 0 \tag{13}$$

The hidden key to the algorithm is the application of the information inequality (Dempster, Laird and Rubin, 1977, lemma 1), which states that $L(\hat{\Psi}^{r+1}|y) \geq L(\hat{\Psi}^{r}|y)$, with equality holding if and only if succesive iterations yield the same value of the maximized expected complete-data log-likelihood, that is, $E[\log L(\hat{\Psi}^{r+1}|y,z)|\hat{\Psi}^{r},y] = E[\log L(\hat{\Psi}^{r}|y,z)|\hat{\Psi}^{r},y]$.

4. MARKET-IMPLIED SEVERITY: A MODEL BASED ON A BETA-COMPONENT MIXTURE

This study uses data from Bloomberg Financials on CDS spreads for Spanish industrial quoted companies with liquid traded CDS, which belong to the EUROSTOXX 50 index, for the period from January 2004 to October 2007. They are low credit risk companies and are Repsol, Iberdrola, Telefonica and Endesa. The data consist on a CDS spread curve with maturities from 1 to 10 years for each company and each day.

For each of the 935 days in the sample and each company we compute the term structure of forward recovery rates by applying Das and Hanouna (2009) methodology. Then, we obtain ten implied recovery rate distributions for each company, one for each maturity. To examine the shape of the implied recovery rate distribution we will restrict ourselves to the most frequently traded CDS maturity of 5 years, other maturities are considerably less liquid.

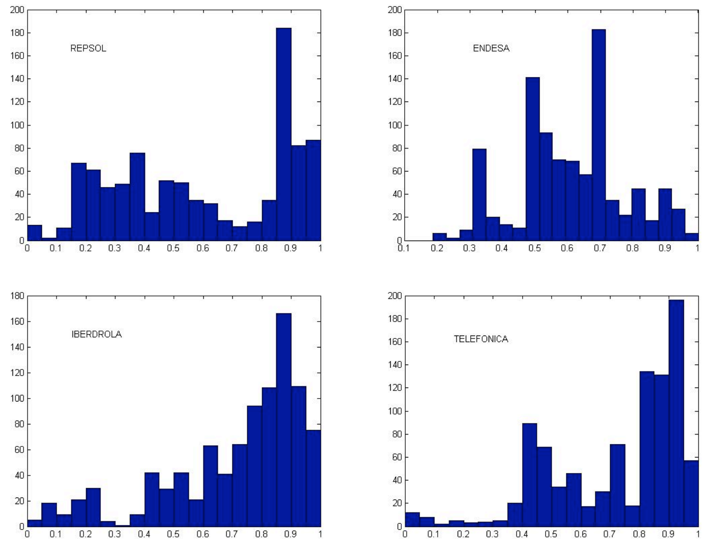
Figure 1 shows the histograms of the 5-year maturity implied recovery rates which we have identified from Das and Hanouna (2009) methodology.

The plots exhibits several local modes. Multimodal distributions have been also reported in Hagmann, Renault and Scaillet (2005) and Schuermann (2005). They appear to be skewed right for Repsol, Iberdrola and Telefonica, while it is approximately bell-shaped for Endesa. The beta distribution is unable to capture the pattern of the histograms and the best practice choice is a beta-component mixture.

In fitting mixture models there is no a priori information regarding the number of components. Therefore, an issue that requires careful consideration is the choice of the number of component densities (Mclachlan and Basford, 1987). Different approaches have been designed to assess the number of components in mixture models (Oliveira-Brochado and Martins, 2005). Alternatively, Whitaker and Lee (2007) proposed a method termed PURE, constructed by combining the "plug-in" principle and the unbiased risk estimation technique. The steps of the PURE method are:

i) Estimate the unknown parameters of the postulated mixture and obtain the estimate $\hat{f}_g$ of f_g.

FIG. 1. Histograms of implied recovery rates



(a). Histogram of implied recovery rates for Repsol; (b). Histogram of implied recovery rates for Endesa; (c). Histogram of implied recovery rates for Iberdrola; (d). Histogram of implied recovery rates for Telefonica.

ii) Compute a pilot estimate $\hat{f}_p$ of f_g using Akaike's information criterion. The bias term is thus computed as $(\hat{f}_p - \hat{f}_g)^2$.

iii) Using nonparametric bootstrap and B bootstrap samples, calculate the bootstrap estimate of the variance of $\hat{f}_g$, $var_{bs}(\hat{f}_g)$. The resulting risk estimator is

$$risk(g) = \sum_{i=1}^{n} \{var_{bs}(\hat{f}_g(x_i)) + (\hat{f}_p(x_i) - \hat{f}_g(x_i))^2\} \tag{14}$$

iv) Choose g as the minimizer of $risk(g)$.

Whitaker and Lee (2007) tested numerically the practical performance of their proposal and compared it with some information criteria, including the commonly used Akaike information criterion (AIC) and the Bayesian information criterion (BIC). In information criteria the estimation of the order of a mixture model is considered by using a penalized form of the log-likelihood function. As the likelihood increases with the addition of a component to a mixture model, the information criteria attempt to balance the increase in fit obtained against the larger number of parameters estimated for models with more components.

In their simulation experiments, the PURE method appeared to be the best method to correctly identify the number of mixture components. It performed particularly well, especially when the true number of components g was large. They showed also that the PURE method gave good performance when the mixture components were not clearly separated. It should be borne in mind that, when information criteria are used, it is important to ensure that the components are well-separated. Unfortunately, in recovery risk modeling, the distance between the means of the components is usually small. For instance, Hagmann, Renault and Scaillet (2005) reported the mean of recovery rates by seniority for data extracted from the Standard & Poor's/PMD database from 1981-1999. The mean recovery rates were 56.31%, 46.74%, 35.35% and 35.03% for senior secured, senior unsecured, subordinated and junior subordinated, respectively. Futhermore, the AIC and BIC had been shown to be inadequate for deciding the number of components in the beta mixture model (Ji, Wu, Liu, Wang and Coombes, 2005), a reliable method of modeling multimodal distributional shapes in credit risk practice. Then, the PURE method can be extremely useful in credit risk practice.

Results of testing the number of beta components are reported in Table 1.

TABLE 1.

PURE method results

Companies	$risk(g = 1)$	$risk(g = 2)$	$risk(g = 3)$	$risk(g = 4)$
REPSOL	465.0154	77.3713	70.5020	72.3120
ENDESA	96.1412	98.5352	49.0323	49.4477
IBERDROLA	613.8715	81.4069	56.2009	58.1822
TELEFONICA	629.9442	451.2719	93.1379	93.6812

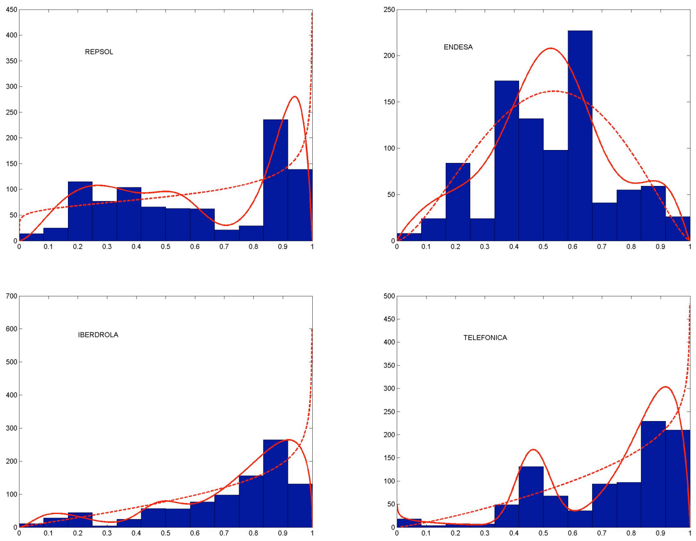
The selected number of components g is the number which minimizes the risk estimator $risk(g)$.

Looking at Table 1, one can observe that the risk estimator is minimized for $g = 3$ in all cases. Then, the null hypothesis of three-beta components is accepted in all cases at 1%, 5% and 10% significance levels.

Figure 2 compares the fitting of the beta distribution to the observed data with respect to the one of the three beta-component mixture.

Figure 2 shows that the beta distribution does not capture the observed behaviour of the implied recovery rates in any case while the three beta-component mixture does it absolutely well.

FIG. 2. Graphical comparison between a beta and a three-beta component mixture



(a). Solid line: a three-beta component mixture. Dashed line: a beta distribution.

In order to provide a formal judgement about whether a beta or a beta-component mixture distribution is adequate to describe the observed behaviour of the implied recovery rates, we provide results of a goodness-of-fit test based on the Cramer-von Mises (CVM) test statistic. The test statistic is constructed substituting the unknown vector of parameters $\theta \in R^s$ appearing in the postulated null distribution $F(. : \theta)$, either a beta or a mixture distribution, by an estimate. Owing to θ is a vector of unknown parameters, the CVM test statistic is defined as

$$\hat{W}_n^2 = \sum_{j=1}^{n} \{F_n(Y_j) - F(Y_j, \hat{\theta})\}^2 \tag{15}$$

where $F_n(.)$ is the empirical distribution function and $\hat{\theta}$ is the maximum likelihood estimate of θ.

Bootstrap methodology is applied to implement this type of goodness-of-fit tests because the tabulated asymptotic critical values have been deduced for the case in which the postulated null distribution is totally known and the observations are independent and identically distributed random variables (Shorack and Wellner, 1986). Nevertheless, those asymptotic critical values are no longer valid when the CVM test statistic is constructed substituting the unknown parameters by their maximum likelihood estimates because it is no distribution-free. The bootstrap procedure works as follows:

1. Let $Y_1, Y_2, \ldots, Y_n$ be a sequence of recovery rates.

2. Considering that the null distribution is a beta distribution $B(.; p, q)$ estimate p and q by maximum likelihood. In this way, $B(.; \hat{p}, \hat{q})$ is obtained. Draw the empirical distribution of Y_j, $j = 1, 2, \ldots, n$, and evaluate $\hat{W}_n^2$ using $B(.; \hat{p}, \hat{q})$.

3. Repeat B times the following: Generate a sample of random variables $Y_1^*, Y_2^*, \ldots, Y_n^*$ from $B(.; \hat{p}, \hat{q})$. Using $\hat{p}$, $\hat{q}$ and Y_j^* calculate new maximum likelihood estimates $\hat{p}^*$, $\hat{q}^*$. Evaluate the test statistic using Y_j^* and $B(.; \hat{p}^*, \hat{q}^*)$. It is denoted $\hat{W}_n^{2*}$. In this way, a sample of B independent (conditional on the original sample) observations of $\hat{W}_n^2$ is obtained, $\hat{W}_{n1}^{2*}, \hat{W}_{n2}^{2*}, \ldots, \hat{W}_{nB}^{2*}$.

4. Let $\hat{W}_{n(1-\alpha)B}^{2*}$ the $(1-\alpha)B$-th order statistic of the sample $\hat{W}_{n1}^{2*}, \hat{W}_{n2}^{2*}, \ldots, \hat{W}_{nB}^{2*}$, given a significance level α. Reject the null hypothesis at the significance level α if $\hat{W}_n^2 > \hat{W}_{n(1-\alpha)B}^{2*}$.

5. Compute the bootstrap p-value as $p_B = card(\hat{W}_{nb}^{2*} \geq \hat{W}_n^2)/B$, $b = 1, \ldots, B$.

Also the procedure above is repeated when the null distribution is a mixture of three-beta components. Results based on $B = 500$ bootstrap samples are shown in Table 2.

TABLE 2.

Goodness-of-fit test results

p-value	REPSOL	ENDESA	IBERDROLA	TELEFONICA
$g = 1$	0.000	0.000	0.000	0.000
$g = 2$	0.020	0.000	0.000	0.000
$g = 3$	0.953	0.611	0.647	0.999

Bootstrap p-values of testing: (a). H_0: "The recovery rate distribution is a beta (g=1)" versus H_1: "The recovery rate distribution is not a beta"; (b). H_0: "The recovery rate distribution is a two-beta component mixture (g=2)" versus H_1: "The recovery rate distribution is not a two-beta component mixture"; (c). H_0: "The recovery rate distribution is a three-beta component mixture (g=3)" versus H_1: "The recovery rate distribution is not a three-beta component mixture".

The beta assumption is rejected in all cases while the three-beta mixture is accepted at 1%, 5% and 10% significance levels.This is not unexpected given the results reported for the PURE method and the graphical analysis.

5. BETA-COMPONENT MIXTURE MODEL ROLE IN CREDIT VAR ESTIMATION

From the viewpoint of credit VaR users it is absolutely relevant to asses the degree of precision in the reported VaR. The severity distribution is essential to estimate credit VaR and, obviously, the systematic use of the beta distribution to estimate it can lead to mismeasured credit VaR quantity.

In this section simulation experiments are carried out to show that the assumption of a beta-component mixture produces much more accurate measures of credit VaR than the commonly used beta distribution. Some simulation results are presented to illustrate the effects of computing credit VaR in the case where a beta distribution is wrongly assumed. In these simulation experiments, the number of risk exposures of a portfolio is 1000, the credit loss is computed as one minus the recovery rate of those exposures at default, the binomial default probability is low and equal to 0.5% because we are interested in low probability of default companies and the pairwise default probability is 2.8%. Four scenarios are considered: data are generated from four different beta-component mixtures, specifically, from the estimated three-beta component mixtures fitted in the previous section. Under each scenario the difference between $CVaR_{beta}$ and the corresponding credit VaR, $CVaR_{mixture}$, is computed. $CVaR_{beta}$ is calculated using the beta distribution to estimate the data distribution.

Table 3 illustrates the different scenarios.

TABLE 3.

Random variable generation from three-beta component mixtures

$\sum_{k=1}^{3} \lambda_k b(.; p_k, q_k)$	Scenario 1	Scenario 2	Scenario 3	Scenario 4
p_1	2.784	1.931	2.906	0.743
q_1	5.879	3.453	13.911	3.721
p_2	20.994	9.165	27.652	34.860
q_2	17.489	8.126	30.001	40.068
p_3	16.505	11.650	6.097	9.943
q_3	1.989	2.358	1.450	1.817
λ_1	0.445	0.307	0.094	0.042
λ_2	0.113	0.546	0.095	0.248
λ_3	0.440	0.145	0.809	0.708

Each scenario is given by a three-beta component mixture whose parameter values are the estimated in Section 4.

The steps of the procedure to compute the diffence between $CVaR_{beta}$ and $CVaR_{mixture}$ are the following (Arvanitis, Browne, Gregory and Martin, 1998):

1. Generate default indicator functions X_j, $j = 1, 2, \ldots, 1000$, by drawing correlated standard normal random variables Y_j:

$$Y_j \to N(0, \Omega) \tag{16}$$

where, $\Omega = \begin{pmatrix} 1 & \cdots & \lambda_{ij} \\ \vdots & \ddots & \vdots \\ \lambda_{ij} & \cdots & 1 \end{pmatrix}$ and λ_{ij} is the pairwise default probability, $i, j = 1, 2, \ldots, 1000$.

The covariance matrix can be factorised as $C = AA'$, for some A (by Cholesky factorisation or ortogonal diagonalisation on C). If u is a multivariate process with components independently drawn from the standard normal distribution, then the vector $v = Au$ has the required matrix C. Having determined the correlation between the normal random variables, the default indicator function X_j is defined as

$$X_j = I(v_j < z) \tag{17}$$

where v_j is the j-element of the vector v and $z = N_{-1}(d_j)$, being $d_j = 0.5\%$ the binomial default probability for all j.

2. Compute $\sum_{j=1}^{1000} X_j$. This value gives the number of portfolio assets which present default.

3. Add the recovery rates corresponding to the assets which present default in each portfolio (which are those with $X_j = 1$).

4. Repeat the procedure above 10000 times to compute the corresponding credit loss distribution. Given a confidence level $(1-\alpha)$, compute the corresponding $(1-\alpha)$-quantile or credit VaR, denoted by $q_{1-\alpha}$.

Finally, bootstrap methodology is used to test if the difference $T_1 = q_{1-\alpha}^{beta} - q_{1-\alpha}^{mixture}$ is statistically significant. The steps of the bootstrap procedure are:

1. Generate a sequence o recovery rates $Y_1, Y_2, \ldots, Y_n$ from the estimated mixture distribution.

2. Under the null distribution, estimate the unknown parameters by maximum likelihood. In this way, the estimated of the null distribution $\sum_{k=1}^{3} \hat{\lambda}_k b(.; \hat{p_k}, \hat{q_k})$ is obtained.

3. Repeat $B = 500$ times the following: Generate a sample of random variables from $\sum_{k=1}^{3} \hat{\lambda}_k b(.; \hat{p_k}, \hat{q_k})$ to obtain a new sequence of recovery rates Y_j^*, $j = 1, 2, \ldots, n$. Compute $q_{1-\alpha}^{*beta}$ and $q_{1-\alpha}^{mixture*}$ using Y_j^* following Arvanitis, Browne, Gregory and Martin (1998). Calculate $T_1 = q_{1-\alpha}^{beta*} - q_{1-\alpha}^{mixture*}$ to obtain a sample of B independent (conditional on the original sample) observations of T_1, say $T_{11}^*, \ldots, T_{1B}^*$.

4. Let $T^*_{1(1-\alpha)B}$ be the $(1-\alpha)B$-th order statistic of the sample $T^*_{11}, \ldots, T^*_{1B}$ given a significance level α. Reject the null hypothesis at the significance level α if $T_1 > T^*_{1(1-\alpha)B}$.

5. Compute the bootstrap p-value as $p_B = card(T^*_{1b} \geq T_1)/B,\ b = 1, \ldots, B$.

Table 4 reports the results of this bootstrap test.

TABLE 4.

Credit VaR estimation

	SCENARIO 1		SCENARIO 2		SCENARIO 3		SCENARIO 4	
	$q_{0.95}$	$q_{0.99}$	$q_{0.95}$	$q_{0.99}$	$q_{0.95}$	$q_{0.99}$	$q_{0.95}$	$q_{0.99}$
Mixture	0.7100	0.8036	0.7011	0.7986	0.6021	0.7428	0.5913	0.7086
Beta	0.7272	0.8310	0.6902	0.7831	0.5829	0.6979	0.5707	0.6888
$q^{beta}_{1-\alpha} - q^{mixture}_{1-\alpha}$	0.0171	0.0273	-0.0109	-0.0155	-0.0192	-0.0448	-0.0205	-0.0198
p-value	0.0879	0.0119	0.1078	0.0299	0.0459	0.0019	0.0519	0.0759

$q_{0.95}$ and $q_{0.99}$ denote the credit VaR at 95% and 99% confidence level, respectively. The default probability is 0.5%. For each scenario credit VaR values are reported under a three-beta component mixture and a beta. The difference between them and the bootstrap p-values are computed.

Table 4 shows that the usual practice of approximating the recovery rate distribution through a beta distribution can lead to underestimation of credit VaR (scenarios 2, 3 y 4). It can be observed that credit VaR differences at 95% loss probability level are statistically significant at 10% significance level for almost all cases. At higher loss probability level (99%) credit VaR measures are significant different too: at 5% significant level in scenarios 1, 2 y 3, and at 10% significant level in scenario 4.

6. CONCLUSIONS

In this paper beta-component mixtures have been proposed to model implied recovery rates in order to capture the observed multimodality. The empirical analysis reveals that the beta distribution is rejected as a correct specification for implied recovery rates while a beta-component mixture is accepted. This analysis is based on implied recovery rates which previously have been extracted and identified from CDS spreads versus using defaulted bonds. This allows us to identify recovery rates for companies which are blue chips. In addition, it has been proved an excellent performance of beta-component mixtures in measuring credit VaR accurately once the number of beta components is fixed. We found significant differences in credit VaR estimates at 95% and 99% significance levels and 1% and 0.5% default probabilities. Accordingly, the beta distribution assumption should therefore be considered with caution for credit VaR estimation.

To sum up, this paper provides a framework to estimate credit VaR accurately using implied recovery rates extracted from CDS spreads.

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