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发表论文
1. 朱平芳、张征宇、姜国麟，2011，FDI与环境规制：基于地方分权视角的实证研究，《经济研究》第6期。
2. 张征宇、朱平芳，2010，地方环境支出的实证研究，《经济研究》第5期。
A Pairwise Difference Estimator for Partially Linear Spatial Autoregressive Models

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Su and Jin (2010) develop for partially linear spatial autoregressive (PL-SAR) model a profile quasi-maximum likelihood based estimation procedure. More recently, Su (2011) proposes for this model a semiparametric GMM estimator. However, both of them can be computationally challenging for applied researchers and are not easy to implement in practice. In this article, we propose a computationally simple estimator for the PL-SAR model in the presence of either heteroscedastic or spatially correlated error terms. This estimator blends the essential features of both the GMM estimator for linear SAR model and the pairwise difference estimator for conventional partially linear model. Limiting distribution of the proposed estimator is established and consistent estimator for its asymptotic CV matrix is provided. Monte Carlo studies indicate that our estimator is attractive particularly when one is interested in estimating the finite-dimensional parameters in the model.

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1. INTRODUCTION

In this article, we consider semi-parametric estimation of a class of partially linear spatial autoregressive (PL-SAR) models, specified as

\[ y_i = \lambda_0 \sum_{j=1}^{n} w_{ij} y_j + x'_i \beta_0 + m(x_{2i}) + u_i, \ i = 1, \cdots, n. \]  

In model (1), \( n \) is the number of total cross sectional units; \( y_i \) is the outcome variable for \( i \); \( \lambda_0 \) is the SAR parameter; \( [w_{ij}]_{i,j=1,\cdots,n;i\neq j} \)’s are termed as the spatial weights, which determine the structure of neighborliness among cross sectional units; \( x_{li}, l = 1, 2, \) is the \( p \times 1 \) -dimensional column vector of exogenous regressors (excluding a constant term); \( \beta_0 \) is a \( p \times 1 \) -dimensional vector of slope coefficients; \( m(\cdot) \) is a unknown function mapping from \( \mathbb{R}^{p \times 2} \) to the real line; \( u_i \)'s are zero mean error terms that are not necessarily either identically or independently distributed and allowed to exhibit a general form of spatial dependence or heteroscedasticity.\(^2\)

In the case \( m(\cdot) \equiv 0 \), model (1) is simplified to a conventional linear SAR model, which has already found its widespread application in empirical analysis related to urban, environmental economics, industrial organization and regional convergence. Meanwhile, a number of methods have been proposed to estimate the parametrically specified SAR model, including the method of moments by Kelejian and Prucha (1999, 2010), the method of quasi-maximum likelihood estimation by Lee (2004), the method of two-stage least squares by Kelejian and Prucha (1998), Lee (2003), Zhang and Zhu (2010) and the generalized method of moments by Lee (2007), Lin and Lee (2010), and Liu et al. (2010). Although the SAR model and its various parametric forms have attracted a great deal of attention both theoretically and empirically, until just recently, researchers have started addressing the importance of nonparametric modeling in spatial econometrics (Baltagi and Li 2001; Pace et al. 2004; Yang et al. 2006). Su and Jin (2010) provide a detailed discussion on motivations for including both nonlinearity and nonparametric component in specifying a SAR model. In comparison with its parametric version, PL-SAR model has a more flexible functional form, thus effectively lessening the possibility of either inconsistent estimate or misleading statistical inference due to mis-specification.\(^3\)

Several estimation procedures have been developed for model (1). Su and Jin (2010) propose a profiled quasi-maximum likelihood based estimator (PQMLE) and demonstrate that the rates of consistency for the finite-dimensional parameters in the model depend on some general features of the spatial weighting schemes. A major drawback for Su and Jin (2010)’s PQMLE seems to be that the error term \( u_i \)'s in their model are required to be both homoscedastic and independently distributed. In the presence of heteroscedasticity, Lin and Lee (2010) have demonstrated that the

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\(^1\)Throughout the paper, any parameter with subscript zero represents the true parameter that generates the data.

\(^2\)See Assumption 2 in Section 3.

\(^3\)Other theoretical studies for regression models with partially linear structure include Robinson (1988), Chen and Khan (2001), Lee (2003) and Sun (2005), to mention a few.
QML-based estimator is usually inconsistent. More recently, Su (2011) proposes for this PL-SAR model a semiparametric GMM estimator (SPGMME). His estimation strategy contains two stages. The first stage treats the finite-dimensional parameters as given, and uses local instruments to estimate $m(\cdot)$ locally as a function of these finite-dimensional parameters. In the second stage, he uses global instruments to estimate the finite-dimensional parameters by profiling out the nonparametric component, and then recovers the estimate of the nonparametric component. As emphasized by Su (2011), SPGMME has several advantages relative to the earlier PQMLE: First, SPGMME achieves usual $\sqrt{n}$ consistency rate for the parametric component in the model whereas the consistency rate of PQMLE depends on some general features of the spatial weight matrix. Second, SPGMME is robust against both heteroscedasticity of unknown form and certain form of spatial dependence in the error terms. Third, the procedure of SPGMME can be readily extended to semi-parametric SAR panel data models or semiparametric models with endogeneity.

However, both estimators mentioned above seem to be computationally challenging for applied researchers and are not easy to implement in practice. In this article, an alternative computationally simple procedure is proposed for consistently estimating PL-SAR model (1), particularly, for simple estimation of the finite-dimensional parameters in the model. We build our estimator on a type of pairwise difference estimator (PDE) that involves pairwise differences of observations for which the regressors in the nonparametric component of the regression function are approximately equal. See Ahn and Powell (1993), Honore and Powell (2005, 2007) and Chen and Zhou (2005) for applications of PDE in various nonlinear models and microeconometric models.

Compared with other applications of PDE, the setting of our SAR model complicates the econometric analysis in two ways. First, for any SAR model, the spatially lagged variable $\bar{y}_i = \sum_{j \neq i} w_{ij} y_j$ is generally correlated with the error term, which precludes any OLS-based estimation procedures. In this position, our suggested PDE for PL-SAR model should combine the essential features of both the GMM estimator for linear SAR model and the pairwise difference estimator for conventional partially linear model. Second, for many applications of PDE, say, in the field of microeconometrics, observations are assumed to follow an i.i.d. sampling scheme. By contrast, for SAR model the observed variables, due to the presence of spatial dependence, are unlikely to be independently distributed across $i$, thus some technical modification is needed to adapt the asymptotic analysis that was developed for conventional PDE to this spatial context.

It is also instructive to compare our PDE procedure with Su (2011)’s SPGMME. It can be argued that our PDE shares all the above-mentioned advantages (closed form expression, $\sqrt{n}$-convergence, robustness against heteroscedasticity and correlated error terms, extendibility) of Su (2011)’s SPGMME relative to PQMLE. But while Su’s SPGMME “profiles” out the nonparametric component as a function of the exogenous regressors before applying instrumental variables (IV) to estimate the finite-dimensional parameters, our PDE “eliminates” the nonparametric component by pairwise differencing, and then applies IV method to estimate the finite-dimensional parameters. By “eliminating” instead of “profiling out” the nonparametric component, our estimator seems to
be computationally simpler than SPGMME particularly when one is interested in estimating the finite-dimensional parameters in the model.

The rest of the paper is organized as follows: Section 2 introduces the pairwise difference estimator and its large sample properties are established in Section 3. Section 4 reports some Monte Carlo results and Section 5 concludes. Technical details are collected in an Appendix.

2. THE ESTIMATOR

The pairwise difference estimator is based on a comparison of pairs of observations for which the regressors in the nonparametric component of the regression function, i.e., \( x_2 \), are close to each other. In an ideal situation, for two indices \( i \) and \( j \), satisfying \( x_2^i = x_2^j \), differencing the corresponding dependent variable \( y_i \) and \( y_j \) gives

\[
\Delta y_{ij} = y_i - y_j = \lambda_0(W_n i - W_n j)Y_n + (x_{1i} - x_{1j})'\beta_0 + u_i - u_j,
\]

where \( Y_n = (y_1, \cdots, y_n)' \) and \( W_n i \) is the \( i \)-th row of \( W_n \). Observe that for (2), the spatial lag term \((W_n i - W_n j)Y_n\) is generally correlated with the error term \( u_i - u_j \), which implies that any OLS-based estimator will be inconsistent. Now consider GMM estimation based on orthogonality of some instruments and the error terms. Let \( q_{ij}, i, j = 1, \cdots, n, i < j \), be a sequence of well chosen \( p_q(\geq p_{x1} + 1) \)-dimensional non-stochastic instruments, such that we have the moment condition

\[
E(q_{ij}(u_i - u_j)) = 0.
\]

(3)

Bearing the condition \( x_2^i = x_2^j \) in mind, the empirical analogue to the moment condition (3) for the whole sample can be written as

\[
\frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{j>i}^{n} 1(x_{2i} = x_{2j})E(q_{ij}\Delta y_{ij}(\theta_0)) = 0,
\]

(4)

where \( \theta = (\lambda, \beta)' \), \( \Delta y_{ij}(\theta) = \Delta y_{ij} - \lambda(W_n i - W_n j)Y_n - (x_{1i} - x_{1j})'\beta \). When \( x_{2i} = x_{2j} \), \( \Delta y_{ij}(\theta_0) = \Delta y_{ij} = u_i - u_j \). Then the GMM estimator of \( \theta \) based on the moment condition (4) can be viewed as the solution to the following minimization problem,

\[
\min_{\theta} \left[ \sum_{i=1}^{n} \sum_{j>i}^{n} 1(x_{2i} = x_{2j})q_{ij}\Delta y_{ij}(\theta) \right]' A_n \left[ \sum_{i=1}^{n} \sum_{j>i}^{n} 1(x_{2i} = x_{2j})q_{ij}\Delta y_{ij}(\theta) \right],
\]

(5)

where \( A_n \) is some \( k_q \times k_q \) nonnegative definite matrix and is assumed to converge to a constant matrix \( A \). This design corresponds to Hansen (1982)'s GMM setting, which can be used to illustrate the optimal weighting issue.

However, the estimation scheme (5) can not be directly implemented. When \( x_{2i} \) is continuously distributed, there are no two such indices \( i, j \) that \( x_{2i} = x_{2j} \) with probability one. A usual manner to cope with this issue is to assign each pair with indices \( (i, j) \) a weight, which declines to zero as
the magnitude of the difference $\|x_{2i} - x_{2j}\|$ increases. This idea can be straightforwardly translated into replacing the indicator function $1(\cdot)$ in (5) with its smoothed counterpart, a symmetric kernel function. To formally introduce our estimator, let $k(\cdot)$ be a univariate symmetric kernel satisfying $\int k(v)dv = 1$ and $k(v) = k(-v)$. $h_n = (h_{1n}, \cdots, h_{p_{x2}n})$ is a vector of smoothing parameters. For a $p_{x2}$-dimensional vector $s = (s_1, \cdots, s_{p_{x2}})$ with all its components being absolute continuous with Lebesgue measure, define $k_{hi}(\cdot) = h_{in}^{-1}k(\cdot/h_{in})$, $K_h(s) = \prod_{i=1}^{p_{x2}} k_{hi}(s_i)$ and $K_h^{x2}$ to be the $[n(n-1)/2] \times [n(n-1)/2]$ square matrix diag $\{K_h(x_{2i} - x_{2j})\}_{i,j=1,\cdots,n,i<j}$. For example, when $n = 3$, $K_h^{x2}$ = diag $\{K_h(x_{21} - x_{22}), K_h(x_{21} - x_{23}), K_h(x_{22} - x_{23})\}$. Furthermore, denote $\Delta_h^u(\theta)$ to be the $n(n-1)/2$-dimensional column vector of $\Delta_{ij}^u(\theta)$’s, $i, j = 1, \cdots, n, i < j$. For example, when $n = 3$, $\Delta_h^u = (\Delta_{12}^u(\theta), \Delta_{13}^u(\theta), \Delta_{23}^u(\theta))'$. Similarly, let $Q_h$ be the $\binom{n(n-1)}{2} \times p_q$ matrix by stacking the row vector $q_{ij}$’s. With these notations, our pairwise difference estimator is the solution to the following minimization problem,

$$\hat{\theta}_n = \arg\min_{\theta} \left[ Q_n' K_h^{x2} \Delta_h^u(\theta) \right]' A_n \left[ Q_n' K_h^{x2} \Delta_h^u(\theta) \right].$$

(6)

Given $\Delta_h^u(\theta)$ being linear in $\theta$, $\hat{\theta}_n$ has the explicit representation,

$$\hat{\theta}_n = \left[ Z_n' K_h^{x2} Q_n A_n Q_n' K_h^{x2} Z_n \right]^{-1} Z_n' K_h^{x2} Q_n A_n Q_n' K_h^{x2} \Delta_h^u,$$

(7)

where $\Delta_h^u$ is the $n(n-1)/2$-dimensional column vector of $y_i - y_j$’s, $i, j = 1, \cdots, n, i < j$, $Z_n = [\Delta_h^u \cdot Y_n, \Delta_h^{z1}]$, $\Delta_h^w$ is the $\binom{n(n-1)}{2} \times n$ matrix by stacking the row vector $(W_{ni} - W_{nj})$’s and $\Delta_h^{z1}$ is the $\binom{n(n-1)}{2} \times p_{x1}$ matrix by stacking the row vector $(x_{li} - x_{lj})$’s. Note that in the case $p_q = p_{x1} + 1$, $Z_n' K_h^{x2} Q_n$ is a square matrix and we get a simpler expression for $\hat{\theta}$,

$$\hat{\theta}_n = \left[ Z_n' K_h^{x2} Q_n \right]^{-1} Q_n' K_h^{x2} \Delta_h^u,$$

(8)

which does not depend on $A_n$.

For the moment, no specific form of the IV matrix $Q_n$ has been given. As choice of such $Q_n$ is not unique, in Section 3, we develop a general asymptotic theory for this PDE as long as $Q_n$ meets some regularity conditions. According to our experience in the Monte Carlo experiment, one may set $q_{ij} = (W_{ni} - W_{nj})Q_{1n}$, where the $n \times p_q$ matrix $Q_{1n}$ is composed of linearly independent $p_q$ columns chosen from $[X_{1n}, W_{n}X_{1n}, W_{n}X_{2n}, W_{n}^2X_{1n}, W_{n}^2X_{2n}, \cdots]$, $X_{ln} = (x_{l1}, \cdots, x_{ln})'$ for $l = 1, 2$. It should be noted that here our strategy of choosing the instrument $q_{ij}$’s, is essentially similar to Su (2011) and other prior literature on choosing IV for a parametrically specified SAR model (e.g., Kelejian and Prucha 1998; Lin and Lee 2010). We will study the optimal choice of $A_n$ for the given choice of $Q_n$, while the optimal choice of instruments is beyond the scope of this paper.

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4See Assumption 3-(ii) in Section 3.
5For identification, $q_{ij}$’s should also be chosen to meet some rank conditions, as summarized by Assumption 6 in Section 3.
6Like Su (2011), the best selection of instruments $q_{ij}$’s is not readily available partly because both heteroscedasticity and spatial autocorrelation of unknown form is allowed in the error terms.
3. LARGE SAMPLE PROPERTIES

To analyze the asymptotic properties of our estimator, let’s introduce the following regularity conditions. It should be emphasized that among these assumptions, most are also maintained by Su and Jin (2010) and Su (2011) (except Assumption 3-(ii) and 6).

ASSUMPTION 1. (i) The spatial weights matrix \( W_n = [w_{ij}]_{i,j=1,\ldots,n} \) has zero diagonals. (ii) The matrix \( S_n = I_n - \lambda_0 W_n \) is nonsingular. (iii) The row and column sums of \( S_n^{-1} \) and \( W_n \) are uniformly bounded in absolute value.\(^7\)

Assumption 1 concerns the essential features of spatial weights matrix. Assumption 1-(i) implies that each unit is not a neighbor of itself. Assumption 1-(ii) implies that the model considered here is well defined, that is, the dependent variable \( Y_n \) is uniquely determined in terms of the disturbances conditional on the regressors. The uniform boundedness condition on \( W_n \) and \( S_n^{-1} \) is originated in a series of papers by Kelejian and Prucha, see, e.g., Kelejian and Prucha (1998, 1999), in order to limit the spatial dependence across units to a manageable degree.

ASSUMPTION 2. \( U_n = R_n \tau_n, \) where the components of \( \tau_n = (\epsilon_1, \ldots, \epsilon_n)' \) are i.i.d. with zero mean, unit variance and some finite absolute moment of order three. \( R_n \) is a \( n \times n \) constant matrix whose row and column sums are uniformly bounded.

Existence of the absolute moment of order three for \( \epsilon_i \)'s is assumed so that some central limit theorem of simple form can be applied. This assumption can be relaxed to the existence of some absolute moment of order higher than two. Assumption 2 allows for not only heteroscedasticity but also spatial dependence across \( u_i \)'s. When \( R_n \) is a scalar matrix or a diagonal matrix, we have either i.i.d. \( u_i \)'s or independently distributed but heteroscedastic \( u_i \)'s. For a general \( R_n \), it allows the error term \( u_i \)'s to exhibit various scenarios of spatial dependence. For example, when \( R_n = I_n + \rho_0 M_n \), \( U_n \) is assumed to follow a spatial moving average process; when \( R_n = (I_n - \rho_0 M_n)^{-1} \), then \( U_n \) follows a SAR process. Note that here \( M_n \) is not necessarily required to be the same as \( W_n \), as long as the row and column sums of \( M_n \) or \( (I_n - \rho_0 M_n)^{-1} \) are uniformly bounded, as paralleling Assumption 1 for \( W_n \) and \( (I_n - \lambda_0 W_n)^{-1} \).

ASSUMPTION 3. (i) \( X_n = [X_{1n}, X_{2n}] \) is a non-stochastic, \( n \times (p_{x1} + p_{x2}) \) regressor matrix whose elements are uniformly bounded. (ii) There exists a sequence of \( p_q \)-dimensional non-stochastic \( q_i \)'s, \( i = 1, \ldots, n \), such that \( q_{ij} = q_i - q_j \), where the elements of \( q_i \)'s are uniformly bounded by a constant.

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\(^7\)The row and column sums of an \( n \times n \) matrix \( P_n \) are said to be uniformly bounded if we have for all \( n \), there exists a positive constant \( c \) independent of \( n \) with \( \max \sum_{j=1}^{n} |P_{n,ij}| < c \) and \( \max_{i} \sum_{j=1}^{n} |P_{n,ij}| < c \). This notion of uniform boundedness can be defined in terms of some matrix norms. The maximum column sum matrix norm \( \| \cdot \|_1 \) of an \( n \times n \) matrix \( P_n \) is defined as \( \| P_n \|_1 = \max_{i} \sum_{j=1}^{n} |P_{n,ij}| \), and the maximum row sum matrix norm \( \| \cdot \|_\infty \) is defined as \( \| P_n \|_\infty = \max_{j} \sum_{i=1}^{n} |P_{n,ij}| \). Thus the uniform boundedness of \( \{P_n\} \) in column or row sums is equivalent to the sequence \( \{\|P_n\|_1\} \) or \( \{\|P_n\|_\infty\} \) being bounded.
The non-stochastic design assumption of $X_n$ is made for several reasons. First, it parallels that of Kelejian and Prucha (1998, 1999, 2001, 2010), Lee (2004, 2007) and Lin and Lee (2010). Second, it allows us to avoid the use of trimming factors (e.g. Robinson 1988). As noted by Lee (2007), non-stochastic regressor design and its uniform boundedness condition are made for technical convenience. If the elements of $X_n$ are stochastic and have unbounded ranges, conditions in Assumption 3-(i) can be replaced by some finite moment conditions. Assumption 3-(ii) provides the regularity conditions for a general $Q_n$ to meet. Given the uniform boundedness condition of $X_n$ and $W_n$ by Assumption 1 and 3-(i), it is reasonable to construct $q_i$’s based on some combinations of $X_n$ and $W_n$. As an example, one may easily verify that the $q_{ij} = (W_{ni} - W_{nj})Q_{1n}$ defined in Section 2 satisfies these conditions, with $q_i = W_{ni}Q_{1n}$.

Assumption 4. Suppose that $x_{2i} \in \mathcal{X}_2 \subset \mathbb{R}^{p_2}$, for $i = 1, \ldots, n$. (i) There exists a scalar function $f(\cdot)$ and a $p_q$-dimensional vector valued function $\varphi(\cdot) = (\varphi_1(\cdot), \ldots, \varphi_{p_q}(\cdot))'$, both of which are defined on $\mathcal{X}_2$, such that

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} q_{il} \nu(x_{2i}) = \int_{s \in \mathcal{X}_2} \nu(s) \varphi_l(s) f(s) ds \quad (9)$$

for any bounded and continuous function $\nu(\cdot)$ defined on $\mathcal{X}_2$, where $q_{il}$ is the $l$-element of $q_i$. Furthermore, if $q_{il} = c$ for some $l$ and for all $i = 1, \ldots, n$, $\varphi_l(\cdot) \equiv c$. Both $f(\cdot)$ and $\varphi(\cdot)$ are bounded, continuous and have bounded first order derivatives on $\mathcal{X}_2$. (ii) The function $m(x)$ is bounded, continuous and has bounded second order derivatives.

Assumption 4-(i) parallels Assumption 3 in Su (2011). Eqn. (9) is frequently seen in spatial econometrics literature involving nonparametric techniques (Su and Jin, 2010; Su, 2011), or more generally, in nonparametric regression literature with fixed regressor design (Linton 1995). Essentially, Eqn. (9) relates the fixed design to an implicit random generation mechanism. For stationary random observations, $f(\cdot)$ can be interpreted as the underlying probability density function that generates $x_{2i}$, $i = 1, \ldots, n$ and $\varphi(\cdot)$ can be regarded as the conditional expectation of $q_i$ given $x_{2i}$. Consider three important special cases implied by this assumption. First, if $q_{il} = c$ for some $l$ and for all $i = 1, \ldots, n$, one may regard $q_{il}$ being generated from a degenerate distribution with all the probability concentrated on a single point $c$. In this case, the expectation of $q_{il}$ conditional on any random variables will be $c$, which has been explicitly assumed above. Second, letting $q_{il} \equiv 1$, we have for any bounded and continuous function $\nu(\cdot)$,

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \nu(x_{2i}) = \int_{s \in \mathcal{X}_2} \nu(s) f(s) ds, \quad (10)$$

which is consistent with the general law of large numbers. Third, if one further lets $\nu \equiv 1$, Eqn. (10) is reduced to the fact that $f(\cdot)$ integrates to one over $\mathcal{X}_2$. In Assumption 4, the components of $x_{2i}$’s are required to be absolute continuous with respect to Lebesgue measure. Like other literature on pairwise difference estimation (e.g., Ahn and Powell 1993; Honore and Powell 2007),
Assumption 6. The limit of the literature. In the case where Assumption 5-(i) automatically. Like Su (2011), the conditions in Assumption 5-(ii) imply that the Gaussian kernel throughout and this kernel can be shown to meet Assumption 5-(ii) concerns the choice of smoothing parameters. In the Monte Carlo simulation, we use the Gaussian kernel throughout and this kernel can be shown to meet Assumption 5-(i) automatically. Like Su (2011), the conditions in Assumption 5-(ii) imply implicitly that \( p_{x_2} \leq 3 \). This is not too restrictive given the “curse of dimensionality” in the nonparametric literature. In the case where \( p_{x_2} \geq 4 \), one can apply a kernel of higher order.

Assumption 5. (i) The kernel function \( k(\cdot) \) is continuous, satisfying \( k(v) = k(-v) \), \( \int k(v)dv = 1 \), \( \int |k(v)v^l|dv < \infty \) for \( l = 1, 2 \) and \( \sup_v |k(v)| < c_k \). (ii) As \( n \to \infty \), \( \max_{1 \leq l \leq p_{x_2}} h_{ln} \to 0 \), \( n h_{1n} \cdots h_{np_{x_2}} \to \infty \), \( n \sum_{l=1}^{p_{x_2}} h_{ln}^2 \to 0 \).

This assumption parallels Assumption 4 in Su (2011). Assumption 5-(i) concerns the choice of kernel function and Assumptions 5-(ii) concerns the choice of smoothing parameters. In the Monte Carlo simulation, we use the Gaussian kernel throughout and this kernel can be shown to meet Assumption 5-(i) automatically. Like Su (2011), the conditions in Assumption 5-(ii) imply that \( p_{x_2} \leq 3 \). This is not too restrictive given the “curse of dimensionality” in the nonparametric literature. In the case where \( p_{x_2} \geq 4 \), one can apply a kernel of higher order.

Assumption 6. The limit of \( J_n = \frac{1}{n} \sum_{i=1}^{n} f(x_{2i})\Xi_i(q_i - \varphi(x_{2i}))' \) exists and has full column rank, where \( \Xi_i = [W_n S_n^{-1}(X_{1i} 0 \circ m(X_{2n})), x_{1i}']' \), \( X_{ln} = [x_{11}, \cdots, x_{ln}]' \) for \( l = 1, 2 \), \( m(X_{2n}) = (m(x_{21}), \cdots, m(x_{2n}'))' \), \( f(\cdot) \) and \( \varphi(\cdot) \) has been defined in Assumption 4.

Essentially, Assumption 6 provides a sufficient condition that allows us to identify \( \theta \) uniquely. Given the expression of \( \hat{\theta}_n \) in (7), it is straightforward to show that \( \theta \) is identifiable as long as the limit of \( \frac{1}{n(n-1)}E(Z_n)'K_n^{1/2}Q_n \) exists and has full column rank. In the Appendix, we show that

\[
\frac{1}{n(n-1)}E(Z_n)'K_n^{1/2}Q_n - J_n = o(1).
\] (11)

Furthermore, Assumption 1, 3-4, together with Lemma 1 imply that \( J_n \) is \( O(1) \).

To establish the limiting distribution of our estimator, let’s introduce more notations. Define \( F_n = \text{diag}\{f(x_{21}), \cdots, f(x_{2n})\} \), \( Q_n^c = [q_1 - \varphi(x_{21}), \cdots, q_n - \varphi(x_{2n})]' \), \( \Omega_n \) in Assumption 6 then can be equivalently expressed as \( \frac{1}{n} \sum_{i=1}^{n} f(x_{2i})Q_n^c \). Similar arguments can be applied to show that \( \Omega_n = \frac{1}{n} \sum_{i=1}^{n} f(x_{2i})Q_n^c \). The following proposition provides the asymptotic distribution of PDE.

Proposition 1. Suppose that Assumption 1-6 hold, then the estimator (7) has the limiting distribution \( \sqrt{n}(\hat{\theta}_n - \theta_0) \rightarrow^d N(0, \Sigma) \) where

\[
\Sigma = \lim_{n \to \infty} \left( J_n A_n J_n' \right)^{-1} J_n A_n \Omega_n A_n J_n' \left( J_n A_n J_n' \right)^{-1}
\] (12)

provided that the limits of \( \Omega_n \) and \( A_n \) exist and have full column rank.

For purposes of inference, we propose a consistent estimator for the limiting covariance matrices above. By (12), it suffices to focus on estimate of \( J_n \) and \( \Omega_n \). A consistent estimate of \( J_n \) is
straightforward. Actually, in the Appendix we have shown that in proving Proposition 1,  
\[
\hat{J}_n = \frac{1}{n(n-1)} Z_n K_n^{x2} Q_n \xrightarrow{p} \lim_{n \to \infty} J_n. 
\]

Similarly, \( \Omega_n \) can be consistently estimated by
\[
\hat{\Omega}_n = \frac{1}{n(n-1)} Q_n' K_n^{x2} [\Delta_n^{\prime} - Z_n \hat{\theta}_n] [\Delta_n^{\prime} - Z_n \hat{\theta}_n]' K_n^{x2} Q_n. 
\]

When \( p_q = p_{x1} + 1, J_n \) is invertible, that is, the CV matrix \( \Sigma \) in (12) has a simpler form
\[
\Sigma = \lim_{n \to \infty} (J_n \Omega_n^{-1} J_n')^{-1}, 
\]
which is independent of \( A_n \). For a general case \( p_q > p_{x1} + 1 \), given the instruments \( q_{ij} \)'s, the optimal choice of a weighting matrix \( A_n \) will be \( \Omega_n^{-1} \) by the generalized Schwartz inequality, yielding \( \Sigma = \lim_{n \to \infty} (J_n \Omega_n^{-1} J_n')^{-1} \) once again.

Finally, given \( \sqrt{n} \)-consistent estimates of the parametric components, a consistent estimate of the nonparametric component seems to be like a usual exercise, that is, \( m(\cdot) \) can be estimated by
\[
\hat{m}_n(x) = \frac{\sum_{i=1}^{n} \left( y_i - \hat{\lambda}_n \sum_{j \neq i} w_{ij} y_j - x_{1i} \beta_{10} + x_{2i} \beta_{20} + m(z_i) + u_i \right)}{\sum_{i=1}^{n} K_h(x - x_{2i})} 
\]

4. MONTE CARLO SIMULATION

In this section, we conduct a small scale Monte Carlo experiment to evaluate the finite sample performance of the suggested PDE for PL-SAR model. A larger Monte Carlo study relating to a wider set of experiments than those described below is left for future research. The DGP is given by
\[
y_i = \lambda_0 \sum_{j=1}^{n} w_{ij} y_j + x_{1i} \beta_{10} + x_{2i} \beta_{20} + m(z_i) + u_i, i = 1, \cdots, n, 
\]
where \( z_i \) is a scalar variable, i.i.d. drawn from the uniform distribution on \([-2, 2]\), \( x_{2i} = 0.2 z_i^2 - z_i + \eta_i \), where \( \eta_i \) is i.i.d. drawn from a standard normal distribution, and \( x_{1i} \) is a 0-1 dummy variable with \( \Pr(x_{1i} = 1) = \Pr(x_{1i} = 0) = 0.5 \). We set \( \beta_{10} = \beta_{20} = 1, \lambda_0 = 0.5 \) and consider two cases for \( m(\cdot) \): (i) \( m_1(z) = \exp(z)/(1 + \exp(z)) \) and (ii) \( m_2(z) = 1 + 0.5 \sin(0.5 \pi z) \), two cases for \( U_n \): (i) \( u_i = \sqrt{1 + z_i^2} \cdot \epsilon_i \) (heteroscedastic error terms) and (ii) \( U_n = (I_n - 0.3 \cdot W_n)^{-1} \epsilon_n \) (errors that are both heteroscedastic and spatially correlated), two cases for the distribution of \( \epsilon_i \)'s: (i) \( \epsilon_i \sim \mathcal{N}(0, 1) \) and (ii) \( \epsilon_i \sim \chi^2(1)/\sqrt{2} \), and two choices of instruments: (i) \( \{q_1, \cdots, q_9\}' = Q^{(1)}_n = [X_{1n}, X_{2n}, W_n X_{1n}, W_n X_{2n}, W_n Z_n] \) and (ii) \( Q^{(2)}_n = [X_{1n}, X_{2n}, W_n X_{1n}, W_n X_{2n}, W_n Z_n, W_n^2 X_{1n}, W_n^2 X_{2n}, W_n^2 Z_n] \).

Like Su and Yang (2011) and Su (2010), we generate the spatial weight matrix \( W_n \) according to the principle of Rook contiguity, by randomly allocating the \( n \) spatial units on a lattice of \( \sqrt{n} \times \sqrt{n} \) squares, finding the neighbors for each unit, and then row normalizing. To implement the estimator, we need to choose a kernel function \( k(\cdot) \) and a bandwidth sequence \( h_n \). Throughout, we will choose a Gaussian kernel. As it is difficult to specify the optimal bandwidth sequence, we
choose the bandwidth by a rule-of-thumb method: $h_n = \sigma_z n^{-1/3.5}$, where $\sigma_z$ is the sample standard deviation of $z_i$'s. In practice, it is recommended that we use this bandwidth as the initial smoothing parameter to obtain a preliminary consistent estimator while in the second step, we conduct the least squares cross validation method to choose the bandwidth. We have applied both methods to compute the estimators but found their performances are similar for the DGP under study. So we only report the simulation results for the former case. We have experimented with $n = 100, 196$ and 400. For each case, there are 1000 repetitions.

Table 1 summarizes the simulation results, empirical bias and empirical standard deviation of the estimators when $u_i$’s are heteroscedastic but independent from each other. There are several main observations:

1. In the presence of heteroscedasticity, the estimators perform fairly well even in small sample $n = 100$, that is, they are computationally easy, only slightly biased while give acceptable precision.
2. Standard deviation of these estimators declines with the sample size. The magnitude of such decline is generally consistent to $\sqrt{n}$-asymptotics, which is consistent with the theoretical predictions.
3. The estimators are only slightly biased and the magnitude of such bias, if any, declines with the growing sample size. When $n = 400$, these estimators are essentially unbiased.
4. The estimators seem quite robust with respect to the distribution of the error terms, functional form of nonparametric components and choice of instruments. Their performance, in terms of either bias or standard deviation, does not seem to change much for different distributions of the error terms, functional forms of nonparametric components and choices of instruments.

Table 2 summarizes the simulation results of the estimators when $u_i$’s are both heteroscedastic and spatially correlated. Overall, the resulting estimators perform similarly to those in Table 1.

5. CONCLUSION

In this paper we propose a semi-parametric pairwise difference estimator of partially linear SAR models where the error term may exhibit either heteroscedasticity or spatial dependence. We derive the limiting distribution of the proposed estimator and suggest a simple way to consistently estimate the asymptotic CV matrix. In comparison with other existing estimation procedures (Su and Jin 2010; Su 2011), our estimator is computationally simpler in obtaining the consistent estimate of the finite-dimensional parameters in the model. For the applied researchers who are mainly interested in estimating the parametric component of the model, our estimator seems to be more attractive. Simulation results indicate that our estimator is promising even in small samples.

APPENDIX

Lemma 1. For any two $n \times n$ matrices $B_{1n}$ and $B_{2n}$ whose row and column sums are bounded uniformly bounded by a constant, (i) the row and column sums of $B_{1n}B_{2n}$ are also uniformly bounded
Proof of Eqn (11). For ease of notation, let $p_{22} = 1$ without loss of generality. Let $\bar{Z}_n = E(Z_n) = [\Delta_n^{\infty} S_n^{-1}(X_{11}\beta_0 + m(X_{2n})), \Delta_n^{\infty}].$ We have by symmetric kernel $k(\cdot)$ and $q_{ij} = q_i - q_j$ that

$$
\frac{1}{n(n-1)} \overline{Z}_n^{\prime} K_n^{\bar{Z}_n^{\prime}} Q_n = - \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{i=1}^{n} \sum_{j=1}^{n} z_{ij} q_{ij} h_n^{-1} k \left( \frac{x_{2i} - x_{2j}}{h_n} \right)
$$

= $$
- \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{j=1}^{n} (z_{i} - z_{j}) q_{ij} h_n^{-1} k \left( \frac{x_{2i} - x_{2j}}{h_n} \right) = - \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{j=1}^{n} z_{ij} q_{ij} h_n^{-1} k \left( \frac{x_{2i} - x_{2j}}{h_n} \right)
$$

= $$
- \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{j=1}^{n} z_{ij} q_{ij} h_n^{-1} k \left( \frac{x_{2i} - x_{2j}}{h_n} \right) - \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{j=1}^{n} z_{ij} q_{ij} h_n^{-1} k \left( \frac{x_{2i} - x_{2j}}{h_n} \right)
$$

= $$
- \frac{1}{n-1} \sum_{i=1}^{n} z_{ij} q_{ij} h_n^{-1} k \left( \frac{x_{2i} - x_{2j}}{h_n} \right) - \frac{1}{n-1} \sum_{i=1}^{n} z_{ij} q_{ij} h_n^{-1} k \left( \frac{x_{2i} - x_{2j}}{h_n} \right)
$$

= $$
- \frac{1}{n-1} \sum_{i=1}^{n} z_{ij} q_{ij} h_n^{-1} \int k \left( \frac{x_{2i} - s}{h_n} \right) f(s) ds - \frac{1}{n-1} \sum_{i=1}^{n} z_{ij} q_{ij} h_n^{-1} \int \varphi'(s) k \left( \frac{x_{2i} - s}{h_n} \right) f(s) ds + o(1)
$$

= $$
- \frac{1}{n-1} \sum_{i=1}^{n} z_{ij} q_{ij} \int k(t) f(h_n t + x_{2i}) dt - \frac{1}{n-1} \sum_{i=1}^{n} z_{ij} \int \varphi'(h_n t + x_{2i}) k(t) f(h_n t + x_{2i}) dt + o(1)
$$

= $$
- \frac{1}{n-1} \sum_{i=1}^{n} z_{ij} q_{ij} \int k(t) \left( f(x_{2i}) + h_n t f^{(1)}(\bar{x}_{2i}) \right) dt
$$

\begin{align*}
&- \frac{1}{n-1} \sum_{i=1}^{n} z_{ij} \int \left( \varphi(x_{2i}) + h_n t \varphi^{(1)}(\bar{x}_{2i}) \right) k(t) \left( f(x_{2i}) + h_n t f^{(1)}(\bar{x}_{2i}) \right) dt + o(1) \\
&= \frac{1}{n} \sum_{i=1}^{n} f(x_{2i}) z_{ij} (q_i - \varphi(x_{2i})) + o(1) = J_n + o(1),
\end{align*}

where $f^{(1)}(\cdot)$ and $\varphi^{(1)}(\cdot)$ are the first derivatives of $f$ and $\varphi$ respectively, both $\bar{x}_{2i}$ and $\bar{x}_{2i}$ lie between $x_{2i}$ and $x_{2i} + h_n t.$
Proof of Proposition 1. For ease of notation, let \( p_{x_2} = 1 \) without loss of generality. We have

\[
\sqrt{n}(\hat{\theta}_n - \theta_0) = \left[ \frac{1}{n(n-1)} Z_n' K_n^{x_2} Q_n A_n \frac{1}{n(n-1)} Q_n' K_n^{x_2} Z_n \right]^{-1} \frac{1}{n(n-1)} Z_n' K_n^{x_2} Q_n A_n \frac{\sqrt{n}}{n(n-1)} Q_n' K_n^{x_2} [\Delta_n^m + \Delta_n^u],
\]

where \( \Delta_n^m \) is the \( n(n-1)/2 \)-dimensional vector of \( \Delta_{ij}^m \)'s, with \( \Delta_{ij}^m = m(x_{2i}) - m(x_{2j}) \), for \( i, j = 1, \ldots, n, i < j \). Then the desired limiting distribution of \( \hat{\theta}_n \) follows if one can establish the following results, that is, (i) \( \frac{1}{n(n-1)} Z_n' K_n^{x_2} Q_n - J_n \toP 0 \); (ii) \( \frac{\sqrt{n}}{n(n-1)} Q_n' K_n^{x_2} \Delta_n^u \toD \mathcal{N}(0, \lim_{n \to \infty} \Omega_n) \); and (iii) \( \frac{\sqrt{n}}{n(n-1)} Q_n' K_n^{x_2} \Delta_n^m \toP 0 \).

For (i), write \( Z_n = \mathbb{E}(Z_n) + [\Delta_n^w S_n^{-1} U_n, 0] \). Observing Eqn.(11), it remains to verify that \( \frac{1}{n(n-1)} (\Delta_n^w S_n^{-1} U_n)' K_n^{x_2} Q_n = o_p(1) \). This follows from Lemma 2, since by similar argument to the proof Eqn.(11) we have \( \frac{1}{n(n-1)} (\Delta_n^w S_n^{-1} U_n)' K_n^{x_2} Q_n = \frac{1}{n} \sum_{i=1}^n f(x_{2i}) \hat{u}_i (q_i - \varphi(x_{2i}))' + o_p(1) = \frac{1}{n} (W_n S_n^{-1} U_n)' \mathcal{F}_n Q_n c + o_p(1) \), where \( \hat{u}_i = W_n S_n^{-1} U_n \).

For (ii), similar arguments to the proof of Eqn. (11) give that

\[
\sqrt{n} \frac{Q_n' \mathcal{F}_n \Delta_n^u}{n(n-1)} = \sqrt{n} \left( \sum_{i=1}^n (q_i - \phi(x_{2i})) f(x_{2i}) u_i + o_p(1) \right) = \frac{1}{\sqrt{n}} Q_n' \mathcal{F}_n R_n \tilde{c}_n + o_p(1) \toD \mathcal{N}(0, \lim_{n \to \infty} \Omega_n)
\]

by Lemma 2.

For (iii),

\[
\frac{\sqrt{n}}{n(n-1)} Q_n' K_n^{x_2} \Delta_n^m = \frac{\sqrt{n}}{n(n-1)} \sum_{i=1}^n \sum_{j=1}^n q_{ij} (m(x_{2i}) - m(x_{2j})) h_n^{-1} k \left( \frac{x_{2i} - x_{2j}}{h_n} \right)
\]

\[
= \frac{\sqrt{n}}{n(n-1)} \sum_{i=1}^n \sum_{j \neq i} q_{ij} m(x_{2i}) h_n^{-1} k \left( \frac{x_{2i} - x_{2j}}{h_n} \right)
\]

\[
= \frac{\sqrt{n}}{n(n-1)} \sum_{i=1}^n \sum_{j=1}^n q_{ij} m(x_{2i}) h_n^{-1} k \left( \frac{x_{2i} - x_{2j}}{h_n} \right) - \frac{\sqrt{n}}{n(n-1)} \sum_{i=1}^n \sum_{j=1}^n q_{ij} m(x_{2i}) h_n^{-1} k \left( \frac{x_{2i} - x_{2j}}{h_n} \right)
\]

\[
= \frac{\sqrt{n}}{n-1} \sum_{i=1}^n q_i \left( m(x_{2i}) \int k(t) f(h_n t + x_{2i}) dt - \int m(h_n t + x_{2i}) k(t) f(h_n t + x_{2i}) dt \right) + o(1)
\]

\[
= -h_n^2 \sqrt{n} \int |k(t)|^2 |dt| \cdot c^2 \cdot \frac{1}{n} \sum_{i=1}^n |q_i| + o(h_n^2 \sqrt{n}) = o(1),
\]
where \( m^{(1)}(\cdot) \), \( m^{(2)}(\cdot) \) and \( f^{(1)}(\cdot) \) are the first derivative, second derivative of \( m(\cdot) \) and the first derivative of \( f(\cdot) \), respectively, both \( \bar{x}_{2i} \) and \( \tilde{x}_{2i} \) lie between \( x_{2i} \) and \( x_{2i} + h\eta t \). The inequality (*) follows from the uniform boundness of \( m^{(1)}(\cdot) \), \( m^{(2)}(\cdot) \), \( f^{(1)}(\cdot) \) and \( f(\cdot) \), e.g., by a positive constant \( c \) and Assumption 4-5.

REFERENCES


Table 1.
PDE of Parametric Components: $u_i = \sqrt{1 + \epsilon_i^2} \cdot \epsilon_i$

<table>
<thead>
<tr>
<th>$n = 100$</th>
<th>$\epsilon_i \sim \mathcal{N}(0, 1)$</th>
<th>$\epsilon_i \sim \chi^2(1)/\sqrt{2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m_1(z)$</td>
<td>$Q_n^{(1)}$</td>
<td>$Q_n^{(2)}$</td>
</tr>
<tr>
<td>$\lambda$ Bias</td>
<td>0.005</td>
<td>0.004</td>
</tr>
<tr>
<td>Std</td>
<td>0.181</td>
<td>0.166</td>
</tr>
<tr>
<td>$\beta_1$ Bias</td>
<td>-0.011</td>
<td>-0.010</td>
</tr>
<tr>
<td>Std</td>
<td>0.299</td>
<td>0.302</td>
</tr>
<tr>
<td>$\beta_2$ Bias</td>
<td>-0.029</td>
<td>-0.023</td>
</tr>
<tr>
<td>Std</td>
<td>0.142</td>
<td>0.160</td>
</tr>
</tbody>
</table>

<table>
<thead>
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<th>$n = 196$</th>
<th>$\epsilon_i \sim \mathcal{N}(0, 1)$</th>
<th>$\epsilon_i \sim \chi^2(1)/\sqrt{2}$</th>
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</thead>
<tbody>
<tr>
<td>$m_1(z)$</td>
<td>$Q_n^{(1)}$</td>
<td>$Q_n^{(2)}$</td>
</tr>
<tr>
<td>$\lambda$ Bias</td>
<td>0.002</td>
<td>0.011</td>
</tr>
<tr>
<td>Std</td>
<td>0.125</td>
<td>0.122</td>
</tr>
<tr>
<td>$\beta_1$ Bias</td>
<td>-0.003</td>
<td>-0.006</td>
</tr>
<tr>
<td>Std</td>
<td>0.216</td>
<td>0.206</td>
</tr>
<tr>
<td>$\beta_2$ Bias</td>
<td>-0.021</td>
<td>-0.012</td>
</tr>
<tr>
<td>Std</td>
<td>0.099</td>
<td>0.114</td>
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</table>

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<th>$n = 400$</th>
<th>$\epsilon_i \sim \mathcal{N}(0, 1)$</th>
<th>$\epsilon_i \sim \chi^2(1)/\sqrt{2}$</th>
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<tr>
<td>$m_1(z)$</td>
<td>$Q_n^{(1)}$</td>
<td>$Q_n^{(2)}$</td>
</tr>
<tr>
<td>$\lambda$ Bias</td>
<td>0.002</td>
<td>0.001</td>
</tr>
<tr>
<td>Std</td>
<td>0.082</td>
<td>0.083</td>
</tr>
<tr>
<td>$\beta_1$ Bias</td>
<td>-0.003</td>
<td>-0.008</td>
</tr>
<tr>
<td>Std</td>
<td>0.149</td>
<td>0.148</td>
</tr>
<tr>
<td>$\beta_2$ Bias</td>
<td>-0.017</td>
<td>-0.007</td>
</tr>
<tr>
<td>Std</td>
<td>0.072</td>
<td>0.079</td>
</tr>
</tbody>
</table>
### Table 2.

PDE of Parametric Components: $U_n = (I_n - 0.3 \cdot W_n)^{-1} \tau_n$

<table>
<thead>
<tr>
<th></th>
<th>$m_1(z)$</th>
<th>$m_2(z)$</th>
<th>$m_1(z)$</th>
<th>$m_2(z)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n = 100$</td>
<td></td>
<td></td>
<td></td>
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</tr>
<tr>
<td>$\lambda$</td>
<td>Bias 0.005 0.003</td>
<td>0.021 0.026</td>
<td>0.013 0.018</td>
<td>0.010 0.025</td>
</tr>
<tr>
<td></td>
<td>Std 0.139 0.135</td>
<td>0.135 0.129</td>
<td>0.135 0.131</td>
<td>0.133 0.130</td>
</tr>
<tr>
<td>$\beta_1$ Bias</td>
<td>-0.008 -0.009</td>
<td>-0.006 -0.004</td>
<td>-0.010 -0.003</td>
<td>-0.013 -0.008</td>
</tr>
<tr>
<td></td>
<td>Std 0.209 0.211</td>
<td>0.220 0.222</td>
<td>0.210 0.216</td>
<td>0.204 0.218</td>
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<tr>
<td>$\beta_2$ Bias</td>
<td>-0.029 -0.018</td>
<td>-0.018 -0.014</td>
<td>-0.025 -0.020</td>
<td>-0.027 -0.019</td>
</tr>
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<td></td>
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<td>0.101 0.117</td>
<td>0.103 0.114</td>
<td>0.102 0.117</td>
</tr>
<tr>
<td>$n = 196$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\lambda$</td>
<td>Bias 0.001 0.016</td>
<td>0.004 0.015</td>
<td>0.008 0.010</td>
<td>0.002 0.010</td>
</tr>
<tr>
<td></td>
<td>Std 0.093 0.097</td>
<td>0.097 0.089</td>
<td>0.091 0.091</td>
<td>0.097 0.091</td>
</tr>
<tr>
<td>$\beta_1$ Bias</td>
<td>-0.003 -0.006</td>
<td>-0.006 -0.003</td>
<td>-0.006 -0.001</td>
<td>-0.002 -0.005</td>
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<tr>
<td></td>
<td>Std 0.154 0.147</td>
<td>0.148 0.148</td>
<td>0.142 0.146</td>
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</tr>
<tr>
<td>$\beta_2$ Bias</td>
<td>-0.019 -0.011</td>
<td>-0.012 -0.012</td>
<td>-0.014 -0.011</td>
<td>-0.011 -0.012</td>
</tr>
<tr>
<td></td>
<td>Std 0.068 0.083</td>
<td>0.071 0.082</td>
<td>0.072 0.079</td>
<td>0.070 0.085</td>
</tr>
<tr>
<td>$n = 400$</td>
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</tr>
<tr>
<td>$\lambda$</td>
<td>Bias 0.001 0.003</td>
<td>0.003 0.003</td>
<td>0.005 0.007</td>
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<tr>
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<td>Std 0.065 0.067</td>
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<td>$\beta_1$ Bias</td>
<td>-0.002 -0.003</td>
<td>-0.002 -0.005</td>
<td>-0.006 -0.001</td>
<td>-0.004 -0.001</td>
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<tr>
<td></td>
<td>Std 0.100 0.103</td>
<td>0.103 0.104</td>
<td>0.103 0.106</td>
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<tr>
<td>$\beta_2$ Bias</td>
<td>-0.010 -0.006</td>
<td>-0.007 -0.001</td>
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<tr>
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<td>Std 0.048 0.059</td>
<td>0.047 0.057</td>
<td>0.049 0.059</td>
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