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# Consistent Estimation of Regression Coefficients in Replicated Data with Non-Normal Measurement Errors

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Working Paper 52

May 2000

Working Paper 52, 1–16

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In this paper we consider a weighted harmonic mean of two inconsistent estimators to propose a new estimator of the coefficient of a linear regression model with measurement errors. The proposed estimator is simple and it does not depend on any unknown quantity. The approximate bias and MSE of the estimator are derived. Further, an empirical application is also presented.

*Key Words*: Measurement Error; Replicated data, Nonnormal erros; Estimation.
*JEL Classification Numbers*: C1

## 1. INTRODUCTION

In any regression analysis of data, when the observations are significantly influenced by the measurement errors, the least square estimators of regres-

* The authors are thankful to Q. Li for helpful comments. First author is grateful to Academic Senate, UCR for financial support.

sion coefficients face the problem of inconsistency and unbiasedness. The traditional measurement error models do not provide any way out for consistent estimation of regression coefficient unless and until some additional information besides the sample observations is available. This additional information may comprise of different forms such as measurement error variances are known or their ratio is known, instrumental variable technique etc.; see, e.g., Cheng and Van Ness (1999), Fuller (1987) and Judge, Griffiths, Hill, Lee and Lütkepohl (1985) for comprehensive exposition. Even in the availability of replicated observations, the estimators of slope parameter arising from the application of least square procedure employing the error-ridden observations and aggregated (cover replications) observations are found to be inconsistent; see Richardson and Wu (1970).

In order to obtain a consistent estimator of slope parameter by considering a linear combination of two inconsistent estimators, the literature is quite rich; see, e.g. Bjørn (1992), Cragg (1999), for application in panel data. Our aim is not to go into the details of work done in this direction. The point to be highlighted here is that all such evolved estimators involve unknown quantities such as measurement error variances. To employ these estimators in practice becomes difficult rather impossible in the absence of knowledge about the correct values of unknown quantities. Such a problem can be overcome by considering a non-linear combination of two inconsistent estimators. An attempt in this direction is made in this article by considering the weighted harmonic mean of two inconsistent estimators. An elegant aspect of such an approach is that the weights to be employed do not depend on any unknown quantities. The ultimate form of the estimator is also very simple to use in practice.

The plan of presentation is as follows. In Section 2, we describe the model and present the estimators for the slope parameter. Their asymptotic properties are analyzed in Section 3 without assuming normality of distributions. Proofs of Theorems are provided in Section 4. Finally, in Section 5 we provide an empirical example.

## 2. THE MODEL AND ESTIMATORS

Let us consider the linear regression relationship between the true values of study variable $(Y)$ and explanatory variable $(X)$ as

$$Y_i = \alpha + \beta X_i \qquad (i = 1, 2, ...n) \tag{1}$$

where $\alpha$ and $\beta$ are the intercept term and slope parameter respectively.

Due to the presence of measurement errors in the observations, instead of $Y_i$ and $X_i$ we have $r$ replicated error-ridden observations $y_{ij}$ and $x_{ij}$

respectively on them which can be expressed as

$$y_{ij} = Y_i + u_{ij} \qquad (j = 1, 2, ..., r) \tag{2}$$

$$x_{ij} = X_i + v_{ij} \tag{3}$$

Here $u_{ij}$ and $v_{ij}$ denotes the measurement errors in $y_{ij}$ and $x_{ij}$ respectively.

Further, $X_1, X_2, \cdots, X_n$ may have different means, say, $m_1, m_2, \cdots, m_n$ so that we may write

$$X_i = m_i + w_i \tag{4}$$

where $w_1, w_2, ..., w_n$ are *i.i.d.* random variables with mean 0 and variance $\sigma_w^2$.

This completes the specification of replicated ultrastructural version of measurement error model on the lines of Dolby (1976). When $m_1 = m_2 = ... = m_n$, we have specification of a structural model whereas when $\sigma_w^2 = 0$, we have the functional model.

It is assumed that $u_{ij}$'s are *i.i.d.* with mean 0 and finite variance $\sigma_u^2$. Similarly, all $v_{ij}$'s are also *i.i.d.* with mean 0 and finite variance $\sigma_v^2$. Further, all $w_i$'s, $u_{ij}$'s and $v_{ij}$'s are assumed to be not necessarily normally distributed but mutally independent of each other. Employing $x_{ij}$'s and $y_{ij}$'s for the estimation of $\beta$ by least squares yields following estimator of $\beta$:

$$b_1 = \frac{\sum_i \sum_j (x_{ij} - \overline{x})(y_{ij} - \overline{y})}{\sum_i \sum_j (x_{ij} - \overline{x})^2} \tag{5}$$

where $\overline{x} = \frac{1}{nr} \sum \sum x_{ij}$ and $\overline{y} = \frac{1}{nr} \sum \sum y_{ij}$.

Alternatively, utilizing $\overline{x}_i$ and $\overline{y}_i$, the averages taken over replications, the least square estimator of $\beta$ is given by

$$b_2 = \frac{\sum_i (\overline{x}_i - \overline{x})(\overline{y}_i - \overline{y})}{\sum_i (\overline{x}_i - \overline{x})^2} \tag{6}$$

In order to study the asymptotic properties, it is assumed that the limiting values of variances of $m_1, m_2, ..., m_n$ as $n$ tends to infinity is $\sigma_m^2$ which is finite. Further, $n$ is assumed to grow large whereas $r$ is asumed to stay fixed.

It can be easily seen that

$$p \lim_{n\to\infty} b_1 = \frac{\sigma_m^2 + \sigma_w^2}{\sigma_m^2 + \sigma_w^2 + \sigma_v^2} \beta \tag{7}$$

$$p \lim_{n\to\infty} b_2 = \frac{\left(\sigma_m^2 + \sigma_w^2\right) r}{\left(\sigma_m^2 + \sigma_w^2\right) r + \sigma_v^2} \beta \tag{8}$$

implying that both $b_1$ and $b_2$ are inconsistent.

Following the popular approach to construct a consistent estimator of $\beta$ by considering a linear combination of two inconsistent estimators of $\beta$, consider the linear combination $[cb_1 + (1-c)b_2]$. Now, choosing $c$ such that it becomes a consistent estimator of $\beta$, we find the choice of $0 \leq c \leq 1$ to be

$$c = -\frac{\sigma_m^2 + \sigma_w^2 + \sigma_v^2}{(r-1)(\sigma_m^2 + \sigma_w^2)}; \quad r \geq 2. \tag{9}$$

Clearly, interestingly enough, such a choice of $c$ involves the unknown variances. So it is difficult to operationalize it in order to employ in practical applications. However, if we consider a non-linear combination of $b_1$ and $b_2$ such as their weighted harmonic mean, we have

$$\frac{1}{b_H} = \frac{1-c^*}{b_1} + \frac{c^*}{b_2} \tag{10}$$

where $0 \leq c^* \leq 1$ is the weight assigned to $b_2$.

This non-linear combination serves as a consistent estimator of $\beta$ when we choose

$$c^* = \frac{r}{r-1}; \quad r \geq 2. \tag{11}$$

This yields the following weighted harmonic mean estimator of $\beta$:

$$b_H = \frac{(r-1)b_1 b_2}{rb_1 - b_2}. \tag{12}$$

It can be readily verified that $b_H$ is a consistent estimator of $\beta$. Interestingly enough, this non-linearly combined estimator does not involve any unknown quantity and has a simple form.

## 3. COMPARISON OF ESTIMATORS

In order to analyze the asymptotic performance properties of the estimators of $\beta$, we assume that the distributions of $u_{ij}$'s, $v_{ij}$'s and $w_i$'s have finite moments at least up to order four. Further, $\gamma_{2v}$ and $\gamma_{2w}$ denote the Pearson's measure of the kurtosis of the distributions of $v_{ij}$'s and $w_i$'s respectively. These are zero when the distributions are normal.

Let us write

$$q = \frac{\sigma_u^2}{\beta^2 \sigma_v^2}; \qquad 0 < q < \infty \tag{13}$$

$$s_m^2 = \frac{1}{n}\sum_i (m_i - \overline{m})^2; \qquad 0 \le s_m^2 \le \infty \tag{14}$$

$$\theta = \frac{\sigma_v^2}{\sigma_v^2 + \sigma_w^2 + s_m^2}; \qquad 0 \le \theta \le 1. \tag{15}$$

$$d = \frac{\sigma_w^2}{\sigma_w^2 + s_m^2}; \qquad 0 \le d \le 1. \tag{16}$$

When d=0, the ultrastructural model reduces to the functional form of the measurement error model. Similarly, when d=1, it reduces to the structural form of the measurement error model.

Now let us compare the two inconsistent estimators $b_1$ and $b_2$ of $\beta$. The following results are proved in the next section.

Theorem 3.1. *The relative biases of $b_1$ and $b_2$ to order $0(n^{-1/2})$ are given by*

$$RB(b_1) = E\frac{b_1 - \beta}{\beta} = -\theta \tag{17}$$

$$RB(b_2) = E\frac{b_2 - \beta}{\beta} = -\frac{\theta}{k} \tag{18}$$

*while their relative variances to order $O(n^{-1})$ are*

$$\begin{aligned} RV(b_1) &= E\frac{b_1 - E(b_1)}{\beta}^2 \\ &= \frac{\theta^2}{nr}[\frac{q}{\theta} + \frac{(1-\theta)}{\theta}(1 - 2\theta + 2\theta^2) + 2rd(2-d)(1-\theta)^2 \\ &\quad +(1-\theta)^2(\gamma_{2v} + rd^2\gamma_{2w})] \end{aligned} \tag{19}$$

$$\begin{aligned} RV(b_2) &= E\frac{b_2 - E(b_2)}{\beta}^2 \\ &= \frac{\theta^2}{nk^4}[\frac{k^3 q}{\theta} + r\theta(1-\theta) + \frac{r^3(1-\theta)^3}{\theta} + 2r^2 d(2-d)(1-\theta)^2 \\ &\quad +r^2(1-\theta)^2(\gamma_{2v} + d^2\gamma_{2w})] \end{aligned} \tag{20}$$

*where*

$$k = \theta + r(1-\theta). \tag{21}$$

From (17) and (18), we observe that the biases in both the estimators $b_1$ and $b_2$ are of $O(1)$ since their $O(n^{-1/2})$ terms are zero, see (31) and (32). Further biases in both $b_1$ and $b_2$ are in negative direction. However, $b_2$ has invariably smaller numerical magnitude of bias in comparison to the estimator $b_1$.

Looking at the expressions (19) and (20), it is interesting to note that the performance of estimators is not influenced by the skewness of the three distributions associated with $w_i$'s and the measurement errors $u_{ij}$'s and $v_{ij}$'s. Only the kurtosis coefficients of the two distrubtions associated with $w_i$'s and $u_{ij}$'s play a role. Further, as long as both the distributions are leptokurtic, the efficiency of both the estimators $b_1$ and $b_2$ declines. On the other hand, when both the distributions are platykurtic, the efficiency of both the estimators increases in comparison to their performance under mesokurtic or more specifically normal distributions.

Comparing (19) and (20) it is seen that $b_2$ is superior to $b_1$ with respect to the criterion of variance to the order of our approximation when

$$q < \frac{(1-\theta)T}{\theta(r-1)k^3} + \frac{(1-\theta)^2}{(r-1)}\left[(k - \frac{r^3}{k^3})\gamma_{2v} + (k - \frac{r^2}{k})\theta r d^2 \gamma_{2w}\right] \tag{22}$$

where

$$T = k^4(1-2\theta+2\theta^2) - r^2\theta^2 - r^4(1-\theta)^2 + 2\theta r d(2-d)(1-\theta)(k^4 - r^2). \tag{23}$$

Now let us consider the consistent estimator $b_H$. If we consider the bias to order $O(n^{-\frac{1}{2}})$, it vanishes unlike the cases of $b_1$ and $b_2$.

THEOREM 3.2. *The relative mean squared error of $b_H$ to order $O(n^{-1})$ is given by*

$$RM(b_H) = \frac{\theta^2}{nr(1-\theta)^2}\left[\frac{q}{r}\left\{\frac{k}{\theta} + \frac{1}{(r-1)(1-\theta)^2}\right\} + \left(\frac{1-\theta}{\theta}\right) + \frac{2+\gamma_{2v}}{r-1}\right] \tag{24}$$

*which is also the relative variance of $b_H$ to order $O(n^{-1})$.*

It is interesting to observe that the asymptotic variance remains the same whether the underlying distributions are skewed or not. It is simply the kurtosis of the distribution of measurement errors associated with explanatory variable in the model. When this distribution is leptokurtic, the estimator has larger asymptotic variance in comparison to the case of mesokurtic or normal distribution. The opposite is true, i.e., the asymptotic variance is smaller in case the platykurtic distribution when compared with the case of normal distribution.

As the estimators $b_1$ and b$_2$ are inconsistent and biased while $b_H$ is consistent and biased, it is not appropriate to compare their variances for analyzing their efficiency properties. The right choice is the mean squared error. If we do so according to the criterion of mean squared error to order $O(n^{-1})$, the leading terms of order $O(1)$ in the mean squared errors of $b_1$ and $b_2$ are $\theta^2$ and $(\frac{\theta}{k})^2$ respectively while it is 0 in case of $b_H$ because it is consistent. Consequently, both inconsistent $b_1$ and $b_2$ are discarded in preference to the proposed consistent estimator $b_H$.

## 4. DERIVATION OF RESULTS

Let us first introduce the following notation:

$$
\begin{aligned}
u &= Col(u_{11}, u_{12}, \cdots, u_{nr}) \\
v &= Col(v_{11}, v_{12}, \cdots, v_{nr}) \\
e_{nr} &= Col(1, 1, \cdots, 1) \\
w &= Col(w_1, w_2, \cdots, w_n) \\
m &= Col(m_1, m_2, \cdots, m_n) \\
e_n &= Col(1, 1, \cdots, 1) \\
A &= I_{nr} - \frac{1}{nr} e_{nr} e'_{nr} \\
B &= \frac{1}{r}\left[(I_n \otimes e'_r) - \frac{1}{n} e_n e'_{nr}\right] \\
C &= I_n - \frac{1}{n} e_n e'_n \\
D &= \frac{1}{r}\left[(I_n \otimes e_r e'_r) - \frac{1}{n} e_{nr} e'_{nr}\right]
\end{aligned}
$$

where, e.g., $u$ represents an $nr \times 1$ column vector and $\otimes$ denotes the Kronecker product operator.

Now, if we write

$$
\begin{aligned}
g_{\varkappa\varkappa} &= \frac{1}{\sqrt{n}\sigma_v^2}\left[2(m+w)'Bv + 2m'cw + (w'cw - n\sigma_w^2)\right] \\
g_{\varkappa y} &= \frac{1}{\sqrt{n}\sigma_v^2}\left[\frac{1}{\beta}(m+w)'Bu + 2m'cw + (m+w)'Bv + (w'cw - n\sigma_w^2)\right] \\
t_{\varkappa\varkappa} &= \frac{1}{\sqrt{n}r\sigma_v^2}(v'Av - nr\sigma_v^2)
\end{aligned}
$$

$$
\begin{aligned}
t_{\varkappa y} &= \frac{u'Av}{\sqrt{n}\beta r\sigma_v^2} \\
t^*_{\varkappa\varkappa} &= \frac{1}{\sqrt{n}r\sigma_v^2}(v'Dv - n\sigma_v^2) \\
t^*_{\varkappa y} &= \frac{u'Dv}{\sqrt{n}\beta r\sigma_v^2}
\end{aligned}
$$

we can express

$$\frac{1}{nr}\sum_i\sum_j(x_{ij}-\overline{x})(y_{ij}-\overline{y}) = \beta\sigma_v^2\left[\frac{(1-\theta)}{\theta}+\frac{1}{\sqrt{n}}(g_{\varkappa y}+t_{\varkappa y})\right] \quad (25)$$

$$\frac{1}{n}\sum_i(\overline{x}_i-\overline{x})(\overline{y}_i-\overline{y}) = \beta\sigma_v^2\left[\frac{(1-\theta)}{\theta}+\frac{1}{\sqrt{n}}(g_{\varkappa y}+t^*_{\varkappa y})\right] \quad (26)$$

$$\frac{1}{nr}\sum_i\sum_j(x_{ij}-\overline{x})^2 = \sigma_v^2\left[\frac{1}{\theta}+\frac{1}{\sqrt{n}}(g_{\varkappa\varkappa}+t_{\varkappa\varkappa})\right] \quad (27)$$

$$\frac{1}{n}\sum_i(\overline{x}_i-\overline{x})^2 = \sigma_v^2\left[\frac{(1-\theta)}{\theta}+\frac{1}{r}+\frac{1}{\sqrt{n}}(g_{\varkappa\varkappa}+t^*_{\varkappa\varkappa})\right] \quad (28)$$

Using these, we obtain from (5) and (6) the following expressions for the relative estimation errors of $b_1$ and $b_2$:

$$
\begin{aligned}
(\frac{b_1-\beta}{\beta}) &= \left[-\theta+\frac{\theta}{\sqrt{n}}(g_{\varkappa y}+t_{\varkappa y}-g_{\varkappa\varkappa}-t_{\varkappa\varkappa})\right] \quad (29)\\
&\quad\times\left[1+\frac{\theta}{\sqrt{n}}(g_{\varkappa\varkappa}+t_{\varkappa\varkappa})\right]^{-1} \\
&= -\theta+\frac{\theta}{\sqrt{n}}\left[g_{\varkappa y}+t_{\varkappa y}-(1-\theta)(g_{\varkappa\varkappa}+t_{\varkappa\varkappa})\right]+O_p(n^{-1})
\end{aligned}
$$

$$
\begin{aligned}
(\frac{b_2-\beta}{\beta}) &= \frac{\theta}{\theta+r(1-\theta)}\left[-1+\frac{r}{\sqrt{n}}(g_{\varkappa y}+t^*_{\varkappa y}-g_{\varkappa\varkappa}-t^*_{\varkappa\varkappa})\right] \quad (30)\\
&\quad\times\left[1+\frac{r\theta}{\sqrt{n}\left\{\theta+r(1-\theta)\right\}}(g_{\varkappa\varkappa}+t^*_{\varkappa\varkappa})\right]^{-1} \\
&= -\frac{\theta}{k}+\frac{r\theta}{\sqrt{n}k}\left[g_{\varkappa y}+t^*_{\varkappa y}-\frac{r(1-\theta)}{k}(g_{\varkappa\varkappa}+t^*_{\varkappa\varkappa})\right]+O_p(n^{-1})
\end{aligned}
$$

where

$$k = [\theta + r(1-\theta)].$$

It can be easily seen that the expectations of the $O(n^{-1/2})$ terms in (29) and (30) are zero, therefore

$$E(\frac{b_1-\beta}{\beta}) = -\theta + O(n^{-1}) \tag{31}$$

$$E(\frac{b_2-\beta}{\beta}) = -\frac{\theta}{k} + O(n^{-1}) \tag{32}$$

which are the results (17) and (18) of Theorem I.

Next, using the results in the Appendix, we observe that the relative variance of $b$ to order $O(n^{-1})$ is

$$\begin{aligned} RV(b_1) &= E\left[\frac{b_1 - E(b_1)}{\beta}\right]^2 \\ &= \frac{\theta^2}{n} E[g_{\varkappa y} + t_{\varkappa y} - (1-\theta)(g_{\varkappa\varkappa} + t_{\varkappa\varkappa})]^2 \\ &= \frac{\theta}{nr}[q + (1-\theta)(1-2\theta+2\theta^2) + 2rd(2-d)(1-\theta)^2\theta] \\ &+ \frac{(1-\theta)^2\theta^2}{nr}(\gamma_{2v} + rd^2\gamma_{2w}) \end{aligned} \tag{33}$$

which is the result (19) of Theorem I.

In a similar manner, employing the results in the Appendix, the result (20) of Theorem I can be easily deduced.

From (12), (25), (26), (27) and (28), we obtain

$$\begin{aligned} &\left(\frac{b_H-\beta}{\beta}\right) \\ &= \frac{rb_1(b_2-\beta) - b_2(b_1-\beta)}{\beta(rb_1-b_2)} \\ &= \left[r\left(1+\frac{b_1-\beta}{\beta}\right)\left(\frac{b_2-\beta}{\beta}\right) - \left(1+\frac{b_2-\beta}{\beta}\right)\left(\frac{b_1-\beta}{\beta}\right)\right] \\ &\quad \left[r\left(1+\frac{b_1-\beta}{\beta}\right) - \left(1+\frac{b_2-\beta}{\beta}\right)\right]^{-1} \\ &= \frac{\theta}{\sqrt{n}(r-1)(1-\theta)^2}[(r-1)(1-\theta)(g_{\varkappa y} - g_{\varkappa\varkappa}) - (t_{\varkappa y} - kt^*_{\varkappa y}) \end{aligned} \tag{34}$$

$$
\begin{aligned}
&+(1-\theta)(t_{\varkappa\varkappa} - rt^*_{\varkappa\varkappa})] \times \left[1 + O_p(n^{-\frac{1}{2}})\right]^{-1} \\
= \; & \frac{\theta}{\sqrt{n}(r-1)(1-\theta)} \left[(r-1)(g_{\varkappa y} - g_{\varkappa\varkappa}) - \left(\frac{1}{1-\theta}\right)(t_{\varkappa y} - kt^*_{\varkappa y})\right. \\
&+(t_{\varkappa\varkappa} - rt^*_{\varkappa\varkappa})] + O_p(n^{-1}).
\end{aligned}
$$

Squaring both sides of (34), then taking their expectations and retaining terms of order $O(n^{-1})$ only, we find the relative mean squared error as

$$
\begin{aligned}
RM(b_H) \; = \; & \frac{\theta^2}{n(r-1)^2(1-\theta)^2} E[(r-1)^2 (g_{\varkappa y} - g_{\varkappa\varkappa})^2 \qquad (35) \\
&+ \left(\frac{1}{1-\theta}\right)^2 (t_{\varkappa y} - kt^*_{\varkappa y})^2 \\
&+(t_{\varkappa\varkappa} - rt^*_{\varkappa\varkappa})^2] \\
= \; & \frac{\theta^2}{n(r-1)^2(1-\theta)^2} [(r-1)^2 \frac{(1-\theta)(q+1)}{\theta r} \qquad (36) \\
&+ \frac{q}{r^2(1-\theta)^2}(r - 2k + k^2) + \left(\frac{r-1}{r}\right)(2 + \gamma_{2v})]
\end{aligned}
$$

which, provides the result stated in Theorem 3.2.

## 5. EMPIRICAL APPLICATION

The effect of devaluation (increase in the exchange rate) on the trade balance is an important question in the trade literature from the policy perspective. Devaluation may decrease a country's expenditure on imports which works in the direction of reducing the country's trade deficit. On the other hand, it may also lead to the fall in the export revenue, which raises the trade deficit. A positive (negative) sign of the coefficient of the regression of trade balance on exchange rate implies that devaluation improves (deteriorates) trade balance. Does devaluation improve trade balance? This issue has been studied extensively in recent years in many empirical studies, e.g., Moffett (1989), Rose (1991), Breda et al (1997), Shirvani et al (1997) among others. Moffett has used time series data for United States to find that trade balance in many sectors deteriorates as a result of a depreciation. Rose did time series analysis for five major OECD countries. His study supports a negative and insignificant relationship between exchange rate and trade balance. Breda et al have considered the case of Turkey to find that devaluation affects the trade balance positively. Shirvani et al have also done time series analysis for U.S. and G-7 bilateral trade to find that exchange rate affects the trade balance in the long run but not in the very short run. Thus some of the studies show that exchange rate affects trade balance positively while some others show that it affects trade balance negatively and insignificantly. We note, however, that all the above studies are based mainly on time series data. The modest objective of this paper is to look into this issue with the help of panel data for two years. In this sense it is the first study to capture the cross-sectional relationship between exchange rate and trade balance. Thus the model considered is as in (1) where $Y_i$ is the trade balance and $X_i$ is the exchange rate of the $i^{th}$ country respectively. $i = 1, \cdots, 68$ countries. But in order to estimate this cross country relationship, we are going to use the replicated observations on both $Y_i$ and $X_i$ for two years (Panel data) which satisfies (2) and (3), that is, $y_{it} = Y_i + u_{it}$, $x_{it} = X_i + v_{it}$ respectively. ($t$ can be considered as $j$), where the two periods are 1977, 1987. Using these replicated observations to study the cross-sectional analysis may improve precision and degrees of freedom. Also replication over time captures the time varying effects (shocks) on the average cross-sectional relationship in terms of means of $Y_{it}$ and $X_{it}$ in this framework. Data sources are World Bank data and Penn-World Table. The results obtained are as follows:

| | $coefficient$ | $S.E.$ | $t$ |
|---|---|---|---|
| $b_1$ | 6.5 | 6.39 | 1.01 |
| $b_2$ | 7.69 | 8 | .96 |
| $b_H$ | 9.4 | 4.5 | 2.1 |

In doing calculations for $b_1$, $b_2$, and $b_H$ above, we have used $x_{it-1}$ instead of $x_{it}$ in order to take care of simultaneity problems. We note that both the inconsistent estimates $b_1$ and $b_2$ are underestimates of the consistent estimator $b_H$. This is consistent with the theoretical results in (17) and (18). Thus after taking care of measurement error and smoothing out time-specific shocks we get a very different result that could not have been captured otherwise. The results for $b_H$ show that devaluation improves trade balance and it has a significant role to play. So raising the exchange rate can be an effective instrument for reducing trade deficit.

## APPENDIX A

Neglecting terms of order $O(n^{-1})$ and higher orders, we have

$$E(g^2_{\varkappa\varkappa}) = 2\left(\frac{1-\theta}{\theta}\right)\left[\frac{2}{r} + \frac{d(2-d)(1-\theta)}{\theta}\right] + \frac{(1-\theta)^2 d^2}{\theta^2}\gamma_{2w} \tag{A.1}$$

$$E(g_{\varkappa y}^2) = \left(\frac{1-\theta}{\theta}\right)\left[\frac{q+1}{r} + \frac{2d(2-d)(1-\theta)}{\theta}\right] + \frac{(1-\theta)^2 d^2}{\theta^2}\gamma_{2w} \tag{A.2}$$

$$E(g_{\varkappa y} g_{\varkappa\varkappa}) = 2\left(\frac{1-\theta}{\theta}\right)\left[\frac{1}{r} + \frac{d(2-d)(1-\theta)}{\theta}\right] + \frac{(1-\theta)^2 d^2}{\theta^2}\gamma_{2w} \tag{A.3}$$

$$E(t_{\varkappa y}^2) = \frac{q}{r} \tag{A.4}$$

$$E(t_{\varkappa y}^{*2}) = \frac{q}{r^2} \tag{A.5}$$

$$E(t_{\varkappa\varkappa}^2) = \frac{1}{r}(2+\gamma_{2v}) \tag{A.6}$$

$$E(t_{\varkappa\varkappa}^{*2}) = \frac{1}{r^2}(2+\gamma_{2v}) \tag{A.7}$$

$$E(t_{\varkappa y} t_{\varkappa y}^*) = \frac{q}{r^2} \tag{A.8}$$

$$E(t_{\varkappa\varkappa} t_{\varkappa\varkappa}^*) = \frac{1}{r^2}(2+\gamma_{2v}) \tag{A.9}$$

Further, the expected values of the products $(g_{\varkappa y} t_{\varkappa y})$, $(g_{\varkappa y} t_{\varkappa y}^*)$, $(g_{\varkappa y} t_{\varkappa\varkappa})$, $(g_{\varkappa y} t_{\varkappa\varkappa}^*)$, $(g_{\varkappa\varkappa} t_{\varkappa y})$, $(g_{\varkappa\varkappa} t_{\varkappa y}^*)$, $(g_{\varkappa\varkappa} t_{\varkappa\varkappa})$, $(g_{\varkappa\varkappa} t_{\varkappa\varkappa}^*)$, $(t_{\varkappa\varkappa} t_{\varkappa y})$, $(t_{\varkappa\varkappa} t_{\varkappa y}^*)$, $(t_{\varkappa y} t_{\varkappa\varkappa}^*)$ and $(t_{\varkappa y}^* t_{\varkappa\varkappa}^*)$ are zero to the given order of approximation.

*Proof.* Suppose that $z$ is a TX1 vector of random variables that are i.i.d. with mean 0, variance 1, third moment $\gamma_1$, and fourth moment $(3+\gamma_2)$. If H denotes a symmetric matrix with nonstochastic elements, we have

$$E(z'Hz) = (trH) \tag{A.10}$$

$$E(z'Hz.z) = \gamma_1(I_T \odot H)e_T \tag{A.11}$$

$$E(z'Hz.zz') = \gamma_2(I_T \odot H) + (trH)I_T + 2H \tag{A.12}$$

where $\odot$ denotes the Hadamard product operator of matrices.

Next we observe that the matrices $A$, $C$, and $D$ are idempotent and

$$B'B = \frac{1}{r}D$$

$$BB' = \frac{1}{r}C$$

$$AD = D$$

$$CB = B$$

$$trA = (nr-1)$$

$$trC \;=\; trD = (n-1)$$

These results are repeatedly used in the derivations that follows now.

From the definition of $g_{\varkappa\varkappa}$ and the stochastic independence of $w$ and $v$, we observe that

$$\begin{aligned}
E(g^2_{\varkappa\varkappa}) &= \frac{1}{n\sigma_v^4} E\left[2\,(m+w)'\,Bv + 2m'cw + \left(w'cw - n\sigma_w^2\right)\right]^2 \qquad \text{(A.13)}\\
&= \frac{1}{n\sigma_v^4} E[4(m+w)'Bvv'B'(m+w) + 4m'cww'c'm \\
&\quad +(w'cw - n\sigma_w^2)^2 + 8(m+w)'Bvw'cm \\
&\quad +4(m+w)'Bv(w'cw - n\sigma_w^2) + 4m'cw(w'cw - n\sigma_w^2)] \\
&= \frac{4\sigma_v^2}{nr\sigma_v^4}[m'cm + \sigma_w^2(trc)] + \frac{4\sigma_w^2}{n\sigma_v^4} m'cm \\
&\quad + \frac{\sigma_w^4}{n\sigma_v^4}[\gamma_{2w} tr(I_n \odot c)c + (trc)(trc+2) - 2n(trc) + n^2] \\
&= \frac{4(1-\theta)}{\theta r} + \frac{4(1-\theta)^2 d(1-d)}{\theta^2} + \frac{(1-\theta)^2 d^2}{\theta^2}(\gamma_{2w}+2) + O(n^{-1})
\end{aligned}$$

which leads to the result **(A.1)**.

Dropping the terms with zero expected values, we see that

$$\begin{aligned}
& E(g^2_{\varkappa y}) \\
= \;& \frac{1}{n\sigma_v^4} E\left[\frac{1}{\beta}(m+w)'Bu + 2m'cw + (m+w)'Bv + (w'cw - n\sigma_w^2)\right]^2 \\
= \;& \frac{1}{n\sigma_v^4} E[\frac{1}{\beta^2}(m+w)'Buu'B'(m+w) + 4m'cww'c'm \\
+ \;& (m+w)'Bvv'B'(m+w) + (w'cw - n\sigma_w^2)^2] \\
= \;& \frac{\sigma_u^2}{nr\sigma_v^4\beta^2}(m'cm + \sigma_w^2 trc) + \frac{4\sigma_w^2}{n\sigma_v^4} m'cm + \frac{\sigma_v^2}{nr\sigma_v^4}(m'cm + \sigma_w^2 trc) \\
& + \frac{\sigma_w^4}{n\sigma_v^4}[\gamma_{2w} tr(I_n \odot c)c + (trc)(trc+2) - 2ntrc + n^2] \\
= \;& \frac{(1-\theta)q}{\theta r} + \frac{4(1-\theta)^2 d(1-d)}{\theta^2} + \frac{(1-\theta)}{r\theta} \\
+ \;& \frac{(1-\theta)^2 d^2}{\theta^2}(\gamma_{2w}+2) + O(n^{-1}) \qquad \text{(A.14)}
\end{aligned}$$

providing the result **(A.2)**.

Similarly, we have

$$E(g_{\varkappa\varkappa} g_{\varkappa y}) \;=\; \frac{1}{n\sigma_v^4} E[4m'cww'c'm + 2(m+w)'Bvv'B'(m+w) \qquad \text{(A.15)}$$

$$
\begin{aligned}
& +(w'cw - n\sigma_w^2)^2] \\
= \; & \frac{4\sigma_w^2}{n\sigma_v^4} m'cm + \frac{2\sigma_v^2}{nr\sigma_v^4}(m'cm + \sigma_w^2 trc) \\
& + \frac{\sigma_w^4}{n\sigma_v^4}[\gamma_{2w} tr(I_n \odot c)c + (trc)(trc+2) - 2ntrc + n^2] \\
= \; & \frac{4(1-\theta)^2 d(1-d)}{\theta^2} + \frac{2(1-\theta)}{r\theta} \\
& + \frac{(1-\theta)^2 d^2}{\theta^2}(\gamma_{2w} + 2) + O(n^{-1})
\end{aligned}
$$

which yields the result **(A.15)**.

Next, consider

$$
\begin{aligned}
E(t_{\varkappa y}^2) &= \frac{1}{n\beta^2 r^2 \sigma_v^4} E(u'Avv'u) \qquad (A.16) \\
&= \frac{\sigma_u^2 trA}{n\beta^2 r^2 \sigma_v^2} \\
&= \frac{q}{r} + O(n^{-1})
\end{aligned}
$$

$$
\begin{aligned}
E(t_{\varkappa y}^{*^2}) &= \frac{1}{n\beta^2 r^2 \sigma_v^4} E(u'Dvv'Du) \qquad (A.17) \\
&= \frac{\sigma_u^2 trD}{n\beta^2 r^2 \sigma_v^4} \\
&= \frac{q}{r^2} + O(n^{-1})
\end{aligned}
$$

$$
\begin{aligned}
E(t_{\varkappa\varkappa}^2) &= \frac{1}{nr^2\sigma_v^4} E(v'Av - nr\sigma_v^2)^2 \qquad (A.18) \\
&= \frac{1}{nr^2}\left[\gamma_{2v} tr(I_{nr} \odot A)A + (trA)(trA+2) - 2nr(trA) + n^2 r^2\right] \\
&= \frac{1}{r}(2 + \gamma_{2v}) + O(n^{-1})
\end{aligned}
$$

$$
\begin{aligned}
E(t_{\varkappa\varkappa}^{*^2}) &= \frac{1}{nr^2\sigma_v^4} E(v'Dv - n\sigma_v^2)^2 \qquad (A.19) \\
&= \frac{1}{nr^2}[\gamma_{2v} tr(I_n \odot D)D + (trD)(trD+2) - 2n(trD) + n^2] \\
&= \frac{1}{r^2}(2 + \gamma_{2v}) + O(n^{-1})
\end{aligned}
$$

which are the results of **(A.4)-(A.7)**.

Similarly, we have

$$
\begin{aligned}
E(t_{\varkappa y}t^*_{\varkappa y}) &= \frac{1}{n\beta^2r^2\sigma_v^4}E(u'Avv'Du) \\
&= \frac{\sigma_u^2 trD}{n\beta^2r^2\sigma_v^2} \\
&= \frac{q}{r^2} + O(n^{-1})
\end{aligned} \tag{A.20}
$$

$$
\begin{aligned}
E(t_{\varkappa\varkappa}t^*_{\varkappa\varkappa}) &= \frac{1}{nr^2\sigma_v^4}E(v'Av - nr\sigma_v^2)(v'Dv - n\sigma_v^2) \\
&= \frac{1}{nr^2}[\gamma_{2v}(I_{nr}\odot A)D + (trA)(trD) + 2(trAD) \\
&- nr(trD) - n(trA) + n^2r] \\
&= \frac{1}{r^2}(2+\gamma_{2v})
\end{aligned} \tag{A.21}
$$

which lead to the results.

In a similar manner, the results related to zero expected values can be easily deduced. ▮

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