Stochastic Growth with the Social-Status Concern: The Existence of a Unique Stable Distribution[1]

Liutang Gong[2]

Guanghua School of Management, Peking University, Beijing, 100871, China

Xiaojun Zhao

Guanghua School of Management, Peking University, Beijing, 100871, China

and

Hengfu Zou

China Economics and Management Academy, CUFE, Beijing, 100081, China

Abstract

This paper extends Kurz's (1968) growth model to a stochastic growth framework with the social-status concern and production shocks. Using the stochastic monotonicity of stochastic dynamic system and methods using in Zhang (2007), the existence and stability of invariant distribution has been investigated. Different from the existence of multiple steady states under certainty, it is shown that there exists a unique stable invariant distribution under uncertainty.

Key Words: Stochastic growth; the Spirit of capitalism; Stochastic dominance; Multiple equilibria.

JEL Classifications: C61; C62; O41.

1 Introduction

In neoclassical growth models wealth accumulation is often taken to be solely driven by one's desire to increase consumption rewards. The representative agent chooses a consumption path to maximize his/her discounted utility, which is defined only on consumption. This motive is important for wealth accumulation. It is, however, not the only motive. Because man is a social animal, he also accumulates wealth to gain prestige, social status, and power in the society; see Frank (1985), Cole, Mailath and Postlewaite (1992, 1995), Fershtman and Weiss (1993), Zou (1994, 1995), Bakshi and Chen (1996), and Fershtman, Murphy and Weiss (1996). Earlier contributions include Duesenberry (1948), Kurz (1968), and Spence (1974). In these wealth-is-status models, the representative agent accumulates wealth not only for consumption but also for wealth-induced status. Another interpretation of these models is in line with the spirit of capitalism in the sense of Weber (1958)

[1]We thank Dr. Xiaoyong Cui, Xuezhi Fan, and Xin Xu for helpful comments. This research was sponsored by the National Science Foundation for Distinguished Young Scholars (70725006) and National Social Science Fundation of China (06BJL021).

[2]Correspondence address: Liutang Gong, Guanghua School of Management, Peking University, Beijing, 100871, China. E-mail: ltgong@gsm.pku.edu.cn.

and Keynes (1971): capitalists accumulate wealth for the sake of wealth. To cite Weber (1958)[3]:

Man is dominated by making of money, by acquisition as the ultimate purpose of his life. Economic acquisition is no longer subordinated to man as the means for the satisfaction of his material needs. This reversal relationship, so irrational from a naive point of view, is evidently a leading principle of capitalism.

Using the wealth-is-status and the-spirit-of-capitalism models, many authors have tried to explain growth, savings, and asset pricing. Cole, Mailath, and Postlewaite (1992) have demonstrated how the presence of social status leads to multiple equilibria in long-run growth. Zou (1994, 1995) has studied the spirit of capitalism and long-run growth and shows that a strong capitalist spirit can lead to unbounded growth of consumption and capital even under the neoclassical assumption of production technology. Bakshi and Chen (1996) have explored empirically the relationship between the spirit of capitalism and stock market pricing and offered an attempt towards the resolution of the equity premium puzzle.

However, with the introduction of the social-status concern, the existence of stationary distribution has not been discussed yet. This paper aims to study a stochastic growth model with the social-status concern and production shocks. Similar to Brock and Mirman (1972) and Zhang (2007), the existence and stability of invariant distribution has been investigated. Different from the existence of multiple steady states under certainty, it is shown that there exists a unique invariant distribution under uncertainty.

The rest of the paper is organized as follows: in Section 2, we give the basic framework of the paper. Section 3 presents the mathematical results for preparation. Sections 4 and 5 analyze an economy under certainty and an economy under uncertainty, respectively. The multiple steady states under certainty has been shown in Section 4, and the unique stable invariant distribution under uncertainty has been examined in Section 5. Section 6 concludes the main finds of this paper.

2 The Framework

Following Kurz (1968), Bakshi and Chen (1996), Gong and Zou (2002), Brock and Mirman (1972) and Stachurski (2002), this section gives the basic framework of the stochastic growth model with the social status concern. To compare the economy under certainty with that under uncertainty, we consider two models: Growth under certainty and growth under uncertainty.

[3]See Cole, Mailath, and Postlewaite (1992); Zou (1994, 1995); and Bakshi and Chen (1996) for more details.

2.1 The agent's optimal problem

At any time t, suppose the representative agent's income and consumption are y_t and c_t, respectively. The budget constraint of the agent can be expressed as

$$y_{t+1} = \varepsilon_{t+1} f(y_t - c_t) + (1 - \delta)(y_t - c_t),$$

where f is the production function. $\{\varepsilon_{t+1}\}$ is an i.i.d (independent and indentical distribution) stochastic process with common measurement μ and represents the production shock. δ is depreciate rate of the capital stock.

As in Kurz (1968), Gong and Zou (2002) and many gorwth models with the social-status concern, we suppose that the agent's instantaneous utility function at time t is $u(c_t, y_t)$. Therefore, the optimal problem for the representative agent is:

$$\max_{c_t, y_t} E[\sum_{t=0}^{\infty} \beta^t u(c_t, y_t)], \tag{1a}$$

subject to

$$y_{t+1} = \varepsilon_{t+1} f(y_t - c_t) + (1 - \delta)(y_t - c_t), \quad 0 \leq c_t \leq y_t. \tag{1b}$$

with the given initial condition y_0.

Letting ε_{t+1} be 1, optimization problem deduce to an economy under certainty, namely,

$$\max_{c_t, y_t} \sum_{t=0}^{\infty} \beta^t u(c_t, y_t), \tag{2a}$$

subject to

$$y_{t+1} = f(y_t - c_t) + (1 - \delta)(y_t - c_t), \quad 0 \leq c_t \leq y_t. \tag{2b}$$

with the given initial condition y_0.

2.2 Assumptions

To derive the conclusion, we summarize assumptions of this paper.

Assumption 1. The production function $f : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is strictly increasing, strictly concave, differentiable and satisfies the Inada condition

$$\lim_{x \to 0} f'(x) = +\infty, \ \lim_{x \to +\infty} f'(x) = 0.$$

Furthermore, $f(0) = 0$.

Assumption 2. The utility function $u : \mathbb{R}_+ \times \mathbb{R}_+ \mapsto \mathbb{R}_+$ is strictly increasing, strictly concave, and differentiable with respect to c, y. Furthermore, it satisfies

$$\lim_{c \to 0} u_c = +\infty, \ \lim_{y \to 0} u_y = +\infty, \ u_{cy} = u_{yc} \geq 0.$$

Assumption 3. The production shock $\{\varepsilon_t\}_{t=0}^{\infty}$ is a sequence of i.i.d. random variables, with the common distribution μ, and

$$0 < E[\varepsilon_t] < \infty, \, 0 < E[1/\varepsilon_t] < \infty.$$

Assumption 1 is the standard assumption of neoclassical production function. The assumption of utility function u is similar to Kurz (1968), Bakshi and Chen (1996), et al. The assumption on ε_t is the same as Zhang (2007), which are unbounded shocks. The assumption on the expectation of $1/\varepsilon_t$ is a technique requirement in the following sections of this paper.

In the next section, we will solve the optimal problems and characterize the equilibrium of the economy, respectively. Before doing this, we give some mathematical results for preparation.

3 Mathematical Results

For any metric space S, we denote the Borel σ-algebra of S by $\mathcal{B}(S)$, and the set of all probability measures on $\mathcal{B}(S)$ by $\Lambda(S)$. If λ is a measure on $\mathcal{B}(S)$, then $\|\lambda\|$ is its total variation norm. The *support* of λ (denoted supp(λ)), which is the subset of S satisfying: (1). $\lambda(S \setminus (\lambda)) = 0$; and (2). If G is open and $G \cap (\lambda) \neq \varnothing$, then $\lambda(G \cap (\lambda)) > 0$. If $s \in S$, then δ_s is the probability that puts mass 1 at s. Let $CB(S)$ be the set of all bounded continuous functions on S.

A sequence λ_n of elements of $\Lambda(S)$ converges weakly (or in distribution) to some λ in$\Lambda(S)$ if and only if $\int_S f d\lambda_n$ converges to $\int_S f d\lambda$ for all f in $CB(S)$. Using this notion of convergence, we have a topology defined on $\Lambda(S)$ called the weak topology. In this paper, unless otherwise specified, we use weak topology when we discuss the convergence of distributions.

A *transition kernel* on $(S, \mathcal{B}(S))$ is a function $Q : S \times \mathcal{B}(S) \mapsto [0, 1]$ such that: (1) for each measurable set $A \in \mathcal{B}(S)$, the real valued function $Q(\cdot, A)$ is $\mathcal{B}(S)$ measurable; and (2) for each point $s \in S$, the set function $Q(s, \cdot)$ is a probability measure on $\mathcal{B}(S)$. The number $Q(s, A)$ should be interpreted as the probability that the economic system will move from state s to some state in the set A during one period of time.

A transition kernel defines a linear operator T from bounded measurable functions to bounded measurable functions via the formula

$$(Tf)(x) = \int f(y) Q(x, dy).$$

The adjoint $T^* : \Lambda(S) \mapsto \Lambda(S)$ of operator T is defined by the formula

$$(T^*\lambda)(A) = \int Q(x, A) \lambda(dx).$$

A transition kernel Q is said to have the Feller property if one of the following equivalent conditions are satisfied:

1.*Tf is bounded and continuous whenever f is.*

2.*$T^*\lambda_n$ converges to $T^*\lambda$ whenever λ_n converges to λ.*

For any two probability measures λ, τ in $\Lambda(\mathbb{R})$, λ is (first-order) *stochastically dominates* τ if $\int f d\lambda \geq \int f d\tau$ for all bounded and increasing functions f. If λ dominates τ, we write $\lambda \succeq \tau$ or $\tau \preceq \lambda$. It is known that

$$\lambda \succeq \tau \Leftrightarrow F_\lambda(x) \leq F_\tau(x), \qquad x \in \mathbb{R},$$

where $F_\lambda(\cdot)$ and $F(\cdot)$ are distribution functions of λ and τ, respectively. A transition kernel Q on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is called monotonic if it satisfies any of the following equivalent conditions:

1. *Tf is bounded and increasing if f is.*
2. *$T^*\lambda \succeq T^*\tau$, if $\lambda \succeq \tau$.*
3. *$Q(x, \cdot) \succeq Q(y, \cdot)$, if $x \geq y$.*

Let $M \subseteq \Lambda(S)$ be a subset of probability measures. Then M is tight if for any $\varepsilon > 0$, there exists a compact subset $K \subseteq S$ such that $\lambda(K) \geq 1 - \varepsilon$ for all λ in M. If S is complete and separable, then M is tight if and only if the closure of M is compact under weak topology.

Using the above notations and mathematical results, now we solve the optimal problem (1) and (2) respectively.

4 The Economy under Certainty

In this section, we focus on solving the growth model (2a) and (2b) under certainty by utilizing the dynamic programming technique. Denote $V(y)$ and $g(y)$ the value function and policy function of problem (2), we have

Theorem 1 *Let u and f satisfy Assumptions 1 and 2, then following results hold.*

1. The value function $V(y)$ is finite and satisfies the Bellman equation

$$V(y) = \max_{0 \leq c \leq y} \{u(c, y) + \beta V(f(y - c) + (1 - \delta)(y - c))\}. \tag{3}$$

2. The value function $V(\cdot)$ is strictly increasing, concave, and differentiable with respect to y.

3. The optimal policy function $c = g(y)$ exists and satisfies

$$V'(y) = u_c(g(y), y) + u_y(g(y), y). \tag{4}$$

Proof. See Appendix A.

From Theorem 1, it is easy to derive the dynamical accumulation equations for consumption and capital stock

$$y_{t+1} = f(y_t - c_t) + (1-\delta)(y_t - c_t), \tag{5}$$

$$u_c(c_t, y_t) = \beta[u_c(c_{t+1}, y_{t+1}) + u_y(c_{t+1}, y_{t+1})][f'(k_t - c_t) + 1 - \delta]. \tag{6}$$

To characterizes the steady state explicitly, we specify the utility function and production function as

$$u(c, y) = \frac{c^{1-\sigma_1} - 1}{1-\sigma_1} + \frac{y^{1-\sigma_2} - 1}{1-\sigma_2}, \text{ and } f(k) = k^{\alpha}, \tag{7}$$

where σ_1, σ_2, and α are positive constants.

Therefore, the steady state (c^*, y^*) for the above systems satisfies

$$y^* = (y^* - c^*)^{\alpha} + (1-\delta)(y^* - c^*), \tag{8}$$

$$c^{*-\sigma_1} = \beta[c^{*-\sigma_1} + y^{*-\sigma_2}][\alpha(y^* - c^*)^{\alpha-1} + 1 - \delta]. \tag{9}$$

Kurz (1968) has presented numerical solutions to show the existence of multiple steady states. Here, we also present a numerical solution to illustrate the existence of multiple steady states.

Given parameters: α= 0.3, $\delta = 1$, and $\sigma_1 = \sigma_2 = 2$, Figure 1 shows that there exists three steady states: A, B, and C. The associated income level and consumption level are (1.7326, 0.2132), (3.4452, 2.2101), and (7.3799, 5.4000), respectively. It is easy to prove that the steady states A and C are saddle-point stable, and the steady state B is unstable.

Therefore, under the certainty environment, there may exist multiple steady states with the consideration of social-status concern. Therefore, just as what Benhabib and Farmer (1999) have pointed that the existence of multiple steady states can be used to understand why the economies of two countries differ even if they have the similar endowments. In the next section, we will focus on existence of steady-state distribution for the economy under uncertainty.

5 The Economy under Uncertainty

In this section, we will solve the growth model (1a) and (1b) under uncertainty and show the existence and stability of the stationary distribution. First, we need some characterizations of transition kernel.

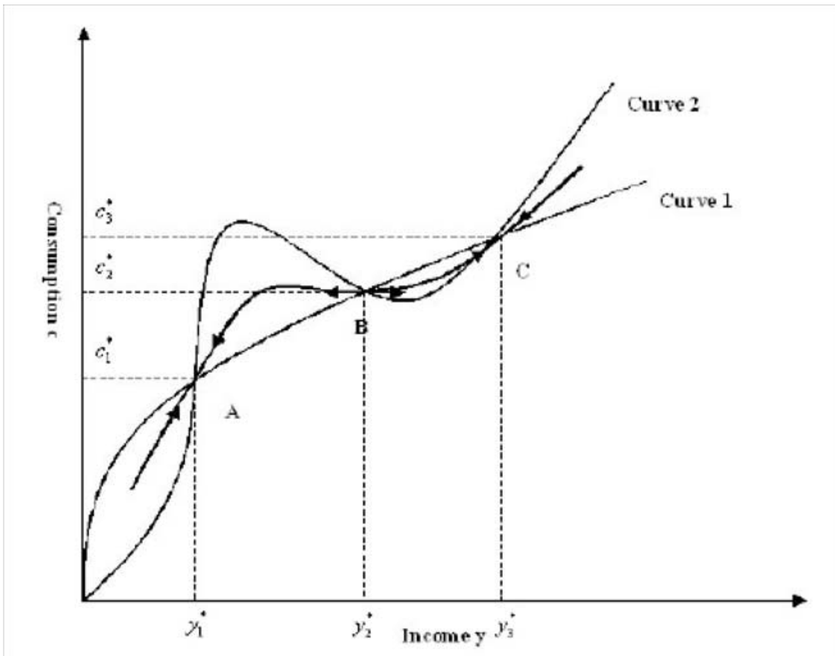

Figure 1: The dynamics of consumption and income with multiple equilibria.

5.1 Transition kernel

To define the tansition kernel, we define the conditional distribution of next-period output y' given current output y and consumption c as

$$Pr(y' \in B) = \mu(\frac{B - (1-\delta)(y-c)}{f(y-c)}), \quad \text{and } y - c > 0, \text{ for all } B \text{ in } \mathcal{B}(\mathbb{R}_+), \tag{10}$$

where $\mathcal{B}(\mathbb{R}_+)$ is the Borel σ-algebra of $\mathbb{R}_+$, and

$$\frac{B - (1-\delta)(y-c)}{f(y-c)} = \{\frac{y' - (1-\delta)(y-c)}{f(y-c)} : y' \in B\}.$$

Note that if $y - c = 0$, we have $Pr(y' = 0) = 1$, therefore, 0 is an absorbing state.

Therefore, the transition kernel $Q(y, c; B)$ is defined as the probability with the next-period output in B, given that the current income is y and consumption is c, which is defined as.

$$Q(y, c; B) = \begin{cases} \mu\left(\frac{B-(1-\delta)(y-c)}{f(y-c)}\right), & y - c > 0, \\ 1, & y - c = 0, 0 \in B, \\ 0, & y - c = 0, 0 \notin B. \end{cases} \tag{11}$$

5.2 The optimal policy

Now we turn to solve the optimization problem (1). Similar to the Theorem 1, we denote $V(y)$ and $g(y)$ the value function and policy function, respectively, and we have

Theorem 2 *Let u, f, and μ satisfy Assumptions 1-3, and $u(c,y) = u_1(c) + u_2(y)$; then following results hold.*

1. The value function $V(y)$ is finite and satisfies the Bellman equation

$$V(y) = \max_{0 \le c \le y} \{u(c,y) + \beta \int_0^\infty V(y')Q(y,c;dy')\}. \tag{12}$$

2. There exists a unique optimal policy g, such that

$$V(y) = u(g(y),y) + \beta \int_0^\infty V(y')Q(y,g(y);dy'). \tag{13}$$

3. The value function is strictly increasing, concave, differentiable, and satisfies

$$V'(y) = u_1'(g(y)) + u_2'(y). \tag{14}$$

4. The optimal policy function g is continuous and satisfies that $0 < g(y) < y$, for all $y > 0$. Furthermore, both $y \mapsto g(y)$ and $y \mapsto y - g(y)$ are strictly increasing (i.e., both savings and consumption are increasing with income).

Proof. See Appendix B.

From Theorem 2, we obtain some properties for the transition kernel $Q(y,c;B)$. Substituting the optimal policy $c_t = g(y_t)$ into the constraints of (2b) yields

$$y_{t+1} = \varepsilon_{t+1} f(y_t - g(y_t)) + (1-\delta)(y_t - g(y_t)). \tag{15}$$

Theorem 2 tells that the saving function $s(y) = y - g(y)$ is a strictly increasing and continuous function with $s(0) = 0$. For simplicity, we use $Q(y,B)$ to denote $Q(y,g(y);B)$. Since s and f are continuous, the transition kernel $Q(\cdot,\cdot)$ satisfies the Feller property. From Stokey et al. (1989), it is also easy to prove that $Q(\cdot,\cdot)$ is monotonic.

For a stochastic growth model, the state of the economic system can be represented by a probability distribution of wealth y_t. Suppose the distribution of y_t is λ_t, then the distribution of y_{t+1} is $T^*\lambda_t$, where T^* is defined in section 3 through the transition kernel $Q(\cdot,\cdot)$. From any λ_0, the trajectory of λ_0 by operator T^* is a sequence $\{\lambda_t\}_{t=1}^\infty$ and is defined by $\lambda_{t+1} = T^*\lambda_t$.

5.3 Existence of the invariant distribution

The invariant (or stationary) distribution is important to characterize the stochastic dynamic system. It is in fact a fixed point of the operator T^*, i.e., it is a distribution $\lambda \in \Lambda(\mathbb{R}_+)$ satisfies that $\lambda = T^*\lambda$. To study the invariant distribution, we firstly characterize some poperties of operator T^*.

Lemma 1 *δ_0 is an invariant probability distribution.*

Proof: Since $Q(0, \{0\}) = 1$, it is obviously that $T^*\delta_0 = \delta_0$. *Q.E.D.*

Lemma 2 *If $\mu(\{0\}) > 0$, then δ_0 is the unique invariant probability distribution, and for any initial distribution λ_0, $\lim_{t\to\infty} \lambda_t(\{0\}) = 1$.*

Proof: Because $\lambda_t((0,\infty)) = \lambda_0((0,\infty))(1-\mu(\{0\}))^t$, the second statement is easy to prove. We can check $\lim_{t\to\infty} ||\lambda_t - \lambda_0|| = 0$, which ensures that δ_0 is the unique fixed point. *Q.E.D.*

Obviously, if $\mu(\{0\}) > 0$, then the economy will die out with probability one. This is not an interesting case for our sake. From now on, we impose an assumption on μ.

Assumption 4. $\mu(\{0\}) = 0$.

Because $\mu(\{0\}) = 0$, we can restrict the operator T^* on $\Lambda(\mathbb{R}_{++})$. This is an implication of part 4 in Theorem 2

Lemma 3 *If Assumption 3 holds, there exists $s > 0$, such that $E(s/\varepsilon) = 1$, $\mu((0, s]) > 0$, and $\mu([s, \infty)) > 0$.*

Proof: The proof is similar to Zhang (2007) and we omit it.

If we define $\varepsilon^* = \varepsilon/s$, $f^*\cdot = sf(\cdot)$ and $h(y) = sf(s(y))$, then

$$y_{t+1} = \varepsilon^*_{t+1} h(y_t) + s(y_t),$$

with $E(\varepsilon^{*-1}) = 1$. So, without loss of generality, we assume $E(\varepsilon^{-1}) = 1$.

Lemma 4 *There exists $\underline{y} > 0$, such that $h(y) > y$ for all $y \in (0, \underline{y}]$.*

Proof: The first-order condition for the optimization problem (2a)-(2b) is

$$u_1'(g(y)) = \beta[f'(f^{-1}(h(y))) + 1 - \delta] \int_0^\infty V'(\varepsilon h(y) + (1-\delta)s(y))\varepsilon\mu(d\varepsilon).$$

On the other hand, from the envelope theorem, we have

$$V'(y) = u_1'(g(y)) + u_2'(y).$$

Therefore,

$$V'(y) = \beta[f'(f^{-1}(h(y))) + 1 - \delta] \int_0^\infty V'(\varepsilon h(y) + (1-\delta)s(y))\varepsilon\mu(d\varepsilon) + u_2^{'}(y) \tag{16}$$

Because $u_2^{'} > 0$, we have

$$V'(y) \geq \beta f'(f^{-1}(h(y))) \int_0^\infty V'(\varepsilon h(y) + (1-\delta)s(y))\varepsilon\mu(d\varepsilon). \tag{17}$$

Because $0 < s(y) < y$, we have $\lim\limits_{y\to 0} s(y) = 0$ and $\lim\limits_{y\to 0} s(y)/f(s(y)) = 1/f'(0) = 0$. Therefore, there exists $y_1 > 0$ and $0 < \varepsilon_0 < 1$ such that for all $y \in (0, y_1]$ and $\varepsilon \in (\varepsilon_0, 1]$,

$$\frac{s(y)}{f(s(y))} = \frac{s(y)}{h(y)} \leq \frac{1-\varepsilon}{1-\delta}.$$

namely,

$$\varepsilon h(y) + (1-\delta)s(y) \leq h(y). \tag{18}$$

Furthermore, from Lemma 3, we can choose ε_0 such that $\mu((\varepsilon_0, 1]) > 0$. Therefore, we obtain

$$\begin{aligned}
\frac{V'(y)}{V'(h(y))} &\geq \frac{\beta f'(f^{-1}(h(y))) \int_0^\infty V'(\varepsilon h(y) + (1-\delta)s(y))\varepsilon\mu(d\varepsilon)}{V'(h(y))} \\
&\geq \frac{\beta f'(f^{-1}(h(y))) \int_{\varepsilon_0}^1 V'(\varepsilon h(y) + (1-\delta)s(y))\varepsilon\mu(d\varepsilon)}{V'(h(y))} \\
&\geq \beta f'(f^{-1}(h(y))) \int_{\varepsilon_0}^1 \varepsilon\mu(d\varepsilon).
\end{aligned}$$

Since $\lim\limits_{y\to 0} f'(f^{-1}(h(y))) = \infty$ and $\int_{\varepsilon_0}^1 \varepsilon\mu(d\varepsilon) > 0$, there exists $y_2 > 0$ such that $V'(y) > V'(h(y))$ for $y \leq y_2$. Taking $\underline{y} = \min\{y_1, y_2\}$, we conclude that $h(y) > y$ when $0 < y \leq \underline{y}$. *Q.E.D.*

Lemma 5 *If $z \in (0, \underline{y}]$ and τ_z is the uniform distribution on the interval $(0, z]$. Then we have $T^*\tau_z \succeq \tau_z$.*

Proof: It is sufficient to show that for any $m \in (0, z]$, $(T^*\tau_z)((0, m]) \leq \tau_z((0, m])$.

By the definition of T^*, from Lemma 3, we have

$$(T^*\tau_z)((0,m]) = \int_0^z \mu((0, \frac{m-(1-\delta)s(y)}{h(y)}])\tau_z(dy) \leq \int_0^z \mu((0, my^{-1}])\tau_z(dy).$$

Since $\tau_z([z, \infty)) = 0$, we further obtain

$$\int_0^z \mu((0, my^{-1}])\tau_z(dy) = \int_0^\infty \mu((0, my^{-1}])\tau_z(dy) = \int_0^z \int_0^{my^{-1}} \mu(dx)\tau_z(dy).$$

From the Fubini's Theorem, we get

$$\int_0^z \int_0^{my^{-1}} \mu(dx)\tau_z(dy) = \int_0^z \int_0^{mx^{-1}} \tau_z(dy)\mu(dx) \leq \int_0^\infty mx^{-1}z^{-1}\mu(dx).$$

Because $E(\varepsilon^{-1}) = 1$, we have

$$\int_0^\infty mx^{-1}z^{-1}\mu(dx) = mz^{-1} = \tau_z((0,m]).$$

Therefore, $T^*\tau_z \succeq \tau_z$. *Q.E.D.*

Define a subset B_z of $\Lambda(\mathbb{R}_{++})$ as

$$B_z = \{\tau \in \Lambda(\mathbb{R}_{++}) : \tau \succeq \tau_z\}.$$

By the monotonicity of T^*, we have $T^*(B_z) \subseteq B_z$, furthermore, B_z is closed under the norm of total variation, which is proved in Theorem 6.1 in Torres (1990).

So far we have found the lower boundary of the dynamic system. This means that the wealth will be pushed to higher level under the operator T^*. If there is no upper boundary, the wealth will go to infinity with probability one. To find the upper boundary of the system, we focus on the case of full depreciation, i.e., $\delta = 1$.

Remind that $h(y) = f(s(y)) < f(y)$ and the Inada conditions of f ensures $\lim_{y\to\infty} h(y)/y = 0$. Therefore, there is a $\overline{y} > 0$, such that

$$h(y) > y(E(\varepsilon))^{-1} \text{ for all } y \geq \overline{y}. \tag{19}$$

For any $z \in [\overline{y}, \infty)$, let λ^z be the probability distribution with density function at

$$y = \begin{cases} zy^{-2}, & y \geq z, \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 6 *For any $z \in [\overline{y}, \infty)$, $\lambda^z \succeq T^*\lambda^z$.*

Proof: Choosing $m \in [z, \infty)$, it is sufficient to prove that $T^*\lambda^z([z,\infty)) \leq \lambda^z([z,\infty))$. Using the fact $h(y) > y(E(\varepsilon))^{-1}$, we find that

$$\begin{aligned} T^*\lambda^z([z,\infty)) &= \int_0^\infty \mu([\frac{m}{h(y)}, \infty))\lambda^z(dy) \\ &\leq \int_0^\infty \mu([mE(\varepsilon)y^{-1}, \infty))\lambda^z(dy) \\ &= \int_0^\infty \int_{mE(\varepsilon)y^{-1}}^\infty \mu(dx)\lambda^z(dy). \end{aligned}$$

From the Fubini's Theorem, we obtain

$$\begin{aligned}
\int_0^\infty \int_{mE(\varepsilon)y^{-1}}^\infty \mu(dx)\lambda^z(dy) &= \int_0^\infty \int_{mE(\varepsilon)x^{-1}}^\infty \lambda^z(dy)\mu(dx) \\
&\leq \int_0^\infty \int_{mE(\varepsilon)x^{-1}}^\infty zy^{-2}dy\mu(dx) \\
&= \int_0^\infty xz(mE(\varepsilon))^{-1}\mu(dx) = zm^{-1} = \lambda^z([z,\infty)).
\end{aligned}$$

Hence $T^*\lambda^z \preceq \lambda^z$. *Q.E.D.*

Similar to the proceeding Lemmas, we define another subset B^z of $\Lambda(\mathbb{R}_{++})$ for $z \in [\overline{y}, \infty)$.

$$B^z = \{\lambda \in \Lambda(\mathbb{R}_{++}) : \lambda \preceq \lambda^z\}.$$

We can also prove $T^*(B^z) \subseteq B^z$ and B^z is closed subset of $\Lambda(\mathbb{R}_{++})$. Here we must cite a well known result introduced in Torres (1990): Choose $z_1 \in (0, \underline{y}]$ and $z_2 \in [\overline{y}, \infty)$, then $B_{z_1} \cap B^{z_2}$ is a non-empty, convex and compact subset of $\Lambda(\mathbb{R}_{++})$. Now we can prove the existence of the invariant distribution.

Proposition 1 *Under the Assumptions 1-4, there exists an invariant distribution in* $\Lambda(\mathbb{R}_{++})$.

Proof: Choose B_{z_1} and B^{z_2} as above, from Lemmas 5 and 6, we can infer that T^* is invariant on the subset $B_{z_1} \cap B^{z_2}$.

On the other hand, since $Q(.,.)$ satisfies the Feller property, T^* is a continuous operator under the weak topology. The compactness of $B_{z_1} \cap B^{z_2}$ encourages us to apply the Brower-Schauder-Tychnoff Theorem. Hence, T^* has at least a fixed point in $B_{z_1} \cap B^{z_2}$. *Q.E.D.*

Figure 2 describes the above process of the existence of invariant distribution. When the initial distribution is τ_y, then T^* pushes τ_y to the distribution whose density function is green dashed line. But when come from λ^z, T^* will draw back the initial distribution to the distribution whose density function is the blue dashed line. So there exists an invariant distribution between τ_y and λ^z, we draw the invariant distribution whose density function is the bold line in figure 2.

5.4 The uniqueness of the invariant distribution

In this section, we will use the "inverse Markovian chain" introduced by Brock and Mirman (1972) to complete the proof of the uniqueness of invariant distribution.

Recall that $y_{t+1} = \varepsilon_{t+1}h(y_t)$, so $y_t = h^{-1}(y_{t+1}/\varepsilon_{t+1})$. The "reverse Markovian chain" is described by the following formula.

$$\widetilde{y}_{t-1} = h^{-1}(\frac{\widetilde{y}_t}{\varepsilon_t}), \qquad \widetilde{y}_t \in (0,\infty), \qquad t \leq 0. \tag{20}$$

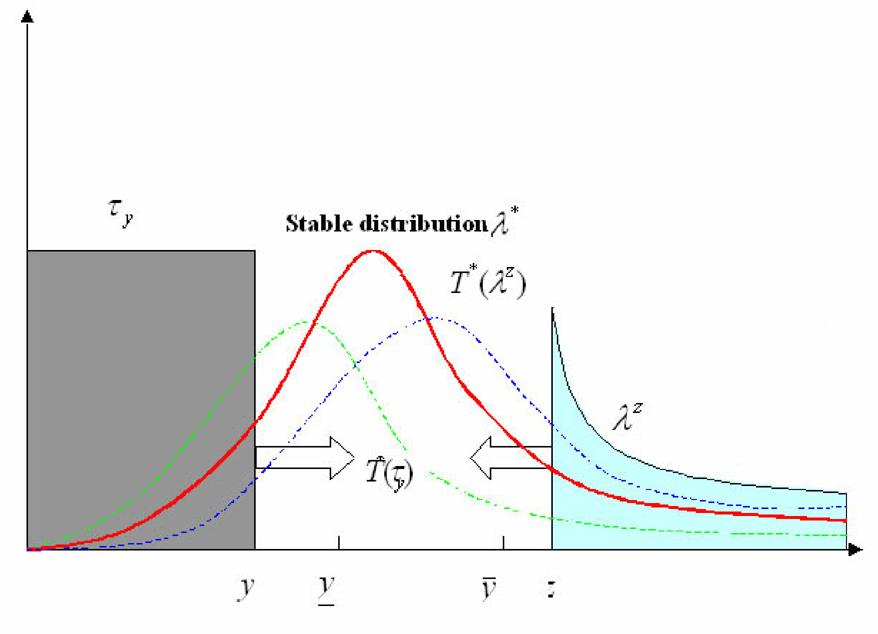

Figure 2: The evolution of the distribution of wealth.

We show that starting from any initial value $\widetilde{y}_0$, the "reverse Markovian chain" will almost surely converge to 0 or ∞. This feature provides us with a contradiction when we assume the existence of more than one invariant distribution.

With the notation of $\underline{y}$ and the corresponding properties of it, for any $z \in (0, \underline{y}]$, we consider the reverse process, which contained in $(0, z]$ according to the following evolution law:

$$\hat{y}_{t-1} = \begin{cases} h^{-1}(\min(\frac{\hat{y}_t}{\varepsilon_t}, h(z))), & \hat{y}_t < z, \\ z, & \hat{y}_t = z. \end{cases} \tag{21}$$

Where $\hat{y}_0 \in (0, z]$.

Note that the above Markovian process is a modification of the "reverse Markovian chain" defined in (20). Once $\widetilde{y}_t$ is larger than or equal to z, it is redefined to stay at z for ever.

Lemma 7 *The reverse process defined in (21) is contained in* $(0, z]$ *and is a super-martingale, that is*

$$E[\hat{y}_{t-1}|\hat{y}_t, \ldots, \hat{y}_0] = \hat{y}_t. \tag{22}$$

Proof: Because we have $E(\hat{y}_{t-1}|\hat{y}_t) = z = \hat{y}_t$ when $\hat{y}_t = z$, we only need to prove

$$E[h^{-1}(\min(\frac{y}{\varepsilon}, h(z)))|y] \leq y, \text{ for any } y < z.$$

Using the fact $h^{-1}(y) < y$ and $E(1/\varepsilon) = 1$, we find that

$$E[h^{-1}(\min(\frac{y}{\varepsilon}, h(z)))|y] < E_\varepsilon[\min(\frac{y}{\varepsilon}, h(z))|y] \leq E_\varepsilon[\frac{y}{\varepsilon}|y] = y.$$

Q.E.D.

Lemma 8 *For any $z \leq \underline{y}$ and any initial random variable $\hat{y}_0$ taking values in $(0, z)$ with probability 1, we have*

$$\Pr(\hat{y}_t < z, \forall t \leq 0) \geq \Pr(\lim_{t\to-\infty} \hat{y}_t < z) \geq \mu([1, \infty)). \tag{23}$$

Proof: Note that $\{y_t\}_{t=0}^{-\infty}$ is a non-negative super martingale contained in $(0, z]$. By the martingale convergence theorem, the sequence converges with probability 1. So, we have

$$\Pr(\hat{y}_t < z, \forall t \leq 0) \geq \Pr(\lim_{t\to-\infty} \hat{y}_t < z).$$

Let $x = h^{-1}(\min(\frac{z}{\varepsilon}, h(z)))$, where ε has distribution μ. For any $\hat{y}_0 \in (0, z)$, let $\hat{y} = \lim_{t\to-\infty} \hat{y}_t$, and λ_t, λ_x be the distributions of $\hat{y}_t$ and x, respectively. We have

$$\lambda_0((0, z)) = 1.$$

Therefore, $\lambda_0 \preceq \delta_z$, then in turn $\lambda_{-1} \preceq \lambda_x$.We can recursively prove $\lambda_t \preceq \lambda_x$, $t \leq -1$. Thus, because $\lambda_{\hat{y}} \preceq \lambda_x$,

$$\Pr(\hat{y}_t < z) \geq \Pr(x < z) \geq \mu([1, \infty)).$$

Therefore

$$\Pr(\lim_{t\to-\infty} \hat{y}_t < z) \geq \mu([1, \infty)).$$

Q.E.D.

Now we consider the upper modification of the reverse Markovian process. For $y \geq \overline{y}$, we have $y > E(\varepsilon)h(y)$, or equivalently, for $y \geq h(\overline{y})$, we have $h^{-1}(y) > E(\varepsilon)y$. For $z > \overline{y}$, define the modified Markovian process as follows:

$$\widetilde{y}_{t-1} = \begin{cases} h^{-1}\left(\max\left(\frac{\widetilde{y}_t}{\varepsilon_t}, h(z)\right)\right), & \widetilde{y}_t > z, \\ z, & \widetilde{y}_t = z. \end{cases} \tag{24}$$

Taking the transform $x_t = 1/\breve{y}_t$, we get the corresponding Markovian process on $(0, 1/z]$.

$$x_{t-1} = \begin{cases} \frac{1}{h^{-1}(\max((x_t\varepsilon_t)^{-1}, h(z)))}, & x_t < 1/z, \\ 1/z, & x_t = 1/z. \end{cases} \tag{25}$$

Lemma 9 *The reverse Markovian process $\{x_t\}$ defined as equation (25) is a super martingale.*

Proof: It is sufficient to show that $E(1/h^{-1}(\max\left((x\varepsilon)^{-1}, h(z)\right))|x) \leq x$ for all $x < 1/z$. Because $h^{-1}(y) > E(\varepsilon)y$ for any $y \geq h(\overline{y})$, we obtain that

$$\begin{aligned} E[\frac{1}{h^{-1}\left(\max\left((x\varepsilon)^{-1}, h(z)\right)\right)}|x] &\leq E_\varepsilon[\frac{1}{E(\varepsilon)\max\left((x\varepsilon)^{-1}, h(z)\right)}|x] \\ &\leq E_\varepsilon[\frac{1}{E(\varepsilon)(x\varepsilon)^{-1}}|x] = x. \end{aligned}$$

Therefore, we complete the proof of the Lemma. *Q.E.D.*

Now, for the same reason, the reverse Markovian process $\{x_t\}$ abide by the following rule

$$\Pr(x_t < 1/z, \forall t \leq 0) \geq \mu((0,1]). \tag{26}$$

Returning to the original process $\{\widetilde{y}_t\}$, we have

$$\Pr(\widetilde{y}_t > z, \forall t \leq 0) \geq \mu((0,1]). \tag{27}$$

From now on, we assume the production shock ε_t is unbounded, the bounded case can be proved similar to Brock and Mirman (1972).

Lemma 10 *Suppose the production shock ε_t is unbounded, then for any $z_1 \in (0, \underline{y}]$ and $z_2 \in [\overline{y}, \infty)$, there is a $\pi > 0$, such that for all initial $\widetilde{y}_0$,*

$$\Pr(\widetilde{y}_t \in (0, z_1) \cup (z_2, \infty), \forall t \leq -1) \geq \pi, \tag{28}$$

where $\{\widetilde{y}_t\}_{t=0}^{-\infty}$ is the reverse Markovian process defined in (20).

Proof: Because μ is unbounded, that is, for any $N > 0$, we have $\mu([N,\infty)) > 0$. Let

$$\pi = \min\{\mu((0,1]), \mu([1,\infty)), \mu([\frac{z_2}{h(z_1)}, \infty))\} \times \min\{\mu((0,1]), \mu([1,\infty))\} > 0.$$

Here, we will consider three cases:

Case 1. $\widetilde{y}_0 \in (0, z_1)$. Under this case, we have

$$\begin{aligned} \Pr(\widetilde{y}_t &\in (0, z_1) \cup (z_2, \infty), \forall t \leq -1) \geq \Pr(\widetilde{y}_t \in (0, z_1), \forall t \leq -1) \\ &\geq \mu([1,\infty)) \geq \pi. \end{aligned}$$

Case 2. $\widetilde{y}_0 \in (z_1, z_2)$. Under this case, we have

$$\begin{aligned} \Pr(\widetilde{y}_t &\in (0, z_1) \cup (z_2, \infty), \forall t \leq -1) \geq \Pr(\widetilde{y}_t \in (0, z_1), \forall t \leq -1) \\ &\geq \Pr(\widetilde{y}_{-1} \in (0, z_1)) \times \mu([1,\infty)) \\ &\geq \mu((\frac{z_2}{h(z_1)}, \infty)) \times \mu([1,\infty)) \geq \pi. \end{aligned}$$

Case 3. $\widetilde{y}_0 \in (z_2, \infty)$. Under this case, we have

$$\begin{aligned}\Pr(\widetilde{y}_t &\in (0, z_1) \cup (z_2, \infty), \forall t \leq -1) \geq \Pr(\widetilde{y}_t \in (z_2, \infty), \forall t \leq -1) \\ &\geq \mu((0,1]) \geq \pi.\end{aligned}$$

Therefore, for any initial value $\widetilde{y}_0$, the reverse Markovian process will go to $(0, z_1)$ and (z_2, ∞) with positive probability.

Q.E.D.

Lemma 11 *For any $z_1 \in (0, \underline{y}]$ and $z_2 \in [\overline{y}, \infty)$, the reverse Markovian process $\widetilde{y}_t$ satisfies*

$$\Pr(\exists \bar{t} < 0, \textit{such that } \widetilde{y}_t \in (0, z_1) \cup (z_2, \infty), \forall t \leq \bar{t}) = 1. \tag{29}$$

That is, the reverse Markovian process will converge to 0 or ∞ with probability 1.

Proof: We will prove the equivalent result

$$\Pr(\widetilde{y}_t \in [z_1, z_2],\ i.o.) = 0.$$

By the Borel-Cantelli Lemma, we need to show that

$$\sum_{t=-1}^{-\infty} \Pr(\widetilde{y}_t \in [z_1, z_2]) < \infty.$$

In fact, suppose that $\sum_{t=-1}^{-\infty} \Pr(\widetilde{y}_t \in [z_1, z_2]) = \infty$. Define

$$A_t = \{\widetilde{y}_t \in [z_1, z_2], \widetilde{y}_s \in (0, z_1) \cup (z_2, \infty), \forall s \leq t-1\},$$

and from Lemma 10, we have

$$\Pr(A_t) \geq \sum_{t=-1}^{-\infty} \Pr(\widetilde{y}_t \in [z_1, z_2])\pi = \infty,$$

which contradicts to the fact that for any $t \leq -1$, A_t are disjoint sets. Hence, the Lemma 11 holds. *Q.E.D.*

Now we prove our main proposition.

Proposition 2 *Suppose ε_t is an unbounded shock, and Assumptions 1-4 hold, the invariant distribution is unique.*

Proof: Suppose there are two invariant distributions F_1 and F_2, for any $\widetilde{y}_0 > 0$. Then, we have

$$F_i(\widetilde{y}_0) = \int F_i(h^{-1}(\widetilde{y}_0/\varepsilon)), \quad i = 1, 2.$$

Recall the shock $\{\varepsilon_t\}_{t=0}^{-\infty}$ is an i.i.d. sequence, and $\{\widetilde{y}_t\}$ is a Markovian process, we obtain

$$F_i(\widetilde{y}_0) = \int F_i(\widetilde{y}_{-1})\mu(d\varepsilon_0) = ... = \int F_i(\widetilde{y}_t)\mu(d\varepsilon_{t+1})...\mu(d\varepsilon_0), \quad i = 1, 2.$$

Since $\lim_{t\to-\infty} \widetilde{y}_t$ is 0 or ∞, from the Lebesgue Dominated Convergence Theorem, we get

$$\begin{aligned} F_1(\widetilde{y}_0) - F_2(\widetilde{y}_0) &= \int (F_1 - F_2)(\widetilde{y}_t)\mu(d\varepsilon_{t+1})...\mu(d\varepsilon_0) \\ &= \lim_{t\to-\infty} \int (F_1 - F_2)(\widetilde{y}_t)\mu(d\varepsilon_{t+1})...\mu(d\varepsilon_0) \\ &= \int [\lim_{t\to-\infty} (F_1 - F_2)(\widetilde{y}_t)]\Pi_{t=0}^{-\infty}\mu(d\varepsilon_t) \\ &= 0. \end{aligned}$$

Therefore, the invariant distribution is unique. *Q.E.D.*

5.5 The stability of the invariant distribution

In the proceeding subsections, we have shown the existence and uniqueness of the invariant distribution λ^*. In this subsection, we will study the stability of λ^*. There are two types of stability: one is the local stability, the other is the global stability.

Definition 1 *Let λ^* be the invariant distribution of T^*, if there is a neighborhood U of the λ^*, such that for every $\lambda \in U$, $\lim_{t\to\infty} T^{*t}\lambda = \lambda^*$. Then we call λ^* is a locally stable distribution of T^*.*

Definition 2 *Under the above definition, if U can be chosen to be $\Lambda(\mathbb{R}_{++})$, then we call λ^* is a globally stable distribution of T^*.*

In this section, we will prove the invariant distribution λ^* derived in the proceeding subsections is a globally stable distribution of T^*.

Lemma 12 *For any $z_1 \in (0, \underline{y}]$ and $z_2 \in [\overline{y}, \infty)$, the sequences $\{T^{*t}\tau_{z_1}\}$ and $\{T^{*t}\lambda^{z_2}\}$ converge to λ^*.*

Proof: From the previous arguments, $\{T^{*t}\tau_{z_1}\}$ and $\{T^{*t}\lambda^{z_2}\}$ are monotonic sequences in the compact set $B_{z_1} \cap B^{z_2}$. Then from Proposition 6.7 in Torres (1990), the two sequences converge. Hence the limit distribution must be the invariant distribution of T^*. The uniqueness of the invariant distribution tells us the two sequences converge to λ^*. *Q.E.D.*

Proposition 3 *Under Assumptions 1-4, λ^* is globally stable.*

Proof: From Lemma 12, we know that for every $\lambda \in B_{z_1} \cap B^{z_2}$, $T^{*t}\lambda$ converges to λ^*. For every $\lambda \in \Lambda(\mathbb{R}_{++})$, define $\lambda_{z_1}^{z_2} \in B_{z_1} \cap B^{z_2}$ as follows:

$$\lambda_{z_1}^{z_2}(A) = \begin{cases} 0, & A = (0, z_1) \text{ or } A = (z_2, \infty), \\ \lambda((0, z_1]), & A = \{z_1\}, \\ \lambda([z_2, \infty)), & A = \{z_2\}, \\ \lambda((a, b)), & A = (a, b) \text{ if } z_1 < a < b < z_2. \end{cases}$$

Obviously, $\lambda_{z_1}^{z_2} \in B_{z_1} \cap B^{z_2}$. We can also choose z_1 small enough and z_2 large enough such that $||\lambda_{z_1}^{z_2} - \lambda||$ can be arbitrarily small.

Now for every $\phi \in CB(\mathbb{R}_{++})$, we want to show

$$\lim_{t \to \infty} \int \phi(y)(T^{*t}\lambda)(dy) = \int \phi(y)\lambda^*(dy).$$

For any small $\delta > 0$, choose $z_1 \in (0, \underline{y}]$ and $z_2 \in [\overline{y}, \infty)$ such that $||\lambda_{z_1}^{z_2} - \lambda|| \leq \delta(2||\phi||)^{-1}$. Then, we can choose N, such that $t \leq N$ implies

$$|\int \phi(y)(T^{*t}\lambda_{z_1}^{z_2})(dy) - \int \phi(y)\lambda^*(dy)| \leq \delta/2.$$

If $t \geq N$, then we have

$$\begin{aligned} |\int \phi(y)(T^{*t}\lambda)(dy) - \int \phi(y)\lambda^*(dy)| &\leq |\int \phi(y)(T^{*t}\lambda)(dy) - \int \phi(y)(T^{*t}\lambda_{z_1}^{z_2})(dy)| \\ &\quad + |\int \phi(y)(T^{*t}\lambda_{z_1}^{z_2})(dy) - \int \phi(y)\lambda^*(dy)| \\ &\leq |\int \phi(y)(T^{*t}\lambda)(dy) - \int \phi(y)(T^{*t}\lambda_{z_1}^{z_2})(dy)| + \delta/2 \\ &\leq ||T^t\phi|| \times ||\lambda_{z_1}^{z_2} - \lambda|| + \delta/2 \\ &\leq ||\phi|| \times ||\lambda_{z_1}^{z_2} - \lambda|| + \delta/2 \leq \delta. \end{aligned}$$

Therefore, the invariant distribution λ^* is globally stable.

6 Conclusions

This paper studies stochastic growth in an economy with social status concern and production shocks. The existence and stability of invariant distribution have been investigated in this paper. Different from the probability of existence of multiple steady states, it is found that there always exists a unique stable invariant distribution with production shocks.

The introduction of production shocks makes the economy from multiple steady states to an unique stable steady state. This result can be explained that, under the consideration of production shocks, the probability that the economy stays in the equilibrium is zero under uncertainty even if there exists multiple steady states under certainty. Hence, the uncertainty can help to understand the multiple equilibria in two reasons: the existence of multiple equilibria is not the essence of an economy, the uncertainty can be deemed as the multiple equilibria in the sense that every state can constitute a special equilibrium. But the probability that the economy stays at a certain equilibrium is zero, thus a certain stable distribution must be introduced to characterize the properties of the economy in the long run.

Although we have compared the economy under certainty with that under uncertainty, the assumption of complete depreciation of capital is too strong. We guess that when δ is large enough, the economy will still have the unique stable distribution. In fact, we have proved when $0 < \delta < 1$, T^* will push the initial distribution to higher level, but we have not yet given the upper bound the T^* in the case of $0 < \delta < 1$. This is partly because the technique difficulties, but what fear us more is that there is no upper bound of T^*, that is, the economy will tends to infinity with probability 1.

Future research may focus on studying other macroeconomic topics on this framework, such as the effects of fiscal policies and monetary policies on growth, asset pricing, etc. Also, note that our solutions rely on the assumption of i.i.d. production shocks, therefore, future research should study the stochstic growth with other production shocks, such as AR(1) process in RBC models.

7 Appendix A: Proof of Theorem 1

Proof: We use the notation introduced in Stokey, et al. (1989), let **SP** be the sequence problem of (1), and **FE** be the functional equation defined in (3).

1). It is sufficient to prove the equivalence of **SP** and **FE** under Assumptions 1-2. This is obvious because we can check the Assumptions 4.1 and 4.2 in Stokey, et al. (1989) hold. Thus, Theorem 4.2 tells the equivalence of **SP** and **FE.**

2). First, we prove the existence of the value function $V(y)$. In fact, V is the fixed point of the transform $\Gamma : C(\mathbb{R}_+) \to C(\mathbb{R}_+)$,

$$(\Gamma V)(y) = \max_{0 \le c \le y} \{u(c, y) + \beta V(f(y - c) + (1 - \delta)(y - c)\},$$

where $C(\mathbb{R}_+)$ is the set of all continuous functions on $\mathbb{R}_+$. To make $C(\mathbb{R}_+)$ be a complete metric space, we construct a metric d as follows.

Let $\{K_n\}$, $n \ge 1$ be a sequence of increasing sets of $\mathbb{R}_+$, such that $\bigcup_{n=1}^{\infty} K_n = \mathbb{R}_+$. For every f, $g \in C(\mathbb{R}_+)$, define

$$d_n(f, g) = \max_{y \in K_n} |f(y) - g(y)|$$

and

$$d(f, g) = \sum_{n=1}^{\infty} \frac{\min\{d_n(f, g), 1\}}{2^n}.$$

It is easy to prove that d is a metric, such that $(C(\mathbb{R}_+), d)$ is a complete metric space. We can further prove Γ is a contraction, so there must uniquely exists a fixed point of Γ. Hence, we have proved the existence and uniqueness of the value function V. From the hints of Stokey, et al. (1989), we can easily find a proof that $V(.)$ is strictly increasing, concave and differentiable with respect to y.

3). From Assumptions 1 and 2, we know 0 and y are not the optimal consumption policy, so $c = g(y)$ is determined by the first-order condition of (3), that is

$$u_c = \beta V'(f(y - c) + (1 - \delta)(y - c))(f' + 1 - \delta). \tag{A1}$$

Taking differential on the above equation, we further obtain

$$\frac{dc}{dy} = \frac{\beta[V''(f' + 1 - \delta)^2 + V'f''] - u_{cy}}{u_{cc} + \beta[V''(f' + 1 - \delta)^2 + v'f'']}.$$

Because $V' > 0$, $V'' < 0$, $f' > 0$, $f'' < 0$, $u_{cc} < 0$, and $u_{cy} > 0$, we know $\frac{dc}{dy} > 0$. From the implicit function theorem, the above shows that (A1) uniquely determines $c = g(y)$, and g is differentiable with $g' = \frac{dc}{dy}|_{c=g(y)} > 0$.

Substituting $c = g(y)$ into equation (3), we reach

$$V(y) = u(g(y), y) + \beta V(f(y - g(y)) + (1 - \delta)(y - g(y))).$$

Taking differential on above equation, we get

$$V'(y) = u_c(g(y), y) + u_y(g(y), y).$$

So we complete the proof of Theorem 1.

Appendix B: Proof of Theorem 2

Proof: Utilizing the same metric space $(C(\mathbb{R}_+), d)$, we can prove the existence of the value function V, and also can prove that V is strictly increasing, concave and differentiable with respect to y. In the bellow, we only focus on the envelope theorem and monotonicity of consumption function and savings function.

Similar to the proof of Theorem 1, we can exclude 0 and y to be the optimal consumption policy, so equation (12) becomes

$$V(y) = \max_{0<c<y} \{u(c, y) + \beta \int_0^{\infty} V(\varepsilon f(y - c) + (1 - \delta)(y - c))\mu(d\varepsilon)\}. \tag{B1}$$

The first-order condition for (B1) is

$$u_c = \beta \int_0^\infty V'(\varepsilon f' + 1 - \delta)\mu(d\varepsilon).$$

Taking differential on the above equation, we reach

$$\frac{dc}{dy} = \frac{\beta \int_0^\infty [V''(\varepsilon f' + 1 - \delta)^2 + \varepsilon f'']\mu(d\varepsilon) - u_{cy}}{u_{cc} + \beta \int_0^\infty [V''(\varepsilon f' + 1 - \delta)^2 + \varepsilon f'']\mu(d\varepsilon)}.$$

From Assumptions 1-3, the monotonicity and concavity of V, we conclude $\frac{dc}{dy} > 0$. Hence, for the same reason, there exists uniquely $c = g(y)$ and $g' = \frac{dc}{dy}|_{c=g(y)} > 0$.

Moreover, because $s(y) = y - g(y)$, we have

$$s'(y) = 1 - g'(y) = \frac{u_{cc} + u_{cy}}{u_{cc} + \beta \int_0^\infty [V''(\varepsilon f' + 1 - \delta)^2 + \varepsilon f'']\mu(d\varepsilon)}.$$

Therefore, if $u(c, y) = u_1(c) + u_2(y)$, then $u_{cy} = 0$. Hence, $s'(y) > 0$ for all $y \in \mathbb{R}_+$.

References

[1] Brock, W.A. and L. Mirman, 1972. Optimal economic growth and uncertainty: The discounted case. *Journal of Economic Theory* 4, 479-513.

[2] Bakshi, G.S., and Z. Chen, 1996. The spirit of capitalism and stock-market prices. The American Economic Review 86: 133-157.

[3] Benhabib, J. and R.E.A. Farmer, 1999. Indeterminacy and sunspots in Macroeconomics. In: *Handbook of Macroeconomics*. Volume 1A, 387-448.

[4] Cole, H., G. Mailath, and A. Postlewaite, 1992. Social norms, saving behavior and growth. *Journal of Political Economy* 100: 1092-125.

[5] Cole, H., G. Mailath, and A. Postlewaite, 1995. Incorporating concern for relative wealth into economic models. *Quarterly Review, Federal Reserve Bank of Minneapolis*, vol.19: 12-21.

[6] Duesenberry, J., 1948. *Income, Saving and the Theory of Consumer Behavior*. Harvard University Press.

[7] Fershtman, C. and Y. Weiss, 1993. Social status, culture and economic performance. *Economic Journal* 103: 946-59.

[8] Fershtman, C., K. Murphy, and Y. Weiss, 1996. Social status, education, and growth. *Journal of Political Economy* 106: 108-132.

[9] Frank, R., 1985. *Choosing the Right Pond: Human Behavior and the Quest for Status.* Oxford University Press.

[10] Gong, L. and H. Zou, 2002. Direct preferences for wealth, the risk premium puzzle, growth, and policy effectiveness. *Journal of Economic Dynamics and Control* 26(2):243-270.

[11] Hopenhayn, H.A. and E.C. Prescott, 1992. Stochastic monotonicity and stationary distributions for dynamic economies. *Econometrica* 60 (6), 1387-1406.

[12] Kamihigashi, T., 2005. Stochastic optimal growth with bounded or unbounded utility and with bounded or unbounded shocks. *RIEB*, Kobe University, Kobe, Japan.

[13] Keynes, J.M., 1971. *The Economic Consequences of the Peace.* London: St. Martin's Press.

[14] Koopmans, T., 1965. On the concept of optimal economic growth. *Pontificiae Academiae Scientiarum Scripta Varia.* 28, 225-300.

[15] Kurz, M., 1968. Optimal economic growth and wealth effects. *International Economic Review* 9, 348-57.

[16] Mirman, L.J. and I. Zilcha, 1975. On optimal growth under uncertainty. *Journal of Economic Theory* 11, 329-339.

[17] Meyn, S.P. and R.L. Tweedie, 1993. *Markov Chains and Stochastic Stability.* Springer, London.

[18] Nishimura, K. and J. Stachurski, 2005. Stability of stochastic optimal growth models: A new approach. *Journal of Economic Theory* 122; 100-118.

[19] Spence, M., 1974. *Market Signaling: Information Transfer in Hiring and Related Screening Processes.* Harvard University Press.

[20] Stachurski, J., 2002. Stochastic optimal growth with unbounded shock. *Journal of Economic Theory* 106; 40-65.

[21] Stokey, N.L., R.E. Lucas, and E.C., Prescott, 1989. *Recursive Methods in Economic Dynamics.* Harvard University Press,Cambridge MA.

[22] Torres, R., 1990. Stochastic dominance. *Discussion Paper* 905, Center for Mathematical Studies in Economics and Management Science, Northwestern University.

[23] Weber, M., 1958. *The Protestant Ethic and the Spirit of Sapitalism.* Charles Scribner's Sons, New York.

[24] Zhang, Y., 2007. Stochastic optimal growth with a non-compact state space. *Journal of Mathematical Economics* 43, 115-129.

[25] Zou, H., 1994. The spirit of capitalism and long-run growth. *European Journal of Political Economy* 10: 279-93.

[26] Zou, H., 1995. The spirit of capitalism and savings behavior. *Journal of Economic Behavior and Organization* 28: 131-143.