Capital Accumulation And Present-biased Preference

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Very Preliminary Version

Abstract: This paper reexamines the capital accumulation within a neo-classical growth model under the assumption of hyperbolic discounting as well as endogenous preference, finding that 1) two kinds of Naifs' behavior coincides under log utility; 2) increasing marginal impatience due to capital accumulation itself will negatively affect the steady state locus of consumption and capital; 3) the effect of hyperbolic settings through effective rate of preference is still ambiguous; 4) we prove the saddle-point equilibrium property for the steady state under various assumptions about individual's preference. Our model also justifies Max Weber's idea that although spirit of capitalism is an engine to capital accumulation, the subsequent growing wealth will damage this engine.

Key words: hyperbolic discounting, time-inconsistent, capital accumulation, spirit of capitalism

JEL classification: D90, E13

1. Introduction

The neoclassical theory of optimal growth assumes that people have stationary time preferences in that they discount the future with a constant exponential rate. However, recent studies (Ainslie [1992]) suggest that people are highly impatient about consuming between today and tomorrow but are much more patient about choices advanced further in the future. Motivated by such findings, Laibson have done a series of works examining intertemporal choices of hyperbolic consumers (Laibson [1994], [1996], [1997], [1998]).

The main problem associated with hyperbolic individuals is the fundamental asymmetry between the present and future selves, which is called the time-inconsistent problem. Individuals are assumed to be composed of conflicting selves --- current self and future selves. Each self is tied to choices of all other selves. At an equilibrium, each self choose optimal strategies given the strategies of all other selves.

Barro [1999] incorporates hyperbolic discounting into the standard Ramsey model, with an re-examination of individual choices under different commitment assumptions. He proves that in the case of no commitment and log utility, the equilibrium features a constant effective rate of time preference and is observationally equivalent to the standard Ramsey model. First he guesses the solution with an undetermined parameter, and then solves the parameter under an intra-personal Nash equilibrium. Note that the first-best choice, characterized by the conventional Hamiltonian system, is never a stable one, since future selves will intrinsically not obey the plans made by current self; instead, all future selves have a tendency to deviate from what previous self has planned because they have better choices under their own beliefs. In the sense of such facts, the only stationary choice (or enforceable consumption plan) is given by an intra-personal Nash equilibrium as in Barro [1999]. The method he uses will be summarized in Section 2, which we will use throughout this paper.

Another problem related to hyperbolic representatives is the uniqueness of the Nash equilibrium, to which Barro [1999] refers as a footnote after he works out the time-consistent solution. Laibson [1996] has proved the uniqueness of the solution in a discrete-time model, given that the utility function is concave, not just for log utility.

In Barro's analysis, however, the long-run discount rate was assumed to a strictly positive constant, and thus could not explain why different countries have various preference structures. For example, empirical studies imply that people in wealthy countries tend to have a higher discount rate than those in poor countries, and wealthy people are more impatient than poor people. Motivated by such evidence, we assume that the long-run discount rate is endogenously determined by capital. Raising the level of real assets increases the rate of time preference and future consumption. This does not contradict the accepted intuition that savings are a decreasing function of financial wealth as described by the Mundell-Tobin effect. Epstein and Hynes [1983] first offered the intuition for using wealth effects to transform time preference into an endogenous function, but it received only a footnote. They argue that monetary growth raises the opportunity cost of holding real balances, which shifts a positively sloped rate of time preference function down along a negatively sloped marginal product of capital locus. This reduces the real interest rate and increases steady state capital according to the Mundell-Tobin effect.

In our model, we characterize two kinds of tendencies as time forwards: first although the impatience increases as time interval expands, the marginal impatience decreases; on the other hand, the marginal impatience increases over time as a result of capital accumulation --- so called increasing marginal impatience. Our theory of increasing marginal impatience originates from Max Weber's perception in his *The Protestant Ethic and the Spirit of Capitalism*. Although capitalism is one engine for capital accumulation, since people with capitalism continually accumulate wealth for its own sake, rather than for the material rewards that it can serve to bring, Weber further points out that as the wealth level increases, people intrinsically tend to have less religion and thus less self-control of his current behavior in terms of the increased desire along with declined ability to resist the temptation. To sum up, wealth accumulation impairs its own engine. Before closing his book, Weber cites John Wesley (the founder of the Methodist Church)'s words to further this idea:

"I fear, wherever riches have increased, the essence of religion has decreased in the same proportion…For religion must necessarily produce both industry and frugality, and these cannot but produce riches. But as riches increase, so will pride, anger, and love of the world in all its branches…For the Methodists in every place grow diligent and frugal; consequently they increase in goods. Hence they proportionately increase in pride, in anger, in the desire of the flesh, the desire of the

eyes, and the pride of life. So, although the form of religion remains, the spirit is swiftly vanishing away…" (John Wesley in Weber, 175).

In our settings people become less patient toward future consumption as wealth accumulates. In addition, the intertemporal marginal rate of substitution is positively related to capital accumulation. In this paper, we do not discuss the effect of capitalism, but concentrate on the pure wealth effect towards person's impatience, which is characterized by our partly marginal increasing impatience model. Further, in sense of Rabin, et al [1999], we suppose people are either naïve or sophisticated about two things: future self-control problems caused by time-inconsistent preference $\phi(t-\tau)$ as in Barro [1999], and captital-related discount factor denoted by $\rho(k_t)$, with $\rho'(\cdot) \geq 0$ [1].

The structure of the paper is as follows. The next section, section 2, describes the basic model. Section 3 employs dynamic analysis and characterizes the steady state under various assumptions. We close in section 4 with concluding remarks and some suggestions for future work. A brief mathematical appendix is also included.

2. The Model

2.1 Basic model with endogenous time preference

We consider a perfectly competitive decentralized economy where the households maximize the discounted value of their dynastic utility over infinite horizon. The households are identical in tastes and preferences as well as in terms of initial endowments. A single commodity is produced using two factors of production --- capital and labor, and at every point of time, there is full employment of both factors. The final commodity can be used as consumption good as well as investment good in the form of capital. Assume the population is constant, which is normalized to unit throughout this paper. If we assume the instantaneous rate of discounting is given

[1] Here ρ denotes instantaneous rate of discounting, and thus $\rho' \geq 0$ means increasing marginal impatience.

by $\rho(k_t)$ with $\rho'(\cdot) \geq 0$, and the two kind of individual preferences --- naifs and sophisticates --- will take the forms of (N1) and (S1) respectively.

(N1) Individual's problem --- naifs,

$$\max \int_{\tau}^{\infty} u(c) e^{-\rho(k_\tau)(t-\tau)} dt$$

$$s.t.\ \dot{k} = f(k) - c,\ \text{given}\ \ k_0$$

(S1) Individual's problem --- sophisticates,

$$\max \int_{\tau}^{\infty} u(c) e^{-\int_{\tau}^{t} \rho(k_s) ds} dt$$

$$s.t. \quad \dot{k} = f(k) - c,\ \text{given}\ \ k_0$$

Remark: naifs could not predict his evolution of time preference, and thus would believe the current time-preference will persist, which is represented by the factor $\rho(k_\tau)$ rather than what sophisticates predict as $\rho(k_t)$ for time t's instantaneous rate of discounting. Assume individual's income is exogenously determined by the wage rate and interest payment from renting capital, that is, $f(k(t)) = r(t)k(t) + w(t)$. Then the budget constraint would be, $\dot{k} = r(t)k(t) + w(t) - c$.

2.1.1 Naifs' choice

Naifs have no demand for any commitment device since they always think they can commit future choice regardless of how the past choice goes. Self τ chooses consumption flow $\{c^{\tau}(t), t \geq \tau\}$ according to his current belief toward his lifetime utility function, whereas he could only enforce c_{τ}^{τ}. In the eyes of Self τ, his choice would be time-consistent since his time preference is characterized by the constant $\rho(k_\tau)$. The first order conditions are given by,

$$u_c(c_t^{\tau}) = \lambda_t,$$

$$\dot{\lambda}_t / \lambda_t = -[f'(k_t) - \rho(k_\tau)], \qquad (1)$$

where λ_t is the co-state variable associated with the capital stock k_t.

And thus, $\dot{c}^{\tau}_{t} / c^{\tau}_{t} = -\frac{u_c}{u_{cc}c}[f'(k_t) - \rho(k_\tau)]$

The first equation in (1) means that, along an optimal program, naifs perceive that the marginal utility of consumption should be equal to the shadow price of the capital stock; the second equation defines what naifs think as the rate of change in the marginal value of the capital stock. Budget constraint still holds, $\dot{k} = f(k) - c$.

Note that self τ could only enforce his time τ consumption based on current capital stock k_τ. Under log utility assumption, individual's time τ consumption can be expressed as,

$$c^{\tau}_{\tau} = \rho(k_\tau) \cdot [k(\tau) + \tilde{w}(\tau)].$$

As a whole, individual's lifetime strategy is time-consistent[2], which is characterized by,

$$c_t = \rho(k_t) \cdot [k(t) + \tilde{w}(t)]. \quad (2)$$

The budget constraint along with (2) leads the following system of differential equations[3],

$$\dot{c}_t / c_t = f'(k_t) - \rho(k_t) + \frac{\dot{\rho}(k_t)}{\rho(k_t)},$$

$$\dot{k} = f(k) - c. \quad (3)$$

2.1.2 Sophisticates' choice

Sophisticates' problem is intrinsically time-consistent, while is characterized by another dynamic system slightly different from naifs'.

$$\dot{c}_t / c_t = -\frac{u_c}{u_{cc}c}[f'(k_t) - \rho(k_t)],$$

$$\dot{k} = f(k) - c. \quad (4)$$

[2] Note that the instantaneous discount factor ρ does not depend on the locus of time directly.

[3] See Appendix Proposition A1 for detailed proof.

With log instantaneous utility, individual's current consumption would be a fraction of his lifetime wealth, although the coefficient is changing with capital flow[4].

$c(t) = \lambda(K_t)\cdot[k(t) + \tilde{w}(t)]$, where $\tilde{w}(t)$ is present value of wages,

and $\lambda(K_t) = 1/\int_t^{\infty} e^{-\int_t^v \rho(k_s)ds} dv$, and the set $K_t = \{k(s), s \geq t\}$.

We can easily verify that $\dot{c}_t / c_t = f'(k_t) - \rho(k_t)$ is associated with the above consumption function. (5)

2.2 Basic model with hyperbolic discounting

The representative's preference is characterized by endogenously determined factor $\rho(k_t)$ along with intrinsically decreasing marginal impatience, represented by an exogenous factor $\phi(t-\tau)$, with $\phi'(\cdot) \geq 0, \phi''(\cdot) \leq 0$, and $\phi'(v) \to 0$ as $v \to \infty$, as in Barro [1999]. From this assumption, we cannot tell whether an individual's marginal impatience is increasing or decreasing, since there are two opposite directions that jointly determine the preference structure. We will never tell which one dominates until we make further assumptions. Here the two kind of individual preferences will take the forms of (N2) and (S2).

(N2)Individual's problem --- naifs,

$$\max \int_{\tau}^{\infty} u(c) e^{-[\rho(k_\tau)(t-\tau)+\phi(t-\tau)]} dt$$

$$s.t. \quad \dot{k} = f(k) - c$$

(S2)Individual's problem --- sophisticates,

$$\max \int_{\tau}^{\infty} u(c) e^{-[\int_{\tau}^{t}\rho(k_s)ds+\phi(t-\tau)]} dt$$

$$s.t. \quad \dot{k} = f(k) - c$$

2.2.1 Naifs' choice

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[4] See appendix for detailed proof.

We divide naifs' choice into two categories: partially naïve and totally naïve whose implications are given as follows.

Case 1--- partially naïve:

If people are partly naïve only in the sense of the captital-related discount factor, whereas know the future self-control problems caused by time-inconsistent preference, in this case, he acts in the manner described by Barro(1999). But the crucial difference here is that current self's strategy of lifetime choice is time-inconsistent because of his wrong prediction of the long run discount rate, although time-consistent in the sense of Barro given a constant long run discount rate. Again self τ could only commit his time τ consumption.

In Barro [1999], given the log utility function, consumption will be a constant fraction of wealth,

$c(t) = \lambda \cdot [k(t) + \text{present value of wages}]$,

$\lambda = 1/\Omega$, where $\Omega = \int_0^\infty e^{-[\rho v + \phi(v)]} dv$

Here in our case, ρ is replaced by $\rho(k_\tau)$ for self τ. That is,

$$\lambda(k_\tau) = 1/\int_0^\infty e^{-[\rho(k_\tau)v + \phi(v)]} dv$$

Since self τ could only enforce c_τ, individual's lifetime choice is characterized by a time-consistent strategy,

$$\dot{c}_t / c_t = r_t - \lambda(k_t) + \frac{\dot{\lambda}(k_t)}{\lambda(k_t)} \tag{6}$$

in which,

$\dot{\lambda}(k_t) = \dot{\rho}(k_t) \int_0^\infty e^{-[\rho(k_t)v + \phi(v)]} \cdot v dv / [\int_0^\infty e^{-[\rho(k_t)v + \phi(v)]} dv]^2$

Remark: 1) Here under the assumption of partially naïve, we see that although current self's perceived "optimal" plan is intrinsically time-inconsistent in the eyes of future selfs, the individual's behavior as a whole exhibit time-consistent dynamics. The logic behind such findings is that the endogenous discount factor does not depend on the realization of a specific time, but only on the current capital level.

2) Our methodology[5] is clear: First find out what each self will choose under his own belief; then combine all selfs' strategy to obtain individual's lifetime choice. Since naifs have no demand for commitment technology, each self could only enforce his current choice of consumption, which is his only contribution to the lifetime choice of the individual.

Case 2--- totally naïve:

If people are totally naïve both in the captital-related discount factor and the future self-control problems caused by time-inconsistent preference, here self τ 's choice is of the neo-classical kind --- first-order optimality conditions from the current value Hamiltonian.

The first order conditions are given by,

$$u_c = \lambda_t,$$

$$\dot{\lambda}_t / \lambda_t = -[f'(k_t) - \rho(k_\tau) - \phi'(t-\tau)],$$

where λ_t is the co-state variable associated with k_t.

And thus in self τ 's eyes, $\dot{c}_t^\tau / c_t^\tau = -\frac{u_c}{u_{cc}c}[f'(k_t) - \rho(k_\tau) - \phi'(t-\tau)]$

Budget constraint still holds, $\dot{k} = f(k) - c$.

Note that self τ could only enforce c_τ based on current capital stock k_τ. Under log utility, self τ 's consumption is given by[6],

$c_\tau^\tau = \eta(k_\tau)\cdot[k(\tau) + \tilde{w}(\tau)]$, where $\eta(k_\tau) = 1/\int_\tau^\infty e^{-[\rho(k_\tau)(t-\tau)+\phi(t-\tau)]}dt = \lambda(k_\tau)$

Here individual's strategy is still time-consistent, which is characterized by,

$$\dot{c}_t / c_t = r_t - \eta(k_t) + \frac{\dot{\eta}(k_t)}{\eta(k_t)},$$

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[5] The method that we use to calculate individual's lifetime behavior will repeat throughout the remaining part of this paper.

[6] See Appendix Proposition A3 for details.

$$\dot{k} = f(k) - c. \qquad (7)$$

Remark: 1) We do not consider here the third case where individual is naïve only with respect to the factor $\phi(\cdot)$, in which his will derive his first-order optimality conditions from the current value Hamiltonian.

2) It is easily seen that, in case 1 and case 2, the effective rate of time preference coincide --- $\lambda(k_\tau) \equiv \eta(k_\tau)$. We may conclude that whether people are naïve about their intrinsically decreasing marginal impatience, characterized by the item $\phi(t-\tau)$, does not matter for his behavior under log utility. From the reasoning above, we also see that it does not matter to distinguish naifs or sophisticates in Barro [1999] under log utility. However, without the assumption of log utility, our conclusion will change[7].

2.2.2 Sophisticates' choice with no commitment

Suppose people are well-informed about there capital-adjusting preference, as well as his future self-control problem. Following Barro(1999), and using the results under 2.1.2, we derive the optimal strategy of individuals.

In 2.1.2, with log utility function, we have,

$c(t) = \lambda(K_t) \cdot [k(t) + \tilde{w}(t)]$, where $\tilde{w}(t)$ is present value of wages,

and $\lambda(K_t) = 1 / \int_t^\infty e^{-\int_t^v \rho(k_s) ds} dv$, and the set $K_t = \{k(s), s \geq t\}$

Proposition 1. Under log instantaneous utility function, sophisticates' consumption function would be of the form,

$$c(t) = \mu(K_t) \cdot [k(t) + \tilde{w}(t)], \quad \mu(K_t) = 1 / \int_t^\infty e^{-[\int_t^v \rho(k_s) ds + \phi(v-t)]} dv.$$

Remark: Barro's [1999] consumption function is a special case of Proposition 1 with

[7] For general utility function, Barro [1999] only considers the case of sophisticates. Also see Rabin, et al [1999] for examples concerning the different behavior between naifs and sophisticates under various assumptions of costs and rewards.

$\rho(k) \equiv \rho$.

3. Characterization of the Steady State

Define a steady-state equilibrium as $\dot{c} = 0$ and $\dot{k} = 0$.

3.1 Naifs (in Sec 2.1.1)

Note that $\dot{\rho}(k_t) = \rho'(k_t)\dot{k}_t$, so in steady state we also have $\dot{\rho} = 0$. The long run discount rate is given by $\rho = \rho(\bar{k})$, in which $\bar{k}$ is the steady state level of capital stock. Hence, the steady state is characterized by the following set of conditions:

$$f'(\bar{k}) - \rho(\bar{k}) = 0,$$

$$f(\bar{k}) - \bar{c} = 0. \tag{8}$$

Remark: In the case of naifs, (8) holds only for log utility.

3.2 Sophisticates (in Sec 2.1.2)

The steady state is characterized by (8) for general utility function.

3.3 Partially or totally naïve person (in Sec 2.2.1)

Note that $\dot{\lambda}(k_t) = \lambda'(k_t)\dot{k}_t$, so we will have $\dot{\lambda} = 0$ in steady state. The long run effective rate of time preference is given by: $\lambda = \lambda(\bar{k}) = 1/\int_0^\infty e^{-[\rho(\bar{k})v+\phi(v)]}dv$. The steady state is characterized by:

$$f'(\bar{k}) - \lambda(\bar{k}) = 0$$

$$f(\bar{k}) - \bar{c} = 0. \tag{9}$$

Compared to other hyperbolic discounting models, the long run discount rate in this model will interact with steady state capital stock, both of which are jointly determined by the dynamic system (9). Thus the extent to which people raise their marginal impatience will have great effect on the final steady state level of capital stock. To illustrate our logic, we assume $\rho(k)$ takes the form of $\rho(k) = bk$, where b is a positive constant representing the marginal increase of impatience due to one unit increase of capital stock. In steady state we have,

$$f'(\bar{k}) = 1/\int_0^\infty e^{-[b\cdot\bar{k}\cdot v+\phi(v)]}dv, \tag{10}$$

which implicitly determines steady state level of capital stock as a function of b, i.e., $\bar{k} = \bar{k}(b)$. Consider the case that b rises from b_0 to b_1. Initially, the right side of equation (10) will increase, and the capital will move downward to the new steady state, which is driven by the decreasing marginal productivity. At the same time, the right side of (10) will decrease gradually as capital stock moves downward until the two sides of (10) rebalance[8].

Further, we could express steady state level of consumption as $\bar{c}(b) = f(\bar{k}(b))$. Differentiating both sides with respect to b, we easily see that, $d\bar{c}(b)/db = f'(\bar{k}(b))\cdot d\bar{k}(b)/db < 0$.

Remark: 1) The greater the extent of increasing marginal impatience, the lower the steady state level of consumption and capital stock, $(\bar{c}, \bar{k})$. Intuitively, as people get more impatient during the process of capital accumulation, the saving rate will decrease accordingly and they tend to distribute less of the income to investment, and thus lower the long run consumption.

2) Compare the steady state between convention Ramsey model and the hyperbolic discounting model.

In conventional settings, $f'(k^*) = \beta$, where β denotes the constant discount rate.

While in the hyperbolic discounting settings, e.g. Barro [1999], we have, $f'(\bar{k}) = \lambda$, where $\lambda = 1/\int_0^\infty e^{-[\rho v+\phi(v)]}dv$, and $\rho \le \lambda \le \rho + \phi'(0)$. From the concavity of product function, we have the following inequality,

$$k^*(\rho + \phi'(0)) \le \bar{k} \le k^*(\rho) \tag{11}$$

[8] See Proposition A4 in appendix for rigorous mathematical proof.

Based on (11), we see that if β lies to the left of ρ, we will have $k^* \geq \bar{k}$, i.e., hyperbolic settings lower the steady state level of capital stock, and if β lies to the right of $\rho + \phi'(0)$, we will have $k^* \leq \bar{k}$, i.e., hyperbolic settings generate a higher level of long run capital. To compare the steady state level of capital between conventional settings and hyperbolic settings is equivalent to compare the effective rate of time preference, which is β for the former and ρ for the latter. Note that this comparison is exogenously determined by representative's preference structure, while in our endogenous preference setting, the extent of increasing marginal impatience due to capital accumulation itself will have a negative effect on steady state locus $(\bar{c}, \bar{k})$. And thus apart from the issues that a strong capitalist spirit can lead to unbounded growth of consumption and capital[9], we find that a high extent to which capital affects individuals' marginal impatience will damage this engine, which theoretically supports the logic of Weber.

3) Compare the steady state between models in 2.1 and that of 2.2, and we may examine what the introducing of hyperbolic preference will imply for long run capital stock and consumption level.

For the former case, we have, $f'(\bar{k}) = \rho_{TC}(\bar{k})$; and for the latter, we have, $\rho_N(\bar{\bar{k}}) \leq f'(\bar{\bar{k}}) = \lambda(\bar{\bar{k}}) = 1/\int_0^\infty e^{-[\rho_N(\bar{\bar{k}})v + \phi(v)]} dv \leq \rho_N(\bar{\bar{k}}) + \phi'(0)$, where $\bar{k}$ and $\bar{\bar{k}}$ are the respective steady state level of capital, and TC represents the case under time-consistent preference while N represents the case of naifs. Our objective is to compare $\bar{k}$ and $\bar{\bar{k}}$, which are already implicitly determined, and thus could be seen as given constants. Further, we assume that

$$\rho_N(x) \leq \rho_{TC}(x) \leq \rho_N(x) + \phi'(0), x \in [0, \infty).$$

[9] With a capitalist-spirit model by including wealth in utility function, Zou [1994] finds that a strong capitalist spirit can lead to unbounded growth of consumption and capital even though the net marginal product of capital is less than the time discount rate or goes to zero when capital stock increases to infinity. For further discussions, see also Zou [1995] and [1998].

We first examine two polar cases.

3.1. $\rho_N(x) = \rho_{TC}(x)$. We easily see that $f'(\bar{k}) \le f'(\bar{\bar{k}})$, and thus $\bar{k} \ge \bar{\bar{k}}$. Here hyperbolic naïve representatives produce lower level of steady state capital than that of non-hyperbolic individuals.

3.2. $\rho_{TC}(x) = \rho_N(x) + \phi'(0)$. In this case, we have $\bar{k} \le \bar{\bar{k}}$.

For the intermediate case, where $\rho_N(x) < \rho_{TC}(x) < \rho_N(x) + \phi'(0)$, the relationship between $\bar{k}$ and $\bar{\bar{k}}$ is undetermined. In essence, we need to compare the two functionals,

$\rho_{TC}(k)$ and $\lambda(k) \equiv 1/\int_0^\infty e^{-[\rho_N(k)v+\phi(v)]}dv$ --- the effective rate of time preference.

4) For other cases concerning naifs as in section 2.1.1 or sophisticates in section 2.1.2, the same results will obtain.

3.4 Stability of the Steady State

Proposition 3.1: Given the time preference is endogenously determined by capital stock, and given that individual is *sophisticate*, i.e. the setup in section 2.1.2 without the assumption of hyperbolic discounting, we will obtain a saddle point equilibrium around the steady state.

Proof: Linearize (4) --- under log utility --- around the steady state $(\bar{c}, \bar{k})$ using (8) to obtain:

$$\begin{pmatrix} \dot{c} \\ \dot{k} \end{pmatrix} = \begin{pmatrix} 0 & c \cdot [f''(\bar{k}) - \rho'(\bar{k})] \\ -1 & f'(\bar{k}) \end{pmatrix} \cdot \begin{pmatrix} c - \bar{c} \\ k - \bar{k} \end{pmatrix} \tag{12}$$

Suppose λ_1 and λ_2 to be the characteristic roots of the coefficient matrix in (12), and then we have the following equations:

$$\lambda_1 + \lambda_2 = f'(\bar{k}) > 0$$

$$\lambda_1 \cdot \lambda_2 = c \cdot [f''(\bar{k}) - \rho'(\bar{k})] < 0 \quad , \tag{13}$$

which shows the existence of real solutions λ_1 and λ_2 [10], and that λ_1 and λ_2 have opposite sign. The system defined by $(\dot{c}, \dot{k})$ is steady state stable at a saddle point. Q.E.D.

Proposition 3.2: Given the time preference is endogenously determined by capital stock, and given that individual is *naive*, i.e. the setup in section 2.1.1 without the assumption of hyperbolic discounting, we will obtain a saddle point equilibrium around the steady state.

Proof: Linearize (3) --- under log utility --- around the steady state $(\bar{c}, \bar{k})$ using (8) to obtain:

$$\begin{pmatrix} \dot{c} \\ \dot{k} \end{pmatrix} = \begin{pmatrix} -\dfrac{\rho'(\bar{k})c}{\rho(\bar{k})} & c \cdot f''(\bar{k}) \\ -1 & f'(\bar{k}) \end{pmatrix} \cdot \begin{pmatrix} c - \bar{c} \\ k - \bar{k} \end{pmatrix} \tag{14}$$

Suppose λ_1 and λ_2 to be the characteristic roots of the coefficient matrix in (12), and then we have the following equations:

$\lambda_1 + \lambda_2 = f'(\bar{k}) - \dfrac{\rho'(\bar{k})c}{\rho(\bar{k})}$, whose sign is undetermined, and

$$\lambda_1 \cdot \lambda_2 = -\rho'(\bar{k})c + f''(\bar{k})c < 0, \tag{15}$$

which shows that λ_1 and λ_2 have opposite sign. The system defined by $(\dot{c}, \dot{k})$ is still steady state stable at a saddle point. Q.E.D.

Proposition 3.3: Given the time preference is endogenously determined by capital stock, and given that individual is *naive*, i.e. the setup in section 2.2 with the assumption of hyperbolic discounting, we will obtain a saddle point equilibrium around the steady state.

Proof: We turn to the case with present-biased preference structure in section 2.2.

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[10] Note that the characteristic polynomial of A is $|\mu E - A| = \mu^2 - [trace(A)] \cdot \mu + |A|$.

Follow the same step as above using (6) or (7), and we will have[11]:

$$\begin{pmatrix} \dot{c} \\ \dot{k} \end{pmatrix} = A(\bar{c}, \bar{k}) \cdot \begin{pmatrix} c - \bar{c} \\ k - \bar{k} \end{pmatrix}, \tag{16}$$

in which the coefficient matrix is given as in Appendix proposition A5.

Let μ_1 and μ_2 be the two characteristics of matrix $A(\bar{c}, \bar{k})$, we have,

$$\mu_1 + \mu_2 = \lambda(\bar{k})\{1 - \bar{c} \cdot \rho'(\bar{k}) \cdot \int_0^\infty e^{-[\rho(\bar{k})v + \phi(v)]} v dv\}$$

$$\mu_1 \cdot \mu_2 = \bar{c}\{f''(\bar{k}) - \lambda'(\bar{k})\} < 0, \tag{17}$$

which shows that the system defined by $(\dot{c}, \dot{k})$ is steady state stable at a saddle point.

Q.E.D.

4. Concluding Remarks

By reexamining the capital accumulation within a neo-classical growth model under the assumption of hyperbolic discounting as well as endogenous preference, we draw the main conclusions that 1) two kinds of Naifs' behavior coincides under log utility; 2) increasing marginal impatience due to capital accumulation itself will negatively affect the steady state locus of consumption and capital; 3) the effect of hyperbolic settings through effective rate of preference is still ambiguous; 4) we prove the saddle-point equilibrium property for the steady state under various assumptions about individual's preference. Another implications is that although spirit of capitalism is an engine to capital accumulation, the subsequent growing wealth will damage this engine in the sense of Max Weber.

Our next step would be examining the government policies along with their welfare implications under the framework provided in this paper. One way to extend our model is to making further assumptions on the capital-related preference, which will enrich the dynamics to a great extent.[12]

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[11] See Appendix Proposition A5 for detailed proof.

[12] See P. Wang [2003] for detailed examples.

Appendix

First, we note that if consumption could be expressed as a fraction of lifetime wealth, a good proposition will obtain.

Proposition A1. If consumption is a fraction of wealth, $c(t) = \lambda_t \cdot [k(t) + \tilde{w}(t)]$, and the fraction λ_t is time-dependent, the dynamics of consumption flow will be,

$$\dot{c}_t / c_t = r_t - \lambda_t + \frac{\dot{\lambda}_t}{\lambda_t}.$$

Proof: If we define lifetime wealth $W_t \equiv k(t) + \tilde{w}(t)$, where $\tilde{w}(t)$ is present value of wages defined by, $\tilde{w}(t) = \int_t^\infty w(v) e^{-R(v,t)(v-t)} dv$, $R(v,t) \equiv \frac{1}{v-t}\int_b^v r(s)ds$, and then the dynamics of wealth is given by,

$$\dot{W}_t = \dot{k}(t) + \dot{\tilde{w}}(t) = (r_t k_t + w_t - c_t) + [-w_t + r_t \tilde{w}_t] = r_t k_t + r_t \tilde{w}_t - \lambda_t \cdot [k_t + \tilde{w}_t] = (r_t - \lambda_t) W_t,$$

where the second equality follows from the budget constraint, and the third equality follows from our special assumption of c_t.

Differentiate the two sides of consumption-wealth relationship with respect to time, and we have,

$$\dot{c}_t = \lambda_t \cdot \dot{W}_t + \dot{\lambda}_t \cdot W_t = \lambda_t (r_t - \lambda_t) W_t + \dot{\lambda}_t \cdot W_t = (r_t - \lambda_t) c_t + c_t \cdot \dot{\lambda}_t / \lambda_t.$$

Rearrange, and we obtain the dynamics of consumption, $\dot{c}_t / c_t = r_t - \lambda_t + \frac{\dot{\lambda}_t}{\lambda_t}$.

Remark A1: In Barro's [1999] paper, he conjectures the fraction to be a constant, which could be seen as a special case of the above proposition with $\dot{\lambda}_t \equiv 0$, and thus the growth rate of consumption is $\dot{c}_t / c_t = r_t - \lambda$. Q.E.D.

Proposition A2. In the basic time-consistent model of sophisticates in section 2.1, the optimal consumption will be a fraction of lifetime wealth under the log instantaneous utility, which is given by $c(t) = \lambda(K_t) \cdot [k(t) + \tilde{w}(t)]$, where $\tilde{w}(t)$ is present value of wages, and $\lambda(K_t) = 1/\int_t^{\infty} e^{-\int_t^v \rho(k_s)ds} dv$, and the set $K_t = \{k(s), s \geq t\}$.

Proof: We obtain our result from the budget constraint, $\dot{k}_t = r_t k_t + w_t - c_t$, and the Euler equation,

$\dot{c}_t / c_t = r_t - \rho(k_t)$. Rearrange the budget constraint,

$d[e^{-R(v,t)(v-t)} k_v]/dv = e^{-R(v,t)(v-t)}[w_v - c_v]$, and then integrate two sides on the inteval $[t, \infty)$.

We will have $-k_t = \int_t^{\infty} e^{-R(v,t)(v-t)}(w_v - c_v)dv = \tilde{w}_t - \int_t^{\infty} e^{-R(v,t)(v-t)} c_v dv$, (A2.1)

in which we have assumed the Non-Ponzi-Game condition $\lim_{v \to \infty} e^{-R(v,t)(v-t)} k_v = 0$.

From the Euler equation, we have $c_v = c_t e^{R(v,t)(v-t) - \int_t^v \rho(k_s)ds}$. (A2.2)

Combine (A2.1) and (A2.2), and we will have,

$$c(t) = [1/\int_t^{\infty} e^{-\int_t^v \rho(k_s)ds} dv] \cdot [k(t) + \tilde{w}(t)]. \qquad \text{Q.E.D.}$$

Proposition A3. In the time-inconsistent model of totally naïve person in section 2.2.1, the optimal consumption will be a fraction of lifetime wealth under the log instantaneous utility, which is given by $c_\tau^\tau = \eta(k_\tau) \cdot [k(\tau) + \tilde{w}(\tau)]$, where $\eta(k_\tau) = 1/\int_0^{\infty} e^{-[\rho(k_\tau)v + \phi(v)]} dv$

Proof: By the same method as in proposition A2, we will obtain our result from the budget constraint, $\dot{k}_t = r_t k_t + w_t - c_t^\tau$ along with the Euler equation perceived by self τ [13],

$\dot{c}_t^\tau / c_t^\tau = f'(k_t) - \rho(k_\tau) - \phi'(t-\tau)$. Rearrange the budget constraint,

$d[e^{-R(t,\tau)(t-\tau)} k_t]/dt = e^{-R(t,\tau)(t-\tau)}[w_t - c_t^\tau]$, and then integrate both sides on the interval

[13] Note that for naifs, future consumption is perceived as c_t^τ by self τ; thus self τ perceives the budget constraint as $\dot{k}_t = r_t k_t + w_t - c_t^\tau$, although the individual will act along $\dot{k}_t = r_t k_t + w_t - c_t$ for every c_t realized.

$[\tau,\infty)$.

We will have $-k_\tau = \int_\tau^\infty e^{-R(v,\tau)(v-\tau)}(w_v - c_v^\tau)dv = \tilde{w}_\tau - \int_\tau^\infty e^{-R(v,\tau)(v-\tau)} c_v^\tau dv$, (A3.1)

in which we have assumed the Non-Ponzi-Game condition $\lim_{v\to\infty} e^{-R(v,\tau)(v-\tau)} k_v = 0$.

From the Euler equation, we have $c_v^\tau = c_\tau^\tau e^{R(v,\tau)(v-\tau)-\rho(k_\tau)(v-\tau)-\phi(v-\tau)}$. (A3.2)

Combine (A3.1) and (A3.2), and we will have,

$$c_\tau^\tau = [1/\int_\tau^\infty e^{-[\rho(k_\tau)(t-\tau)+\phi(t-\tau)]}dt]\cdot[k(\tau)+\tilde{w}(\tau)].$$

That is,

$$c_\tau^\tau = [1/\int_0^\infty e^{-[\rho(k_\tau)v+\phi(v)]}dv]\cdot[k(\tau)+\tilde{w}(\tau)] \qquad \text{Q.E.D.}$$

Proposition A4. Given that $\rho(k)$ takes the form of $\rho(k)=bk$, where b is a positive constant representing the marginal increase of impatience due to one unit increase of capital stock. And the steady state level of capital is implicitly determined by, $f'(\bar{k}) = 1/\int_0^\infty e^{-[b\cdot\bar{k}\cdot v+\phi(v)]}dv$ (10). If we denote the steady state capital as $\bar{k} = \bar{k}(b)$, the capital stock is a decreasing function of b, i.e. $d\bar{k}(b)/db < 0$.

Proof: Differentiate both sides of (10) with respect to b, and we will get,

$$f''(\bar{k})\cdot d\bar{k}(b)/db = \int_0^\infty e^{-[b\cdot\bar{k}\cdot v+\phi(v)]}(\bar{k}\cdot v + b\cdot v\cdot d\bar{k}(b)/db)dv/[\int_0^\infty e^{-[b\cdot\bar{k}\cdot v+\phi(v)]}dv]^2.$$

Rearrange, and we will have,

$$[f''(\bar{k}) - \int_0^\infty e^{-[b\cdot\bar{k}\cdot v+\phi(v)]}(b\cdot v)dv]\cdot d\bar{k}(b)/db = \int_0^\infty e^{-[b\cdot\bar{k}\cdot v+\phi(v)]}(\bar{k}\cdot v)dv/[\int_0^\infty e^{-[b\cdot\bar{k}\cdot v+\phi(v)]}dv]^2.$$

The right side of the above equation is positive, and the item in the bracket in the left side is negative, so we conclude that $d\bar{k}(b)/db < 0$. Q.E.D.

Proposition A5. Given $\lambda(k) = 1/\int_0^\infty e^{-[\rho(k)v+\phi(v)]}dv$ and $\dot{k} = f(k) - c$, the dynamic system of (6) could be linearized as follows,

$\begin{pmatrix}\dot{c}\\ \dot{k}\end{pmatrix} = A(\bar{c},\bar{k})\cdot\begin{pmatrix}c-\bar{c}\\ k-\bar{k}\end{pmatrix}$, where the matrix $A(\bar{c},\bar{k}) = \begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}$.

The elements in the coefficient matrix are given by,

$$a_{11} = -\lambda(\bar{k}) \cdot \bar{c} \cdot \rho'(\bar{k}) \cdot \int_0^\infty e^{-[\rho(\bar{k})v+\phi(v)]} v dv,$$

$$a_{12} = \bar{c}\{f''(\bar{k}) - \lambda'(\bar{k}) + \lambda(\bar{k}) \cdot \rho'(\bar{k}) \cdot f'(\bar{k}) \cdot \int_0^\infty e^{-[\rho(\bar{k})v+\phi(v)]} v dv\},$$

$$a_{21} = -1,$$

$$a_{22} = f'(\bar{k}) = \lambda(\bar{k}). \qquad \text{(A5.1)}$$

Proof: First, we easily see that, $d\lambda(k)/dk = \lambda^2 \cdot \rho'(k) \cdot \int_0^\infty e^{-[\rho(k)v+\phi(v)]} v dv > 0$, and then we have, $\dot{\lambda} = [d\lambda(k)/dk] \cdot \dot{k} = \lambda^2 \cdot \rho'(k) \cdot [f(k) - c] \cdot \int_0^\infty e^{-[\rho(k)v+\phi(v)]} v dv$. Then (6) becomes,

$$\dot{c}/c = f'(k) - \lambda(k) + \lambda(k) \cdot \rho'(k) \cdot [f(k) - c] \cdot \int_0^\infty e^{-[\rho(k)v+\phi(v)]} v dv$$
$$\dot{k} = f(k) - c \qquad \text{(A5.2)}$$

Linearize (A5.2) around the steady state $(\bar{c}, \bar{k})$, we will have,

$\begin{pmatrix} \dot{c} \\ \dot{k} \end{pmatrix} = A(\bar{c}, \bar{k}) \cdot \begin{pmatrix} c - \bar{c} \\ k - \bar{k} \end{pmatrix}$, where the matrix $A(\bar{c}, \bar{k})$ is defined as in (A5.1). Q.E.D.

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