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Martingale and Relaxation-Projection Methods for Utility Maximization with Portfolio Constraints and Stochastic Income

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The problem of maximizing the expected utility from terminal wealth in the presence of a stochastic endowment and constraints on the portfolio choices is examined. We model short-sale and borrowing constraints, as well as incomplete markets, as special cases of constraints. The existence of optimal policies is established under fairly general assumptions on the security price coefficients and the individual's utility function. This result is obtained by using martingale techniques to reformulate the individual's dynamic optimization problem as an equivalent static one.

Key Words: Portfolio constraints; Stochastic income; Relaxation-projection methods.
JEL Classification Numbers: G11, C61.

1. INTRODUCTION

This paper examines the problem of maximizing the expected utility from terminal wealth in a continuous-time, finite-horizon economy when labor income follows an arbitrary bounded stochastic process and the dollar amounts invested in the traded assets are constrained to take values in a given closed, convex set. Short-sale and borrowing constraints, as well as incomplete markets, can be modeled as special cases of this setting.

The problem of maximizing the expected utility from terminal wealth is well solved in the context of a complete financial market in three steps. First, on the underlying probability space we determine a new measure that discounts the growth inherent in the market; under this measure, the expected value of the final wealth attained by any reasonable portfolio is equal to the initial endowment. Second, among all random variables whose expectation under the new measure is equal to the initial endowment, we determine the most desirable one. Third, it is shown that an optimal portfolio can be constructed, which attains this most desirable random variable as its terminal wealth.

Results on the characterization and existence of optimal consumption and investment policies with a finite horizon and with security prices following general Itô processes have been obtained using martingale and duality techniques by He and Pearson (1991), Karatzas, Lehoczky, and Xu (1991), and Xu and Shreve (1992). They have examined the optimal consumption/investment problem with incomplete markets and/or short-sale constraints. Cvitanic and Karatzas (1992) have considered the more general case in which the *portfolio weights* are constrained to take values in a closed convex subset. All of these papers assume, however, that the agent is only endowed with some nonnegative amount of wealth at the initial date and there is no labor income. Hence, they do not address the problem of a nontraded endowment. Their approaches transform the primal constrained-maximization problem into a dual unconstrained-minimization problem that solves for the individual shadow state prices (intertemporal marginal rates of substitution). Because this dual problem is, in general, not convex in the preference of a nontraded endowment process, the possibility of directly extending their results to models with stochastic income is precluded.

A recent paper by Cuoco (1997) is the only successful application of martingale technique in a continuous-time setting to establish the existence of optimal policies in the presence of stochastic income and constraints on investment policies. Cuoco considers the intertemporal optimal consumption and investment problem in the presence of a stochastic endowment and constraints on the portfolio choices.

In contrast this paper allows for the presence of nontraded stochastic income and portfolio constraints to obtain optimal investment and portfolio rules about the preference from terminal wealth in a very general setting. We emphasize that we focus on the case of constraints on the dollar amounts invested in risky assets, rather than on the portfolio weights: this different formulation is required by the fact that with a nontrival income process, and hence possibly negative wealth, portfolio weights are not defined. It should, however, be easy to see that, with a nonnegative wealth process, our setting includes the case of constraints on the portfolio weights as a special case.

The rest of this paper is organized as follows. In sections 2, 3, and 4, we describe the model with the corresponding notations and definitions. Section 5 obtains a static characterization of the feasible terminal wealth policies as those satisfying a budget constraint with respect to all of the state-price densities consistent with the absence of arbitrage opportunities. Section 6 shows that the primal problem of maximizing the expected utility on the set of feasible terminal wealth policies admits a solution. Section 7 characterizes optimal policies. Section 8 makes some concluding remarks. Proofs of theorems are included in the appendix. Our proof of existence in the primal problem uses the relaxation-projection technique for optimization without compactness introduced by Levin (1976), and further developed by Fougeres (1979).

2. THE ECONOMIC SETTING

We consider a continuous-time economy on the finite time span $[0, T]$, in which an individual endowed with some initial wealth and a stochastic income flow chooses an optimal investment policy. Let λ denote the Lebesgue measure on $[0, T]$.

Information structure. The uncertainty is represented by a filtered probability space $(\Omega, \mathfrak{F}, \mathbb{F}, P)$on which is defined an n-dimensional Brownian motion

$$w = \{(w_1(t), \cdots, w_n(t))^T : t \in [0, T]\}.$$

The filtration $\mathbb{F} = \{\mathfrak{F}_t\}$ is the augmentation under P of the filtration generated by w. We assume that $\mathfrak{F} = \sigma \left\{ \bigcup_{0 \leq t \leq T} \mathfrak{F}_t \right\}$, or that the true state of nature is completely determined by the sample paths of w on $[0, T]$. We interpret the sigma-field $\mathfrak{F}_t$ as representing the information of the individual at time t and the probability measure P as representing his beliefs. All the stochastic processes in the sequel are progressively measurable with respect to $\mathbb{F}$ and all the equalities involving random variables are understood to hold P-*a.s.*.

Securities market. The investment opportunities are represented by $(n+1)$ long-lived securities. The first security, which we term the "bond", is locally riskless and pays no dividends. Its price process, denoted by B, is given by

$$B(t, \omega) = \exp\left(\int_0^t r(\tau, \omega) d\tau\right) \tag{1}$$

for some interest rate process r.

Assumption 1. The interest rate process r is bounded uniformly in $(t, \omega) \in [0, T] \times \Omega : r \leq r_B$ for some $r_B > 0$.

The remaining n assets are risky. Letting $S = (S_1, \cdots, S_n)$ denote their price process and $D = (D_1, D_2, \cdots, D_n)$ their cumulative dividend process, we assume that $S+D$ is an Itô process,

$$\begin{aligned} & S(t,\omega)+D(t,\omega) \\ = \; & S(0)+\int_0^t I_S(\tau,\omega)\mu(\tau,\omega)d\tau+\int_0^t I_S(\tau,\omega)\sigma(\tau,\omega)dw(\tau,\omega), \end{aligned} \quad (2)$$

where $I_S(t)$ denotes the $n\times n$ diagonal matrix with elements $S(t)$ and

$$\int_0^T |I_S(t)\mu(t)|dt+\int_0^T |I_S(t)\sigma(t)|^2 dt<\infty.$$

Assumption 2. The diffusion matrix σ satisfies the nondegeneracy condition

$$x^T\sigma(t)\sigma(t)^T x\geq \varepsilon|x|^2 \quad (3)$$

almost surely for all $(x,t)\in R^n\times[0,T]$ and for some $\varepsilon>0$. Moreover, letting

$$k_0=-\sigma^{-1}(\mu-r\bar{1}), \quad (4)$$

where $\bar{1}=(1,\cdots,1)^T\in R^n$, we have

$$E\left[\exp\left(\frac{1}{2}\int_0^T |k_0(t)|^2 dt\right)\right]<\infty. \quad (5)$$

Condition (3) implies in particular that $\sigma(t)$ has full rank *a.s.* for all $t\in[0,T]$, so that in the absence of portfolio constraints markets are dynamically complete, and that $\sigma(t,\omega)^{-1}$ has an essentially bounded matrix norm, uniformly in $(t,\omega)\in[0,T]\times\Omega$ (Karatzas and Shreve, 1988, Problem 5.8.1). Condition (5) is a "Novikov condition" (Karatzas and Shreve, 1988, Corollary 3.5.13) and is used to guarantee the existence of an equivalent martingale measure.

Trading strategies. Trading takes place continuously and there are no market frictions. An admissible trading strategy is an $(n+1)$-dimensional vector process (α,θ)-where $\alpha(t)$ and $\theta_k(t)$ denote, respectively, the dollar amount invested at time t in the bond and the kth risky asset-satisfying

$$\int_0^T |\alpha(t)r(t)|dt+\int_0^T \left|\theta(t)^T\mu(t)\right|dt+\int_0^T \left|\theta(t)^T\sigma(t)\right|^2 dt<\infty. \quad (6)$$

The set of admissible trading strategies is denoted by Θ.

Preferences and endowments. The agent in our model has a utility function $U:(0,\infty)\to R$ for wealth. U is strictly increasing, strictly concave, continuous and continuously differentiable, and satisfies

$$U'(0)\triangleq\lim_{x\to 0}U'(x)=\infty, \quad U'(\infty)\triangleq\lim_{x\to\infty}U'(x)=0. \quad (7)$$

The (continuous, strictly decreasing) inverse of the function U' will be denoted by $f:(0,\infty)\to(0,\infty)$; by analogy with (7), it satisfies

$$f(0) \triangleq \lim_{y\to 0} f(y) = \infty, \quad f(\infty) \triangleq \lim_{y\to\infty} f(y) = 0. \tag{8}$$

We introduce also the function

$$\overline{U}(y) \triangleq \max_{x>0}[U(x) - xy] = U(f(y)) - yf(y), \quad 0 < y < \infty, \tag{9}$$

which is the convex conjugate of $-U(x)$, with U extended to be $-\infty$ on the negative real axis. The function $\overline{U}$ is strictly decreasing, strictly convex, and satisfies

$$\overline{U}'(y) = -f(y), \quad 0 < y < \infty, \tag{10}$$

$$U(x) = \min_{y>0}[\overline{U}(y) + xy] = \overline{U}(U'(x)) + xU'(x), \quad 0 < x < \infty. \tag{11}$$

The useful inequalities

$$U(f(y)) \geq U(x) + y[f(y) - x], \quad \forall x > 0, y > 0, \tag{12}$$

$$\overline{U}(U'(x)) \leq \overline{U}(y) - x[U'(x) - y], \quad \forall x > 0, y > 0, \tag{13}$$

then follow directly from (9) and (11).

The monotonicity of U and $\overline{U}$ guarantees that the limits

$$U(0) \triangleq \lim_{x\to 0} U(x), \; U(\infty) \triangleq \lim_{x\to\infty} U(x),$$
$$\overline{U}(0) \triangleq \lim_{y\to 0} \overline{U}(y), \; \overline{U}(\infty) \triangleq \lim_{y\to\infty} \overline{U}(y),$$

exist in the extended real-number system. Furthermore,

$$U(0) = \overline{U}(\infty), \; \overline{U}(0) = U(\infty)$$

(The proof of the above equalities is in Karatzas, Shreve, and Xu (1991)).

The agent is endowed with some initial wealth $d_0 \geq 0$ and a nonnegative stochastic income process d with

$$\int_0^T B(t)^{-1} d(t)dt \leq K_d \tag{14}$$

for some $K_d > 0$.

3. PORTFOLIO CONSTRAINT SETS

We fix from now on a nonempty, closed, and convex set $A \subseteq R^{n+1}$ and assume that the agent's portfolio (α, θ) is constrained to take values in A. As we will see shortly, several constraints of practical interest such as short-sale prohibitions, nontradeable assets, or minimum capital requirements, can be modeled as special cases of this class of constraints.

For $v = (v_0, v_-) \in R \times R^n$, let

$$\delta(v) = \sup_{(\alpha,\theta)\in A} -(\alpha v_0 + \theta^T v_-)$$

denote the support function of $-A$ and let

$$\overline{A} = \left\{ v \in R^{n+1} : \delta(v) < \infty \right\}$$

denote its *effective domain*. We note that δ is a positively homogeneous, lower semicontinuous, and proper convex function on R^{n+1} and that $\overline{A}$ is a closed convex cone (the barrier cone of $-A$). We will assume that the constraint set A is such that the following condition is satisfied.

Assumption 3. The function δ is upper semicontinuous and bounded above on $\overline{A}$. Moreover, $v_0 \geq 0$ for all $v \in \overline{A}$.

Remark 3.1. It is easily verified that the set $\left\{ v_0 : v \in \overline{A} \right\}$ must be bounded below by 0 as long as lending and investing nothing in the risky assets is admissible (i.e., as long as $(\alpha, \theta) \in A$ for all α large enough). Also, since δ is positively homogeneous and $\overline{A}$ is a cone, the condition that δ be bounded abov on $\overline{A}$ is equivalent to δ being nonpositive on $\overline{A}$. In particular, this condition is satisfied if A is a cone, in which case $\delta \equiv 0$ on $\overline{A}$.

We now provide some examples of constraint sets A satisfying Assumption 3, together with the associated support functions and dual sets.

(a) *No constraints*:

$$\begin{aligned} &A = R^{n+1}, \\ &\overline{A} = \{0\}, \\ &\delta(v) = 0 \quad \text{for} \quad v \in \overline{A}. \end{aligned}$$

This is the problem studied by Cox and Huang (1989, 1991).

(b) *Nontradeable assets* (incomplete markets):

$$\begin{aligned} &A = \left\{ (\alpha, \theta) \in R^{n+1} : \theta_k = 0, k = m+1, \cdots, n \right\}, \\ &\overline{A} = \left\{ v \in R^{n+1} : v_k = 0, k = 0, \cdots, m \right\}, \\ &\delta(v) = 0 \quad \text{for} \quad v \in \overline{A}. \end{aligned}$$

For the case of no stochastic income ($y \equiv 0$), this problem was originally approached using martingale techniques by He and Pearson (1991), and Karatzas, Shreve, and Xu (1991).

(c) *Short-sale constraints*:

$$\begin{aligned} A &= \left\{(\alpha, \theta) \in R^{n+1} : \theta_k \geq 0, k = m+1, \cdots, n\right\}, \\ \overline{A} &= \left\{v \in R^{n+1} : v_k = 0, k = 1, \cdots, m;\ v_k \geq 0, k = m+1, \cdots, n\right\}, \\ \delta(v) &= 0 \quad \text{for} \quad v \in \overline{A}. \end{aligned}$$

Again assuming no income stream, this problem was examined by Xu and Shreve (1992).

(d) *Buying constraints*:

$$\begin{aligned} A &= \left\{(\alpha, \theta) \in R^{n+1} : \theta_k \leq 0, k = m+1, \cdots, n\right\}, \\ \overline{A} &= \left\{v \in R^{n+1} : v_k = 0, k = 1, \cdots, m; v_k \leq 0, k = m+1, \cdots, n\right\}, \\ \delta(v) &= 0 \quad \text{for} \quad v \in \overline{A}. \end{aligned}$$

(e) *Portfolio-mix constraints*:

$$A = \left\{(\alpha, \theta) \in R^{n+1} : \alpha + \sum_{k=1}^{n} \theta_k \geq 0, \theta \in M\left(\alpha + \sum_{k=1}^{n} \theta_k\right)\right\},$$

where M is any nonempty, closed, and convex subset that contains the origin,

$$\begin{aligned} \overline{A} &= \left\{v \in R^{n+1} : v^T(\alpha, \theta) \geq 0, \forall (\alpha, \theta) \in A\right\}, \\ \delta(v) &= 0 \quad \text{for} \quad v \in \overline{A}. \end{aligned}$$

For the case of no income stream, and hence a nonbinding nonnegativity constraint on wealth, this problem was examined by Cvitanic and Karatzas (1992).

(f) *Minimum capital requirements*:

$$A = \left\{(\alpha, \theta) \in R^{n+1} : \alpha + \sum_{k=1}^{n} \theta_k \geq K\right\},$$

where $K \geq 0$,

$$\begin{aligned} \overline{A} &= \left\{k\overline{1} : k \geq 0\right\}, \\ \delta(v) &= -Kv_0 \quad \text{for} \quad v \in \overline{A}. \end{aligned}$$

Special cases of minimum-capital requirements are the borrowing constraints studied by He and Pages (1993) (where $K = 0$) and the portfolio-insurance constraint studied by Bardhan (1994) and Basak (1995) (where $K > 0$).

4. THE INDIVIDUAL INVESTMENT PROBLEM

Given the price coefficients $\mathfrak{P} = (r, \mu, \sigma)$, a random variable B is said to be feasible terminal wealth if there exists an admissible trading strategy $(\alpha, \theta) \in \Theta$ and a nonnegative, increasing process C such that, letting

$$W(t) = \alpha(t) + \sum_{k=1}^{n} \theta_k(t)$$

denote the value of the agent's portfolio at time t, we have

$$\begin{aligned} W(t) &= d_0 + \int_0^t \Big(\alpha(\tau) r(\tau) + \theta(\tau)^T \mu(\tau)\Big) d\tau \\ &+ \int_0^t \theta(\tau)^T \sigma(\tau) dw(\tau) + \int_0^t (d(\tau)) d\tau - C(t), \end{aligned} \tag{15}$$

$$W(t) \geq -K, \tag{16}$$

$$B = W(T) \geq 0, \tag{17}$$

$$\min\Big(E(U(W(T)))^+, E(U(W(T)))^-\Big) < \infty \tag{18}$$

for all $t \in [0, T]$ and some $K \in R$, where x^+ and x^- denote, respectively, the positive and the negative part of the real number x. The terminal wealth B is said to be A-*feasible* if the above conditions are satisfied and $(\alpha(t), \theta(t)) \in A$ for all $t \in [0, T]$. We will let $\mathfrak{B}(\mathfrak{P}, A)$ denote the set of A-feasible terminal wealth given the price system $\mathfrak{P}$.

The process C in (15) captures the possibility of free disposal of wealth: in other words, the agent is allowed not to reinvest some of his wealth if he chooses to do so. The total amount of wealth "wasted" up to time t is given by $C(t)$. Equation (14) is then the usual dynamic budget constraint: it states that the wealth at any time t equals the initial wealth, plus the trading gains, minus the cumulative net withdrawals. Equations (16) and (17) state that, while the investor is allowed to borrow against future income and thus to have short-term deficits, the final wealth must be sufficient to cover any amount borrowed. Moreover, the wealth process must admit a uniform lower bound: this is sufficient to rule out arbitrage opportunities, such as the doubling strategies discussed by Harrison and Kreps (1979).

The individual investment problem can now be formally stated as that of maximizing the expected utility functional $E(U)$ over the set $\{B : B \in \mathfrak{B}(\mathfrak{P}, A)\}$.

5. STATE PRICES AND FEASIBLE WEALTH PROCESSES

Since security prices and the income stream in this model allow possibly non-Markovian processes, stochastic dynamic programming cannot be applied to analyze the agent's consumption problem. Therefore we will use martingale techniques to transform the dynamic budget constraint in (15) into a set of equivalent static constraints.

In order to motivate the static characterization of feasible consumption plans, consider first the unconstrained case ($A = R^{n+1}$). Define the discount process

$$\beta_0 (t) = B (t)^{-1} = \exp\left(-\int_0^t r (\tau) \, d\tau \right)$$

and the exponential local martingale

$$\xi_0 (t) = \exp\left(\int_0^t k_0 (\tau)^T \, dw(\tau) - \frac{1}{2} \int_0^t |k_0 (\tau)|^2 \, d\tau \right), \tag{19}$$

where k_0 is the process of (4). By (5), ξ_0 is in fact a strictly positive martingale, so that it is possible to define a probability measure Q_0 equivalent to P by $\frac{dQ_0}{dP} = \xi_0 (T)$. Also, it is well-known (and easily verified by Girsanov's theorem) that Q_0 has the property that the discounted gain process

$$G_0 (t) = \beta_0 (t) \, S (t) + \int_0^t \beta_0 (s) \, dD(s)$$

becomes a local martingale under it, and Q_0 is in fact the unique probability measure equivalent to P with this property: it is alternatively known in the finance literature as the *risk-neutral probability* or the *equivalent martingale measure*. The process $\pi_0 = \beta_0 \xi_0$ then identifies the unique *state-price density* for the economy, in the sense that the value at time 0 of any wealth process W satisfying an integrability condition is given by

$$E \left[\pi_0 (T) \, W (T) \right] = E^{Q_0} \left[\beta_0 (T) \, W (T) \right],$$

where E^{Q_0} denotes the expectation operator under Q_0. Without constraints on the set of admissible portfolio policies, it follows that a terminal wealth is feasible if and only if its value does not exceed the value of the individual's endowment, i.e., if only if

$$E^{Q_0} \left[\beta_0 (T) \, W (T) - \int_0^T \beta_0 (t) \, d (t) \, dt \right] \leq d_0. \tag{20}$$

We will refer to (20) as a *static budget constraint.* The dynamic problem of maximizing the expected utility over the terminal wealth $W(T)$ satisfying (15)-(17) can then be restated as the equivalent problem of maximizing the expected utility subject to the single budget constraint in (20).

We will now show that a similar transformation of the optimal investment problem into an equivalent static one is possible in the presence of constraints and that conditions similar to those in Cox and Huang (1991) are sufficient to guarantee the existence of an optimal investment plan. The main differences are that, with unconstrained portfolio policies ($A = R^{n+1}$) π_0 is the unique state-price density consistent with the absence of arbitrage opportunities; but with constrained portfolios ($A \subset R^{n+1}$) there exist infinitely many state-price densities that are consistent with no arbitrage, and a family of static budget constraints needs to be considered in order to ensure feasibility.

Following the lead of Cvitanic and Karatzas (1992, 1993), let $\mathfrak{N}$ denote the set of $\overline{A}$-valued processes v satisfying

$$E\left[\int_0^T |v(t)|^2\,dt\right] < \infty. \tag{21}$$

For each $v \in \mathfrak{N}$, the processes

$$\begin{aligned}
&\beta_v(t) = \exp\left(-\textstyle\int_0^t (r(\tau) + v_0(\tau))\,d\tau\right),\\
&k_v(t) = -\sigma(t)^{-1}\left(\mu(t) + v_-(t) - (r(t) + v_0(t))\,\overline{1}\right),\\
&\xi_v(t) = \exp\left(\textstyle\int_0^t k_v(\tau)^T dw(\tau) - \frac{1}{2}\int_0^t |k_v(\tau)|^2\,d\tau\right),\\
&\pi_v(t) = \beta_v(t)\,\xi_v(t),
\end{aligned}$$

are well-defined, and ξ_v is a strictly positive local martingale. Let $\mathfrak{N}^*$ denote the subset of elements $v \in \mathfrak{N}$ for which ξ_v is in fact a martingale. Note that $\mathfrak{N}^*$ is nonempty, since (5) and the fact that $\overline{A}$ is a cone ensure that we always have $0 \in \mathfrak{N}^*$.

Each ξ_v with $v \in \mathfrak{N}^*$ can be interpreted as the density process corresponding to some probability measure Q_v equivalent to P. Also, it is clear that each π_v with $v \in \mathfrak{N}^*$ can be interpreted as the unique state-price density in a fictitious unconstrained economy with price coefficients $\mathfrak{P}=(r + v_0, \mu + v_-, \sigma)$. More generally, the following result shows that each π_v with $v \in \mathfrak{N}^*$ constitutes an arbitrage-free state-price density in the original economy when the portfolio policies are constrained to be in A, and that the satisfaction of a budget constraint with respect to all of these state-price densities is also sufficient to guarantee the A-feasibility.

THEOREM 5.1. *A random variable B is A-feasible if and only if*

$$E^{Q_v}\left[\beta_v(T)B\right] \le d_0 + E^{Q_v}\left[\int_0^T \beta_v(t)\left(\delta(v(t)) + d(t)\right)dt\right], \quad \forall v \in \mathfrak{N}^*. \tag{22}$$

Proof. Suppose first that B is A-feasible, i.e., that (15)-(18) are satisfied for some investment strategy (α, θ) with $(\alpha(t), \theta(t)) \in A$ for all $t \in [0, T]$. Using Itô's lemma, it is easy to show that (15) implies

$$\begin{aligned}
&\beta_v(t)W(t) - \int_0^t \beta_v(\tau)d(\tau)d\tau \\
\le\ & \beta_v(t)W(t) - \int_0^t \beta_v(\tau)d(\tau)d\tau + \int_0^t \beta_v(\tau)dC(\tau) \\
=\ & d_0 - \int_0^t \beta_v(\tau)\left(\alpha(\tau)v_0(\tau) + \theta(\tau)^T v_-(\tau)\right)d\tau \\
+\ & \int_0^t \beta_v(\tau)\theta(\tau)^T\sigma(\tau)dw_v(\tau) \\
\le\ & d_0 + \int_0^t \beta_v(\tau)\delta(\nu(\tau))d\tau + \int_0^t \beta_v(\tau)\theta(\tau)^T\sigma(\tau)dw_v(\tau),
\end{aligned} \tag{23}$$

for all $v \in \mathfrak{N}^*$, where

$$w_v(t) = w(t) - \int_0^t k_v(\tau)d\tau,$$

is a Brownian motion under Q_v. For each positive integer n, letting

$$\tau_n = T \wedge \inf\left\{t \in [0, T] : \int_0^t \left|\theta(\tau)^T\sigma(\tau)\right|^2 d\tau \ge n\right\},$$

with the usual convention maintained for the remainder of the paper, i.e., $\inf(\varnothing) = \infty$. Since the stochastic integral on the right-hand side of (23) is a Q_v-martingale on $[0, \tau_n]$, taking expectations gives

$$\begin{aligned}
&E^{Q_v}\left[\beta_v(\tau_n)W(\tau_n)\right] - E^{Q_v}\left[\int_0^{\tau_n} \beta_v(t)d(t)dt\right] \\
\le\ & d_0 + E^{Q_v}\left[\int_0^{\tau_n} \beta_v(t)\delta(v(t))dt\right].
\end{aligned} \tag{24}$$

Letting $n \uparrow \infty$, we have $\tau_n \uparrow T$ (because of (6)). Applying the monotone convergence theorem twice and using the fact that

$$E^{Q_v}\left[\int_0^T \beta_v(t)\, d(t)\, dt\right] \le E^{Q_v}\left[\int_0^T \beta_0(t)\, d(t)\, dt\right] \le K_y$$

because of (14) and Assumption 3, shows that

$$\lim_{n\to\infty} E^{Q_v}\left[\int_0^{\tau_n} \beta_v(t)\, d(t)\, dt\right] = E^{Q_v}\left[\int_0^T \beta_v(t)\, d(t)\, dt\right].$$

Applying the dominated convergence theorem and Assumption 3, shows that

$$\lim_{n\to\infty} E^{Q_v}\left[\int_0^{\tau_n} \beta_v(t)\, \delta(v(t))\, dt\right] = E^{Q_v}\left[\int_0^T \beta_v(t)\, \delta(v(t))\, dt\right].$$

As for the first term in (24), we have from (16) and Assumption 3

$$(\beta_v(\tau_n) W(\tau_n))^- \le (\beta_0(\tau_n) W(\tau_n))^- \le K exp\left(\int_0^T |r(t)|\, dt\right) < \infty,$$

for all n. Fatou's lemma then gives

$$\liminf_{n\to\infty} E^{Q_v}[\beta_v(\tau_n) W(\tau_n)] \ge E^{Q_v}[\beta_v(T) W(T)] \ge 0,$$

where the last inequality follows from (17). Therefore, we have

$$\begin{aligned} E^{Q_v}[\beta_v(T) W(T)] &\le \liminf_{n\to\infty} E^{Q_v}[\beta_v(\tau_n) W(\tau_n)] \\ &\le \liminf_{n\to\infty} E^{Q_v}\left[\int_0^{\tau_n} \beta_v(t)\, d(t)\, dt\right] + d_0 \\ &+ \liminf_{n\to\infty} E^{Q_v}\left[\int_0^{\tau_n} \beta_v(t)\, \delta(v(t))\, dt\right] \\ &= E^{Q_v}\left[\int_0^T \beta_v(t)\, (d(t) + \delta(v(t)))dt\right] + d_0. \end{aligned}$$

This establishes (22).

To show the converse, let $\mathfrak{T}$ denote the set of stopping time τ with $\tau \le T$, and for any $\tau \in \mathfrak{T}$ let

$$W(\tau) = \sup_{v\in\mathfrak{N}^*} \beta_v(\tau)^{-1} E^{Q_v}\left[\beta_v(T) W(T) - \int_\tau^T \beta_v(t)\, (d(t) + \delta(v(t)))dt \mid \mathfrak{F}_\tau\right].$$

Since $W(0) \leq d_0$ and the family of random variables

$$X_v(\tau) = \int_\tau^T \exp\left(-\int_\tau^t (r(s) + v_0(s))\, ds\right)(y(t) + \delta(v(t)))dt,$$

is uniformly bounded above (because of (14) and Assumption 3), it follows from the argument used in the proof of Propositions 6.2 and 6.3 in Cvitanic and Karatzas (1993) that W satisfies the equation of dynamic programming

$$\begin{aligned} W(\tau_1) &= \sup_{v \in \mathfrak{N}^*} [E^{Q_v} \int_{\tau_1}^{\tau_2} \exp\left(-\int_{\tau_1}^t (r(s) + v_0(s))\, ds\right)(d(t) + \delta(v(t)))dt \\ &+ \exp\left(-\int_{\tau_1}^{\tau_2} (r(s) + v_0(s))\, ds\right) W(\tau_2) \mid \mathfrak{F}_{\tau_1}] \end{aligned}$$

for all $\tau_1, \tau_2 \in \mathfrak{T}$ with $\tau_1 < \tau_2$, and hence that the process

$$M_v(t) = \beta_v(t)\, W(t) - \int_0^t \beta_v(s)\,(d(s) + \delta(v(s)))ds$$

is a Q_v-supermartingale for all $v \in \mathfrak{N}^*$. By the Doob decomposition and the martingale representation theorem, for each $v \in \mathfrak{N}^*$ there exists an increasing real-valued process A_v with $A_v(0) = 0$ and a R^n-valued process ψ_v with $\int_0^T |\psi_v(t)|^2\, dt < \infty$ such that

$$M_v(t) = W(0) + \int_0^t \psi_v(\tau)^T\, dw_v(\tau) - A_v(t).$$

Since

$$\begin{aligned} &\beta_v(t)^{-1}\left(M_v(t) + \int_0^t \beta_v(s)\,(d(s) + \delta(v(s)))ds\right) \\ &= W(t) = \beta_0(t)^{-1}\left(M_0(t) + \int_0^t \beta_0(s)\, d(s)ds\right), \end{aligned} \tag{25}$$

we must have

$$\beta_v(t)^{-1}\,\psi_v(t) = \beta_0(t)^{-1}\,\psi_0(t)$$

and

$$\begin{aligned} &\int_0^t \left(\delta(v(\tau)) + v_0 W(\tau) + \beta_v(\tau)^{-1}\,\psi_v(\tau)^T\,\sigma(\tau)^{-1}\left(v_-(\tau) - v_0(\tau)\,\bar{1}\right)\right) d\tau \\ &- \int_0^t \beta_v(\tau)^{-1}\, dA_v(\tau) = -\int_0^t \beta_0(\tau)^{-1}\, dA_0(\tau) \end{aligned} \tag{26}$$

for all $v \in \mathfrak{N}^*$ and all $t \in [0,T]$. Now, letting

$$\theta(t)^T = \beta_0(t)^{-1} \psi_0(t)^T \sigma(t)^{-1},$$

and

$$\alpha(t) = W(t) - \sum_{k=1}^{n} \theta_k(t),$$

we intend to show that (α, θ) is an admissible trading strategy.

Conditions (16) and (17) follow immediately from the definition of W, (14), and Assumption 3. Next, observe that the process

$$C(t) = d_0 - W(0) + \int_0^t \beta_0(\tau)^{-1} dA_0(\tau)$$

is nonnegative and increasing, and that we have from (25) and Itô's lemma

$$\begin{aligned} W(t) &= W(0) + \int_0^t r(\tau) W(\tau) d\tau + \int_0^t \beta_0(\tau)^{-1} (dM_0(\tau) + d(\tau)\beta_0(\tau) d\tau) \\ &= d_0 + \int_0^t r(\tau) W(\tau) d\tau + \int_0^t \theta(\tau)^T \sigma(\tau) dw_0(\tau) + \int_0^t d(\tau) d\tau - C(t) \\ &= d_0 + \int_0^t \left(\alpha(\tau) r(\tau) + \theta(\tau)^T \mu(\tau)\right) d\tau + \int_0^t \theta(\tau)^T \sigma(\tau) dw(\tau) \\ &+ \int_0^t d(\tau) d\tau - C(t), \end{aligned}$$

thus (15) is also satisfied, and we are only left to show that (α, θ) take values in A.

By (26) and the definition of (α, θ), we have

$$\begin{aligned} 0 &\le \int_0^t \beta_v(\tau)^{-1} dA_v(\tau) \\ &= \int_0^t \beta_0(\tau)^{-1} dA_0(\tau) + \int_0^t \left(\delta(v(\tau)) + \alpha(\tau) v_0(\tau) + \theta(\tau)^T v_-(\tau)\right) d\tau. \end{aligned}$$

Since $v(\in \mathfrak{N}^*)$ is arbitrary, $\overline{A}$ is a convex cone and δ is positively homogeneous, this implies the existence of a set E having full $(\lambda \times P)$ measure (where λ denotes the Lebesgue measure on $[0,T]$) such that

$$\delta(v) + \alpha(t,\omega) v_0 + \theta(t,\omega)^T v_- \ge 0, \quad \forall (t,\omega) \in E, \quad v \in \overline{A}.$$

By Theorem 14.1 in Rockafellar(1970), this implies $(\alpha, \theta) \in A$, $(\lambda \times P)$-*a.e.*. ▮

6. EXISTENCE OF OPTIMAL POLICIES

Theorem 5.1 implies that it is possible to reformulate the agent's investment problem as

$$\max V \triangleq \max EU(B)$$

s.t.

$$E^{Q_v}\left[\beta_v(T)B\right] \le d_0 + E^{Q_v}\left[\int_0^T \beta_v(t)\left(\delta(v(t)) + d(t)\right)dt\right] \quad \forall v \in \mathfrak{N}^*, \quad (\mathbf{P})$$

$$B \ge 0.$$

This can be regarded as a convex optimization problem over a closed and norm-bounded subset of $L^1(Q_0)$. Unfortunately $L^1(Q_0)$ spaces are not reflexive, and hence the feasible set in (**P**) lacks (weak) compactness. To circumvent this difficulty, all of the existing papers approaching the optimal consumption problem under constraints using martingale techniques have focused on a dual minimization problem.

The motivation for the dual problem is the following. Let B^* denote the optimal final wealth plan, and suppose that at least one of the budget constraints in (**P**) is tight. Since the set $\{\pi_v : v \in \mathfrak{N}^*\}$ is convex, this suggests the existence of a state-price density π_{v^*} and a Lagrangian mutiplier $\psi^* > 0$ such that (B^*, ψ^*, v^*) is a saddle point of the map

$$\mathfrak{L}(B,\psi,v) = E\left[U(B)\right] - \psi E\left[\pi_v(T)B - \int_0^T \pi_v(t)\left(\delta(v(t)) + d(t)\right)dt - d_0\right], \tag{27}$$

where we maximize with respect to B and minimize with respect to (ψ, v). Maximizing (27) with respect to B leads to the dual shadow state-price problem

$$\min_{(\psi,v)\in(0,\infty)\times\mathfrak{n}^*} J(\psi,v) = E\left[\overline{U}(\psi\pi_v(T)) + \psi\left(d_0 + \int_0^T \pi_v(t)\left(\delta(v(t)) + d(t)\right)dt\right)\right]. \quad (\mathbf{P}^*)$$

From (7)-(13), we have the following proposition:

PROPOSITION 6.1. *Suppose that the utility function $U(x)$ satisfies the Inada conditions in* (7) *and that there exist constants $\beta \in [0,1]$ and $\gamma \in (0,\infty)$ such that*

$$\beta U'(B) \ge U'(\gamma B), \quad \forall B \in (0,\infty). \tag{28}$$

If there exists a solution (ψ^*, v^*) *to the dual state price problem* $(\mathbf{P}^*)$ *and*

$$E\left[f\left(\psi^* \pi_{v^*}(T)\right)\pi_{v^*}(T) - \int_0^T \pi_{v^*}(t)\,\delta\left(v^*(t)\right)dt\right] < \infty, \tag{29}$$

then there exists a constrained optimal wealth B^* *and the equality*

$$U'\left(B^*\right) = \psi^* \pi_{v^*}(T) \tag{30}$$

holds for some $\psi^* > 0$*, such that*

$$E\left[\pi_{v^*}(T)\, f\left(\psi^* \pi_{v^*}(T)\right) - \int_0^T \pi_{v^*}(t)\left(d(t) + \delta\left(v^*(t)\right)\right)dt\right] = d_0. \tag{31}$$

Conversely, if (30) *and* (31) *hold for some* $(\psi^*, v^*) \in (0,\infty) \times \mathfrak{N}^*$ *and some* $B^* \in \mathfrak{B}(\mathfrak{P}, A)$*, then* (ψ^*, v^*) *solves the dual problem.*

Proof. See Appendix A. ▌

The difficulty in using the duality approach in the presence of a stochastic income should now be apparent: unless $\delta \equiv 0$, $d \equiv 0$ (or, more generally, d is financed by an admissible trading strategy), and the utility function U has an Arrow-Pratt coefficient of relative risk aversion that is everywhere strictly less than one, the map $v \rightarrow J(\psi, v)$ is not convex. If these rather restrictive assumptions are satisfied, the problem can be relaxed by looking for a solution in $(0,\infty) \times \mathfrak{N}$ (i.e., by allowing the density process to be a local martingale, rather then a martingale), and the existence of a solution to $(\mathbf{P}^*)$ can then be shown using the technique of Cvitanic and Karatzas (1992).

In the present paper, we depart from the previous literature and show the existence of an optimal plan by attacking directly the primal problem $(\mathbf{P})$. We deal with the lack of compactness in the set of feasible terminal wealth by using the so-called technique of relaxation-projection introduced by Levin (1976) for optimization in non-reflexive spaces. The following theorem represents the main result of the paper.

THEOREM 6.1. *Suppose that:*
(i) *there exists a* $B \in \mathfrak{B}(\mathfrak{P}, A)$ *with* $E[U(B)] > -\infty$*;*
(ii) *either* U *is bounded above on* $(0,\infty)$*, or there exist constants* $k \geq 0$*,* $b \in (0,1)$ *and* $p > 1$ *such that*

$$U(x) \leq k\left(1 + x^{1-b}\right), \quad \forall x \in (0,\infty), \tag{32}$$

and

$$\xi_0^{-1}(T) \in L^{\frac{p}{b}}(Q_0). \tag{33}$$

Then there exists a constrained optimal terminal wealth.

Proof. See Appendix B. ▌

7. CHARACTERIZATION OF OPTIMAL POLICIES

While Theorem 6.1 guarantees the existence of an optimal terminal wealth B^*, it gives no indications as to the nature of such a policy. In the unconstrained case (i.e., with a single budget constraint), it follows immediately from the Lagrangian theory of optimization that the optimal consumption plan satisfies the Kuhn-Tucker condition

$$(U'(B^*) - \psi\pi_0(T))\, B^* = 0,$$

P-a.e. for some Lagarangian multiplier $\psi > 0$. In other words, the individual's marginal rates of substitution at the optimum equal the unique state-price density π_0 whenever the nonnegativity constraint on terminal wealth is nonbinding. The following proposition gives a generalization of this result.

PROPOSITION 7.1. *Let B^* denote the optimal terminal wealth and suppose that $B^* \neq 0$ and that there exists a $\gamma \in (0,1)$ such that*

$$E[U'(\gamma B^*)] < \infty. \tag{34}$$

Then there exists a sequence $\{\psi_n \pi_{v_n}\}$ with $\psi_n > 0$ and $v_n \in \mathfrak{N}^*$ for all n such that

$$(U'(B^*) - \psi_n \pi_{v_n}(T))\, B^* \to 0, \quad P\text{-a.e. and in } L^1(P). \tag{35}$$

If in addition

$$\inf_{v \in \mathfrak{N}^*} E[\pi_v(T)\, B^*] > 0, \tag{36}$$

then (35) *holds with $\psi_n = \psi > 0$ for all n.*

Proof. Define the subset M and N in $L^1(P)$ by

$$M = \{\psi\pi_v B^* : \psi > 0, v \in \mathfrak{N}^*\},$$
$$N = \left\{U'(B^*) - m : m \in \overline{M}\right\},$$

where $\overline{M}$ denotes the closure of M in $L^1(P)$. Arguing by contradiction, suppose that there is no sequence $\{\psi_n \pi_{v_n}\}$ such that $\psi_n \pi_{v_n} B^* \to U'(B^*) B^*$ in $L^1(P)$. Then $N \cap \{0\} = \varnothing$. Moreover, it is easily verified that N is convex and closed in $L^1(P)$. Therefore, it follows from the separating hyperplane theorem that there exists a $\varphi \in L^\infty(P)$ such that

$$E[U'(B^*) B^* \varphi] - \psi E[\pi_v(T) B^* \varphi] > 0,$$

for all $v \in \mathfrak{N}^*$ and $\psi \in R_+$. Letting $\overline{\varphi} = B^* \varphi \diagup \|\varphi\|_{L^\infty}$, the above implies

$$E[U'(B^*) \overline{\varphi}] > 0 \geq E[\pi_v(T) \overline{\varphi}], \quad \forall v \in \mathfrak{N}^*. \tag{37}$$

But then for each $\varepsilon \in (0, 1-\gamma)$(where γ is the constant of (34)) the terminal wealth $B_\varepsilon = B^* + \varepsilon \overline{\varphi} \geq \gamma B^*$ is A-feasible, and hence it follows from the optimality of B^* that

$$0 \geq \lim_{\varepsilon \downarrow 0} \frac{V(B_\varepsilon) - V(B^*)}{\varepsilon} = E[U'(B^*) \overline{\varphi}],$$

where the last equality follows from the dominated convergence theorem, the fact that

$$\frac{|U(B_\varepsilon) - U(B^*)|}{\varepsilon} \leq U'(\gamma B^*) \frac{|B_\varepsilon - B^*|}{\varepsilon} \leq U'(\gamma B^*) B^*,$$

and that the last expression is integrable by (34). This contradicts (37), and hence establishes (35) in the part concerning convergence in $L^1(P)$. Almost everywhere convergence can be ensured by passing to a subsequence.

Next, suppose that (36) also holds and let $\{\psi_n \pi_{v_n} B^*\} \subset M$ be such that $\psi_n \pi_{v_n} B^* \to U'(B^*) B^*$ in $L^1(P)$. Since $\psi_n \|\pi_{v_n} B^*\|_{L^1} \to \|U'(B^*) B^*\|_{L^1}$ and $\|\pi_{v_n} B^*\|_{L^1}$ is bounded below away from zero, $\{\psi_n\}$ is bounded. Hence, we can assume (by possibly passing to a subsequence) that $\psi_n \to \psi > 0$ and we are only left to show that $\|\psi \pi_{v_n} B^* - U'(B^*) B^*\|_{L^1} \to 0$. But this follows from the inequalities

$$\begin{aligned}
& \|\psi \pi_{v_n} B^* - U'(B^*) B^*\|_{L^1} \\
= \; & \|(\psi - \psi_n) \pi_{v_n} B^* + \psi_n \pi_{v_n} B^* - U'(B^*) B^*\|_{L^1} \\
\leq \; & \|\pi_{v_n} B^*\|_{L^1} |\psi - \psi_n| + \|\psi_n \pi_{v_n} B^* - U'(B^*) B^*\|_{L^1} \\
\leq \; & (d_0 + K_d) |\psi - \psi_n| + \|\psi_n \pi_{v_n} B^* - U'(B^*) B^*\|_{L^1} \to 0,
\end{aligned}$$

where K_d is the constant in (14). ▮

An immediate implication of the previous proposition is that if the optimal consumption plan is strictly positive (as must be the case with infinite marginal utility at zero), then, after scaling by a constant, the marginal utility process for an optimizing agent must be the pointwise limit of a sequence of state-price densities. We state this result in the next corollary.

COROLLARY 7.1. *If* $B^* > 0$ *a.e. and* (34) *and* (36) *hold, then there exists a* $\psi > 0$ *and a sequence* $\{\pi_{v_n}\}$ *with* $v_n \in \mathfrak{N}^*$ *for all* n *such that*

$$U'(B^*) = \lim_{n\to\infty} \psi \pi_{v_n}(T) \tag{38}$$

for P*-almost all* $\omega \in \Omega$.

8. CONCLUDING REMARKS

This paper studies the existence of optimal investment and portfolio rules given the preference from terminal wealth in the presence of constraints on portfolio choice and stochastic income. We apply the so-called technique of relaxation projection for optimization without compactness. Though we have assumed a state-independent utility function for terminal wealth, the case of state-dependent utility functions is easily accommodated by our existence result.

APPENDIX A

This Appendix is devoted to the proof of Proposition 6.1. The argument is adapted from Cuoco (1997).

Proof (Proposition 6.1). Assume that $(\psi^*, v^*) \in (0,\infty) \times \mathfrak{n}^*$ solves $(\mathbf{P}^*)$, and that (29) holds. In accordance with (30), define the terminal wealth plan B^* by $B^* = f(\psi^* \pi_{v^*}(T))$. In order to prove that B^* is constrained-optimal, we will proceed in two steps: first we will show that $V(B^*) \geq V(B)$ holds for all $B \in \mathfrak{B}(\mathfrak{P}, A)$ and then that $B^* \in \mathfrak{B}(\mathfrak{P}, A)$.

Step 1. Taking $B = f(y)$ in (28), applying $f(\cdot)$ to both sides and iterating, shows that for all $\beta \in (0,\infty)$ there exists a $\gamma \in (0,\infty)$ such that

$$f(\beta y) \leq \gamma f(y) \quad \forall y \in (0,\infty).$$

Since δ is bounded above on $\overline{A}$, (29) then implies

$$E\left[f(\psi \pi_{v^*}(T))\, \pi_{v^*}(T)\right] < \infty, \tag{A.1}$$

for all $\psi \in (0,\infty)$.

By the optimality of ψ^*, we have

$$\begin{aligned} 0 &= \lim_{\varepsilon\to 0}\frac{J(\psi^*+\varepsilon,v^*)-J(\psi^*,v^*)}{\varepsilon} \\ &= E\left[\lim_{\varepsilon\to 0}\frac{\overline{U}\left(\left(\psi^*+\varepsilon\right)\pi_{v^*}(T)\right)-\overline{U}\left(\left(\psi^*\right)\pi_{v^*}(T)\right)}{\varepsilon}\right] \\ &\quad +E\left[d_0+\int_0^T \pi_{v^*}(t)\left(d(t)+\delta\left(v^*(t)\right)\right)dt\right] \\ &= d_0-E\left[\pi_{v^*}(T)B^*-\int_0^T \pi_{v^*}(t)\left(d(t)+\delta\left(v^*(t)\right)\right)dt\right], \quad \text{(A.2)} \end{aligned}$$

where the second equality follows from Lebesgue's dominated convergence theorem, using (A.1) and the fact that

$$\begin{aligned} &\left|\frac{\overline{U}\left(\left(\psi^*+\varepsilon\right)\pi_{v^*}(T)\right)-\overline{U}\left(\left(\psi^*\right)\pi_{v^*}(T)\right)}{\varepsilon}\right| \\ \leq\ & \frac{\overline{U}\left(\left(\psi^*+|\varepsilon|\right)\pi_{v^*}(T)\right)-\overline{U}\left(\left(\psi^*\right)\pi_{v^*}(T)\right)}{|\varepsilon|} \\ \leq\ & \pi_{v^*}(T)\, f\left(\left(\psi^*-|\varepsilon|\right)\pi_{v^*}(T)\right) \\ \leq\ & \pi_{v^*}(T)\, f\left(\frac{\psi^*}{2}\pi_{v^*}(T)\right) \end{aligned}$$

for $|\varepsilon|<\frac{\psi^*}{2}$(because $\overline{U}(\cdot)$ is decreasing and convex, $\overline{U}'(x)=-f(x)$, and $f(x)$ is decreasing). By concavity,

$$U(f(x))-U(B)\geq x\,(f(x)-B)\,,$$

it then follows from (22) and (A.2) that, for any terminal wealth $B\in\mathfrak{B}(\mathfrak{P},A)$

$$\begin{aligned} V(B^*)-V(B) &= E\left[U(B^*)-U(B)\right] \\ &= E\left[U\left(f\left(\psi^*\pi_{v^*}(T)\right)\right)-U(B)\right] \\ &\geq 0. \end{aligned}$$

Hence, B^* must be optimal provided that it is A-feasible.

Step 2. By the continuity of f and π_{v^*}, it is clear $0<B^*<\infty$. Also, from the inequality

$$U(1)-z\leq\max_{B\geq 0}\left[U(B)-zB\right]=U(f(z))-zf(z)\,,$$

we have

$$E\left[U\left(B^{*}\right)^{-}\right] \leq U\left(1\right)^{-} + \psi^{*} E\left[\pi_{v^{*}}\left(T\right)\right] < \infty.$$

Therefore, we are only left to show that there exists an admissible trading strategy (α, θ) satisfying (15)-(17) and $(\alpha\left(t\right), \theta\left(t\right)) \in A$ for $\forall t$.

Define the wealth process W by

$$\begin{aligned} W\left(t\right) &= \pi_{v^{*}}\left(t\right)^{-1} E\left[\pi_{v^{*}}\left(T\right) B^{*} - \int_{t}^{T} \pi_{v^{*}}\left(\tau\right)\left(d\left(\tau\right) + \delta\left(v^{*}\left(\tau\right)\right)\right) d\tau \mid \mathfrak{F}_{t}\right] \\ &= \beta_{v^{*}}\left(t\right)^{-1} E^{Q_{v^{*}}}\left[\beta_{v^{*}}\left(T\right) B^{*} - \int_{t}^{T} \beta_{v^{*}}\left(\tau\right)\left(d\left(\tau\right) + \delta\left(v^{*}\left(\tau\right)\right)\right) d\tau \mid \mathfrak{F}_{t}\right] \end{aligned}$$

(the expectation is finite because of (14) and (29)). Clearly, $W(T) = B^{*}$, and W is bounded below (because of (14) and Assumption 3), so that (16) and (17) are satisfied. Also, since $W(0) = d_0$ (because of (A.2)), it follows from the martingale representation theorem that there exists a process ψ with $\int_0^T \left|\psi\left(t\right)\right|^2 dt < \infty$ *a.s.* such that

$$\beta_{v^{*}}\left(t\right) W\left(t\right) - \int_{0}^{t} \beta_{v^{*}}\left(\tau\right)\left(d\left(\tau\right) + \delta\left(v^{*}\left(\tau\right)\right)\right) d\tau = d_0 + \int_{0}^{t} \psi\left(\tau\right)^{T} dw_{v^{*}}\left(\tau\right), \tag{A.3}$$

where $w_{v^{*}} = w\left(t\right) - \int_0^t k_{v^{*}}\left(\tau\right) d\tau$ is a standard Brownian motion under $Q_{v^{*}}$.

Define the trading strategy $(\alpha, \theta) \in \Theta$ by

$$\theta\left(t\right)^{T} = \beta_{v^{*}}\left(t\right)^{-1} \psi\left(t\right)^{T} \sigma\left(t\right)^{-1}, \tag{A.4}$$

and

$$\alpha\left(t\right) = W\left(t\right) - \sum_{k=1}^{n} \theta_k\left(t\right).$$

Using (A.3) and Itô's lemma shows that

$$\begin{aligned} W\left(t\right) &= d_0 + \int_{0}^{t} \left(r\left(\tau\right) + v_0^{*}\left(\tau\right)\right) W\left(\tau\right) d\tau + \int_{0}^{t} \theta\left(\tau\right)^{T} \sigma\left(\tau\right) dw_{v^{*}}\left(\tau\right) \\ &\quad + \int_{0}^{t} \left(d\left(\tau\right) + \delta\left(v^{*}\left(\tau\right)\right)\right) d\tau \\ &= d_0 + \int_{0}^{t} \Big(\alpha\left(\tau\right)\left(r\left(\tau\right) + v_0^{*}\left(\tau\right)\right) + \theta\left(\tau\right)^{T}\left(\mu\left(\tau\right) + v_{-}^{*}\left(\tau\right)\right)\Big) d\tau \\ &\quad + \int_{0}^{t} \theta\left(\tau\right)^{T} \sigma\left(\tau\right) dw\left(\tau\right) + \int_{0}^{t} \left(d\left(\tau\right) + \delta\left(v^{*}\left(\tau\right)\right)\right) d\tau. \end{aligned}$$

A comparison with (15) then reveals that in order to prove that $B^* \in \mathfrak{B}(\mathfrak{P}, A)$ we are only left to verify that

$$(\alpha(t,\omega), \theta(t,\omega)) \in A, \quad (\lambda \times P)\text{-}a.e., \tag{A.5}$$

and that

$$\alpha(t,\omega) v_0^*(t,\omega) + \theta(t,\omega)^T v_-^*(t,\omega) + \delta(v^*(t,\omega)) = 0, \quad (\lambda \times P)\text{-}a.e.. \tag{A.6}$$

Fix an arbitrary $v \in \mathfrak{N}$ and define the process

$$\begin{aligned}\varsigma(t) &= \int_0^t (v_0^*(\tau) - v_0(\tau))\, d\tau \\ &\quad + \int_0^t \left(\sigma(\tau)^{-1}\left(v_-^*(\tau) - v_-(\tau) - (v_0^*(\tau) - v_0(\tau))\bar{1}\right)\right)^T dw_{v^*}(\tau)\end{aligned}$$

as well as the sequence of stopping times

$$\begin{aligned}\tau_n &= T \wedge \inf\{t \in [0,T] : |\varsigma(t)| + |\pi_{v^*}(t)| + |W(t)| \geq n, \\ &\quad \text{or } \int_0^t \left|\theta(\tau)^T \sigma(\tau)\right|^2 d\tau \geq n, \\ &\quad \text{or } \int_0^t |v_0^*(\tau) - v_0(\tau)|^2 d\tau \geq n, \\ &\quad \text{or } \int_0^t \left|\sigma(\tau)^{-1}\left(v_-^*(\tau) - v_-(\tau) - (v_0^*(\tau) - v_0(\tau))\bar{1}\right)\right|^2 d\tau \geq n\}.\end{aligned}$$

Then $\tau_n \uparrow T$ *a.s.*. Also, letting

$$v_{\varepsilon,n}(t) = v^*(t) + \varepsilon\,[v(t) - v^*(t)]\, 1_{\{t \leq \tau_n\}},$$

for $\varepsilon \in (0,1)$, we have $v_{\varepsilon,n} \in \mathfrak{N}$ (because of the convexity of $\overline{A}$) and

$$\begin{aligned}\pi_{v_{\varepsilon,n}}(t) &= \pi_{v^*}(t)\exp\Bigg(\varepsilon\varsigma(t \wedge \tau_n) \\ &\quad - \frac{\varepsilon^2}{2}\int_0^{t\wedge\tau_n} \left|\sigma(\tau)^{-1}\left(v_-^*(\tau) - v_-(\tau) - (v_0^*(\tau) - v_0(\tau))\bar{1}\right)\right|^2 d\tau\Bigg).\end{aligned}$$

It then follows from the definition of the stopping times τ_n that

$$e^{-2\varepsilon n}\pi_{v^*}(t) \leq \pi_{v_{\varepsilon,n}}(t) \leq e^{2\varepsilon n}\pi_{v^*}(t), \tag{A.7}$$

and

$$e^{-3\varepsilon n}\xi_{v^*}(t) \le \xi_{v_{\varepsilon,n}}(t) \le e^{3\varepsilon n}\xi_{v^*}(t).$$

Therefore, $\xi_{v_{\varepsilon,n}}$ is of class D, and hence $v_{\varepsilon,n} \in \mathfrak{N}^*$ (Jacord and Shiryaev, 1987, Proposition I.1.47).

We will show below that for any $v \in \mathfrak{N}^*$ we have

$$\begin{aligned} &\lim_{\varepsilon \to 0} \frac{J(\psi^*, v^*) - J(\psi^*, v_{\varepsilon,n})}{\varepsilon} \\ \ge\ &\psi^* E\Bigg[\int_0^{\tau_n} \pi_{v^*}(t)\left(\alpha(t)\left(v_0^*(t) - v_0(t)\right) + \right. \\ &\left. \theta(t)^T\left(v_-^*(t) - v_-(t)\right) + \delta(v^*(t)) - \delta(v(t))\right)dt\Bigg]. \end{aligned} \tag{A.8}$$

Since $J(\psi^*, v_{\varepsilon,n})$ reaches a minimum at $\varepsilon = 0$, the left-hand side of (A.8) is nonpositive, and thus so is the right-hand side.

Taking $v = v^* + \rho, \rho \in \mathfrak{N}$, it follows from the fact that $\overline{A}$ is a convex cone that $v \in \mathfrak{N}$, and hence (A.8) gives

$$E\left[\int_0^{\tau_n} \pi_{v^*}(t)\Big(\alpha(t)\rho_0(t) + \theta(\tau)^T\rho_-(t) + \delta(\rho(t))\Big)dt\right] \ge 0,$$

where we have used the subadditivity of δ. Since $\rho \in \mathfrak{N}$ was arbitrary, this implies the existence of a set E having full measure $(\lambda \times P)$ such that

$$\alpha(t,\omega)v_0(t,\omega) + \theta(t,\omega)^T v_-(t,\omega) + \delta(v(t,\omega)) \ge 0, \quad \forall (t,\omega) \in E,\ v \in \overline{A}.$$

By Theorem 14.1 in Rockafellar (1970), the above implies (A.5).

On the other hand, for $v \equiv 0$, (A.8) gives

$$E\left[\int_0^{\tau_n} \pi_{v^*}(t)\left(\alpha(t)v_0^*(t) + \theta(t)^T v_-^*(t) + \delta(v^*(t))\right)dt\right] \le 0,$$

and it then follows, using the fact that $\alpha(t)v_0^*(t) + \theta(t)^T v_-^*(t) + \delta(v^*(t)) \ge 0$ for $(\alpha(t,\omega), \theta(t,\omega)) \in A$ and $v^*(t) \in \overline{A}$, that (A.6) also holds. To show the inequality in (A.8), we start by observing that

$$\left|\frac{\overline{U}\left(\psi^*\pi_{v^*}(T)\right)-\overline{U}\left(\psi^*\pi_{v_{\varepsilon,n}}(T)\right)}{\varepsilon}\right.$$
$$\left.+\psi^*\int_0^T\left(d(t)+\delta\left(v^*\right)\right)\frac{\pi_{v^*}(t)-\pi_{v_{\varepsilon,n}}(t)}{\varepsilon}dt\right|$$
$$\leq\psi^*\left[\frac{f\left(\psi^*e^{-2n\varepsilon}\pi_{v^*}(T)\right)\left|\pi_{v_{\varepsilon,n}}(T)-\pi_{v^*}(T)\right|}{\varepsilon}\right.$$
$$\left.+\int_0^T\left(d(t)+\delta\left(v^*\right)\right)\frac{\left|\pi_{v^*}(t)-\pi_{v_{\varepsilon,n}}(t)\right|}{\varepsilon}dt\right]$$
$$\leq\psi^*\left[K_n\pi_{v^*}(T)\,f\left(\psi^*e^{-2n\varepsilon}\pi_{v^*}(T)\right)+K_n\int_0^T\pi_{v^*}(t)\left(d(t)+\delta\left(v^*(t)\right)\right)dt\right],$$

where

$$K_n=\sup_{\varepsilon\in(0,1)}\frac{e^{2n\varepsilon}-1}{\varepsilon}<\infty,$$

and that

$$\pi_{v_{\varepsilon,n}}(t)\left(\delta\left(v^*(t)\right)-\delta\left(v(t)\right)\right)^-\leq-e^{2n\varepsilon}\pi_{v^*}(t)\,\delta\left(v^*(t)\right).$$

It then follows from (29), (31), Lebesgue's dominated convergence theorem, the convexity of δ, and Fatou's lemma that

$$\lim_{\varepsilon\downarrow 0}\frac{J\left(\psi^*,v^*\right)-J\left(\psi^*,v_{\varepsilon,n}\right)}{\varepsilon}$$
$$=\lim_{\varepsilon\downarrow 0}E[\frac{\overline{U}\left(\psi^*\pi_{v^*}(T)\right)-\overline{U}\left(\psi^*\pi_{v_{\varepsilon,n}}(T)\right)}{\varepsilon}$$
$$+\psi^*\int_0^T\left(d(t)+\delta\left(v^*\right)\right)\frac{\pi_{v^*}(t)-\pi_{v_{\varepsilon,n}}(t)}{\varepsilon}dt$$
$$+\psi^*\int_0^T\pi_{v_{\varepsilon,n}}(t)\frac{\delta\left(v^*(t)\right)-\delta\left(v_{\varepsilon,n}(t)\right)}{\varepsilon}dt$$
$$\geq E\left[\lim_{\varepsilon\downarrow 0}\frac{\overline{U}\left(\psi^*\pi_{v^*}(T)\right)-\overline{U}\left(\psi^*\pi_{v_{\varepsilon,n}}(T)\right)}{\varepsilon}\right]$$
$$+\psi^*E\left[\int_0^T\lim_{\varepsilon\downarrow 0}\left[\left(d(t)+\delta\left(v^*\right)\right)\frac{\pi_{v^*}(t)-\pi_{v_{\varepsilon,n}}(t)}{\varepsilon}\right]dt\right]$$

$$
\begin{aligned}
&+\psi^{*}\lim_{\varepsilon\downarrow 0}E\left[\int_{0}^{T}\pi_{v_{\varepsilon,n}}(t)\,\frac{\delta\left(v^{*}(t)\right)-\delta\left(v_{\varepsilon,n}(t)\right)}{\varepsilon}dt\right]\\
=&E\left[\psi^{*}\pi_{v^{*}}(T)\,B^{*}\varsigma(\tau_{n})-\psi^{*}\int_{0}^{T}\pi_{v^{*}}(t)\,\varsigma(t\wedge\tau_{n})\left(d(t)+\delta\left(v^{*}(t)\right)\right)dt\right]\\
&+\psi^{*}\lim_{\varepsilon\downarrow 0}E\left[\int_{0}^{\tau_{n}}\pi_{v_{\varepsilon,n}}(t)\,\delta\left(v^{*}(t)\right)-\delta\left(v(t)\right)dt\right]\\
\geq&\psi^{*}E\left[\pi_{v^{*}}(\tau_{n})\,W(\tau_{n})\,\varsigma(\tau_{n})-\int_{0}^{\tau_{n}}\pi_{v^{*}}(t)\,\varsigma(t\wedge\tau_{n})\left(d(t)+\delta\left(v^{*}(t)\right)\right)dt\right]\\
&+\psi^{*}E\left[\int_{0}^{\tau_{n}}\pi_{v^{*}}(t)\,\delta\left(v^{*}(t)\right)-\delta\left(v(t)\right)dt\right].
\end{aligned}
\tag{A.9}
$$

On the other hand, using (A.3) and (A.4), Itô's lemma shows that

$$
\begin{aligned}
&\beta_{v^{*}}(\tau_{n})\,W(\tau_{n})\,\varsigma(\tau_{n})-\int_{0}^{\tau_{n}}\beta_{v^{*}}(t)\left(d(t)+\delta\left(v^{*}(t)\right)\right)dt\\
=&\int_{0}^{\tau_{n}}\beta_{v^{*}}(t)\Big(W(t)\sigma(t)^{-1}\left(v_{-}^{*}(t)-v_{-}(t)-\left(v_{0}^{*}(t)-v_{0}(t)\right)\bar{1}\right)\\
&+\varsigma(t)\,\theta(t)^{T}\sigma(t)\Big)\,dw_{v^{*}}(t)\\
&+\int_{0}^{\tau_{n}}\beta_{v^{*}}(t)\left(\alpha(t)\left(v_{0}^{*}(t)-v_{0}(t)\right)+\theta(t)^{T}\left(v_{-}^{*}(t)-v_{-}(t)\right)\right)dt,
\end{aligned}
$$

Since the stochastic integral in the above expression is a $Q_{v^{*}}$-martingale, we have

$$
\begin{aligned}
&E\left[\pi_{v^{*}}(\tau_{n})\,W(\tau_{n})\,\varsigma(\tau_{n})-\int_{0}^{\tau_{n}}\pi_{v^{*}}(t)\left(d(t)+\delta\left(v^{*}(t)\right)\right)dt\right]\\
=\;&E\left[\int_{0}^{\tau_{n}}\pi_{v^{*}}(t)\left(\alpha(t)\left(v_{0}^{*}(t)-v_{0}(t)\right)+\theta(t)^{T}\left(v_{-}^{*}(t)-v_{-}(t)\right)\right)dt\right].
\end{aligned}
$$

Substituting the above expression in (A.9) yields (A.8). This shows that $B^{*}\in\mathfrak{B}(\mathfrak{P},A)$ and hence that B^{*} is optimal.

To prove the converse, suppose that (30) and (31) are satisfied by some $B^* \in \mathfrak{B}(\mathfrak{P}, A)$. By the argument above, we have for all $(\psi, v) \in (0, \infty) \times \mathfrak{N}^*$

$$\begin{aligned} J(\psi, v) &= E\left[\overline{U}(\psi\pi_v(T)) + \psi\left(d_0 + \int_0^T \pi_v(t)(d(t) + \delta(v(t)))\,dt\right)\right] \\ &\geq E\left[\overline{U}(\psi^*\pi_{v^*}(T)) + B^*\psi^*\pi_{v^*}(T)\right. \\ &+ \left.\psi\left(d_0 + \int_0^T \pi_v(t)(d(t) + \delta(v(t)))\,dt\right) - \pi_v(T)B^*\right] \\ &\geq E\left[\overline{U}(\psi^*\pi_{v^*}(T)) + \psi^*\left(d_0 + \int_0^T \pi_{v^*}(t)(d(t) + \delta(v^*(t)))\,dt\right)\right] \\ &= J(\psi^*, v^*), \end{aligned}$$

where the last inequality follows from (22) and (31). This shows that (ψ^*, v^*) solves $(\mathbf{P}^*)$. ▮

APPENDIX B

This appendix is devoted to the proof of Theorem 6.1. The proof is based on the following result, due to Levin (1976) (cf. also Fougeres(1979)).

LEMMA B.1. *Let $F : L^1(S, \Sigma, \mu) \to R \cup \{+\infty\}$ be a convex functional, where (S, Σ, μ) is a measure space with μ finite and nonnegative and Σ complete. If F is lower semicontinuous in the topology τ of convergence in measure, then it attains a minimum on any convex set $K \subset L^1(S, \Sigma, \mu)$ that is τ-closed and norm-bounded.*

Proof. See Levin (1976). ▮

LEMMA B.2. *Let K denote the set of terminal wealth satisfying* (22)

$$K = \left\{B : B \geq 0, E^{Q_v}[\beta_v(T)B] \leq d_0 + E^{Q_v}\left[\int_0^T \beta_v(t)(\delta(v(t)) + d(t))\,dt\right],\ \forall v \in \mathfrak{N}^*\right\}.$$

Then K is a norm-bounded subset of $L^1(Q_0)$.

Proof. $\forall B \in K$, since $0 \in \mathfrak{N}^*$, $\delta \leq 0$ and $r \leq r_B$ by Assumption 1, we have

$$E^{Q_0}[\beta_0(T)B] \leq d_0 + E^{Q_0}\left[\int_0^T \beta_0(t)(\delta(v(t)) + d(t))\,dt\right] \leq K_d + d_0,$$

so

$$E^{Q_0}[B] \leq E^{Q_0}\left[e^{r_B T}\beta_0(T)B\right] \leq e^{r_B T}(K_d + d_0). \quad \blacksquare$$

Lemma B.3. *Under the assumption of Theorem 6.1, K is convex and closed in the topology of convergence in Q_0-measure.*

Proof. We start by noticing that, under the assumptions of Theorem 6.1, the requirement that $\min\left(E\left(U\left(W\left(T\right)\right)\right)^+, E\left(U\left(W\left(T\right)\right)\right)^-\right) < +\infty$ is satisfied by all $B \in L^1_+(Q_0)$, since this implies $B < \infty$ *a.s.*, and

$$\begin{aligned} E\left[U\left(B\right)\right]^+ &\leq kE\left[1 + B^{1-b}\right] \\ &\leq k + kE^{Q_0}\left[\xi_0\left(T\right)^{-1}B^{1-b}\right] \\ &\leq k + k\left(E^{Q_0}\left[\xi_0\left(T\right)^{-\frac{1}{b}}\right]\right)^b\left(E^{Q_0}\left[B\right]\right)^{1-b} \\ &< \infty, \end{aligned} \tag{B.1}$$

for some $k \geq 0$, $b \in (0,1)$ (by Holder's inequality and Lemma B.2). The convexity of K is now immediately verified and the closure follows from Fatou's lemma and the fact that any sequence converging in Q_0-measure has a subsequence converging Q_0-*a.e.*. $\blacksquare$

We next record some properties of the utility functional U.

Lemma B.4. *V is concave:*

$$V\left(\alpha B_1 + (1-\alpha)B_2\right) \geq \alpha V\left(B_1\right) + (1-\alpha)V\left(B_2\right).$$

Lemma B.5. *Let $V(B) \equiv E[U(B)]$, then, under the assumptions of Theorem 6.1, V is bounded above on K and upper semicontinuous with respect to convergence in Q_0-measure: i.e., for every $\{B_n\} \subset K$ and $B \in L^1(Q_0)$ with $B_n \to B$ in measure, we have $V(B) \geq \limsup_{n\to\infty} V(B_n)$.*

Proof. The fact that V is bounded above on K follows from (A1) and the fact that K is bounded in the $L^1(Q_0)$-norm.

Next, suppose that V is not upper semicontinuous on K. Then there exists an $a \in R$, a sequence $\{B_n\} \subset K$, and a $B \in K$ such that $B_n \to B$ in measure and

$$V(B) < a \leq V(B_n), \quad \text{for all } n. \tag{B.2}$$

By passing to a subsequence, we can assume without loss of generality that $B_n \to B$ *a.e.*. We will show that, under the assumptions of Theorem 6.1, the family

$$\left\{\xi_0 (T)^{-1} U (B_n)\right\} \tag{B.3}$$

is Q_0-uniformly integrable. It then follows from Fatou's lemma that

$$\begin{aligned} V(B) &= E^{Q_0}[\xi_0(T)^{-1}U(B)] \\ &\geq \limsup_{n\to\infty} E^{Q_0}\left[\xi_0(T)^{-1}U(B_n)\right] \\ &= \limsup_{n\to\infty} V(B_n). \end{aligned}$$

This contradicts (B.2) and thus establishes the upper semicontinuity of V.

Finally, to prove our claim that the family in (B.3) is Q_0-uniformly integrable, we observe that this is immediate if U is bounded above. Otherwise, we have

$$U\left(B_n^+\right) \leq k\left(1 + B_n^{1-b}\right),$$

and it is enough to show that

$$\sup_n E^{Q_0}\left[\left(\xi_0(T)^{-1}B_n^{1-b}\right)^{\overline{p}}\right] < \infty$$

holds for some $\overline{p} > 1$. Taking $\overline{p} = \frac{p}{b+p(1-b)}$ (where $b \in (0,1)$ and $p(> 1)$ are the constants in (32) and (33)), we have from Holder's inequality

$$\begin{aligned} & E^{Q_0}\left[\xi_0(T)^{\overline{p}}B_n^{(1-b)\overline{p}}\right] \\ \leq\ & \left(E^{Q_0}\left[\xi_0(T)^{-\frac{\overline{p}}{1-\overline{p}(1-b)}}\right]\right)^{1-\overline{p}(1-b)} \left(E^{Q_0}\left[B_n\right]\right)^{\overline{p}(1-b)} \\ =\ & \left(E^{Q_0}\left[\xi_0\left(T\right)^{-\frac{p}{b}}\right]\right)^{1-\overline{p}(1-b)} \left(E^{Q_0}\left[B_n\right]\right)^{\overline{p}(1-b)} \\ <\ & \infty. \quad \blacksquare \end{aligned}$$

Proof (Theorem 6.1). Consider the map $I : L^1(Q_0) \to R \cup \{+\infty\}$ defined by $I(B) = -V(B)$, if $B \in K$; $I(B) =+\infty$, otherwise.

Since K is convex and closed with respect to convergence in measure, it follows from the concavity V and Lemma B.5 that I is convex and lower semicontinuous in measure. Also, K is nonempty and bounded in norm. Therefore, it follows from Lemma B.1 and the fact that $I(B) < \infty$ for some $B \in K$, that there exists a $B^* \in K$ such that $I(B^*) \leq I(B)$ for all $B \in L^1(Q_0)$. This implies that B^* solves (**P**). $\blacksquare$

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