# War, Capital, and the Birth of the State: A Mean Field Theory of Endogenous Political Dominance

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**Abstract**

This paper develops a dynamic stochastic model of state formation from an anarchic condition, integrating insights from Hobbes, Nozick, and Tilly within a mean field game framework. A continuum of agents, each equipped with capital and arms, optimize consumption, investment, and military spending while facing externalities from the militarization of others. Recursive military advantage allows some agents to defeat rivals, seize resources, and transition into dominant protective agencies—emergent states. We formalize war as a capital transfer mechanism and taxation as a means of fiscal consolidation. The model produces endogenous divergence in coercive and productive capacity, demonstrating how military asymmetries, recursive violence, and fiscal extraction lead to the birth and persistence of centralized authority.

Key Words: State formation, mean field games, military asymmetry, capital accumulation, war and taxation, dynamic political economy

## 1 Introduction

How does centralized political authority emerge in a world without it? From Hobbes to Tilly, this question has animated centuries of political theory, economic history, and state-building scholarship. In Leviathan (1651), Thomas Hobbes imagined a world of isolated, self-defending individuals locked in a permanent "war of every man against every man." Without a central coercive authority, security is privately produced, at great cost and with constant threat of appropriation. Robert Nozick's Anarchy, State, and Utopia (1974) offers a decentralized refinement: from a world of mutually hostile protective agencies, a dominant protector arises to become a minimal state. Despite their normative differences, both views point to the same mechanism—the state emerges as the solution to an endogenous security dilemma, born from the strategic asymmetries of coercive power.

This paper develops a rigorous dynamic theory of this process using the tools of stochastic mean field games (MFG). We model a large population of agents, each endowed with capital and arms, choosing consumption, investment, and defense in continuous time. Production is subject to external threats: more militarized neighbors depress output and reduce the marginal value of capital accumulation. The utility of each agent depends positively on consumption and self-protection, but negatively on the average and directed arms stock of others. This structure captures a Hobbesian world of strategic deterrence, insecurity, and mutual exploitation.

Unlike previous models of symmetric interaction, our framework allows for power divergence and institutional emergence. Agents begin symmetrically, but shocks to productivity, nonlinearity in military returns, and feedback effects cause heterogeneous accumulation. Some agents begin to pull ahead in both capital and coercive capacity. Section 2 presents the full model setup: state variables, production, budget constraints, and utility under external military threat. Section 3 derives the capital and military dynamics under optimal control. Section 4 introduces a stochastic regime-switching mechanism, in which agents may transition into a high-productivity state—capturing endogenous sources of divergence.

Section 5, the heart of the paper, studies state emergence. As one agent's military stock exceeds that of others by a sufficient threshold, conflict becomes unprofitable for rivals. The dominant agent effectively becomes a monopolist protective agency: it no longer threatens others, but stabilizes the system and begins to extract output through taxes. We show how this transition from mutual insecurity to unilateral protection emerges as an equilibrium outcome in the mean field game.

Section 6 develops a formal model of war as redistributive conflict, following Charles Tilly's thesis that "war made the state, and the state made war" (Coercion, Capital, and European States, AD 990–1992, 1990). At stochastic intervals, militarized agents engage in conflict. Victors capture the capital and arms of losers, expanding their future capacity to fight and accumulate. This recursive war cycle leads to increasing inequality in capital and coercive strength, ultimately consolidating power in the hands of a few. We show how this process, repeated over time, naturally leads to institutional dominance and the emergence of a coercive hierarchy—what we recognize as the state.

Mathematically, the model comprises coupled Hamilton–Jacobi–Bellman (HJB) and Fokker–Planck–Kolmogorov (FPK) equations with recursive feedback. Agents solve optimal control problems under risk, threat, and resource constraints, while the population distribution evolves endogenously. The equilibrium consists of policy functions, state transitions, and distributional dynamics consistent with rational expectations.

Section 7 introduces taxation by the dominant agent: once power is consolidated, the state extracts fiscal resources from others and reinvests them in capital and arms. This fiscal expansion loop deepens asymmetry and institutionalizes protection. Section 8 models the war cycles that underlie long-run state formation, integrating redistribution, coercion, and threat elimination into a dynamic consolidation process.

Section 9 concludes with historical and theoretical extensions: fragmentation and collapse, endogenous legitimacy, and polycentric military equilibria. Together, the paper formalizes a fundamental insight: the state is not imposed, but emerges endogenously from insecurity, divergence, and recursive consolidation. It is the natural attractor of a decentralized game of survival, accumulation, and coercion.

# 2 Model Setup

We consider a continuum of agents indexed by $i \in [0, 1]$, each evolving over continuous time $t \geq 0$. Each agent is characterized by two key state variables: capital stock $k_i(t)$, which governs productive capabilities, and military stock $m_i(t)$, which confers self-defense and deterrence power but also poses external threats to others. The setting is a formalization of the Hobbesian state of nature, in which no centralized state exists initially. Each individual agent is both a producer and a potential aggressor, fending off or inflicting harm on others in pursuit of self-preservation and accumulation.

All agents are born symmetric but evolve heterogeneously due to stochastic shocks and recursive feedback loops in capital and military accumulation. Over time, some agents will amass significant advantages, potentially becoming monopolistic protective agencies. This endogenous divergence gives rise to the state as an emergent equilibrium object.

## 2.1 Budget Constraint and Production Function

At every instant, agent i chooses consumption $c_i(t)$, capital investment $i_{k,i}(t)$, and military spending $g_i(t)$. The agent's budget constraint is given by:

$$c_i(t) + i_{k,i}(t) + g_i(t) = A_i(t) \cdot k_i(t) + B_i(t) \cdot m_i(t) - D \cdot \bar{m}(t) - E \cdot \tilde{m}_i(t),$$

where:

- $A_i(t) > 0$: the capital productivity of agent i, possibly stochastic or subject to regime switches;
- $B_i(t) > 0$: the military productivity of agent i, capturing how effectively they extract resources from or defend themselves through arms;
- $\bar{m}(t) = \int_0^1 m_j(t)\, dj$: the average militarization across all agents—interpreted as a generalized threat environment;
- $\tilde{m}_i(t) = \int_0^1 m_j(t) \cdot \phi_{ij}(t)\, dj$: the agent-specific, weighted threat level from others, with $\phi_{ij}(t) \in [0, 1]$ denoting the bilateral proximity of coercive influence (geographic, cultural, or political).

This production function implies that output is augmented by both capital and militarization, but also diminished by the hostile arming behavior of the population. The term $B_i(t)m_i(t)$ captures the private productive or extractive value of militarization for agent i, whereas the negative terms $D\bar{m}(t)$ and $E\tilde{m}_i(t)$ reflect generalized and directed threats from others.

## 2.2 Stochastic Dynamics of Capital and Military Stocks

Agents' state variables evolve via the following Itô diffusion processes:

$$dk_i(t) = \big(i_{k,i}(t) - \delta^k k_i(t)\big)\, dt + \sigma_k \, dW_i^k(t),$$
$$dm_i(t) = \left(g_i(t) - \delta^m m_i(t)\right) dt + \sigma_m \, dW_i^m(t),$$

where:
- $\delta^k, \delta^m > 0$: depreciation rates of capital and military stock;
- $\sigma_k, \sigma_m > 0$: standard deviations of idiosyncratic shocks;
- $W_i^k(t), W_i^m(t)$: independent Brownian motions.

These dynamics capture both the intentional investment behavior of agents and the random disturbances that drive heterogeneity over time. The stochastic accumulation of coercive capacity will be central in determining the divergence in military advantage.

## 2.3 Preferences and Threat-Adjusted Utility

Each agent aims to maximize lifetime utility, discounted at rate \rho, with the following instantaneous flow utility function:

$$u_i(t) = a\, c_i(t) - \frac{1}{2} b\, c_i^2(t) + \eta\, m_i(t) - \frac{1}{2}\phi\, m_i^2(t) - \theta_1\, \bar{m}(t) - \theta_2\, \tilde{m}_i(t),$$

leading to the optimization problem:

$$\sup_{\{c_i, i_{k,i}, g_i\}} \mathbb{E}\left[\int_0^\infty e^{-\rho t} u_i(t) dt\right].$$

- The quadratic structure captures decreasing marginal utility from consumption and increasing costs of militarization.
- The parameters $\theta_1, \theta_2 > 0$ encode how militarization by others directly diminishes the well-being of agent $i$, either globally ($\bar{m}(t)$) or selectively ($\tilde{m}_i(t)$).

This utility function integrates both Nozickian ideas (value of arms for self-protection) and Hobbesian dynamics (disutility from generalized fear of others).

## 2.4 Endogenous Divergence and State Emergence

To model the endogenous formation of dominant protective agencies, we assume that capital productivity $A_i(t)$ and military productivity $B_i(t)$ follow independent Markov regime-switching processes:

$$A_i(t) \in \{A^L, A^H\}, \quad B_i(t) \in \{B^L, B^H\}, \quad \text{with upward transition rates } \lambda_A, \lambda_B > 0.$$

These stochastic transitions represent institutional innovation, discovery of efficient weapons technologies, or strategic learning. An agent who simultaneously enters high-capital and high-military productivity regimes gains a compounding advantage in both economic and coercive terms. If unchecked, such an agent will begin to overshadow others in both dimensions, becoming a de facto monopolist in violence and taxation—a political Leviathan.

Once a critical mass of advantage is achieved, further coercion is unnecessary: other agents recognize the dominant position and begin to treat the leading agent as a legitimate protector. Thus, the "state" emerges not by social contract or external imposition, but through the nonlinear dynamics of decentralized accumulation and threat management.

### 2.5 Regime Switching and State Emergence

To endogenize the emergence of the state, we introduce rare but persistent transitions in capital productivity. Each agent's productivity process $A_i(t)$ follows a two-state Markov process:

$$A_i(t) \in \{A^L, A^H\}, \quad \mathbb{P}(A_i(t+dt) = A^H \mid A_i(t) = A^L) = \lambda \, dt,$$

where $A^L < A^H$ and $\lambda > 0$ is a small rate. This structure captures exogenous discoveries, conquest, technological gains, or political consolidation that allow an agent to enter a high-growth regime.

Agents who transition into the high-productivity state accumulate capital more rapidly and—when combined with sustained investment in arms—can attain disproportionate levels of both capital and military resources. Once a threshold of relative dominance is crossed, the strategic behavior of others changes: the superior agent is no longer viewed as a threat but as a de facto provider of protection. We interpret this transition as the endogenous birth of a monopolistic protective agency—the state.

At this point, others rationally reduce their own arms spending, recognizing the futility of rebellion and the stabilizing effect of the dominant power. A new regime emerges, not from collective agreement, but from strategic divergence in resource accumulation under anarchy. This completes the endogenous pathway from Hobbesian war to Nozickian minimal statehood.

## 3 Optimization and Distributional Dynamics

In this section, we characterize the optimal behavior of each agent given his individual capital and military stock levels, stochastic productivity parameters $A_i(t)$ and $B_i(t)$, and exposure to militarization from the broader population. Each agent maximizes his expected lifetime utility subject to the intertemporal budget constraint and capital accumulation dynamics. The solution requires solving a dynamic stochastic control problem, yielding a Hamilton–Jacobi–Bellman (HJB) equation for the value function and a corresponding Fokker–Planck–Kolmogorov (FPK) equation for the evolution of the distribution of agents' states.

### 3.1 Agent Optimization Problem

Each agent $i$ solves:

$$V_i(t,k,m) = \sup_{\{i_{k,i}(t), g_i(t)\}} \mathbb{E}\left[\int_t^\infty e^{-\rho(s-t)} u_i(s)\, ds\right],$$

subject to:

$$\begin{aligned}
c_i(t) &= A_i(t)k_i(t) + B_i(t)m_i(t) - i_{k,i}(t) - g_i(t) - D\bar{m}(t) - E\tilde{m}_i(t), \\
dk_i(t) &= \big(i_{k,i}(t) - \delta^k k_i(t)\big)\, dt + \sigma_k\, dW_i^k(t), \\
dm_i(t) &= (g_i(t) - \delta^m m_i(t))\, dt + \sigma_m\, dW_i^m(t),
\end{aligned}$$

and instantaneous utility:

$$u_i(t) = ac_i(t) - \frac{1}{2}bc_i^2(t) + \eta m_i(t) - \frac{1}{2}\phi m_i^2(t) - \theta_1 \bar{m}(t) - \theta_2 \tilde{m}_i(t).$$

Substituting the expression for $c_i(t)$ into the utility function, the agent's instantaneous utility becomes:

$$\begin{aligned}
u_i = a\left(A_i k_i + B_i m_i - i_{k,i} - g_i - D\bar{m} - E\tilde{m}_i\right) &- \frac{b}{2}\left(A_i k_i + B_i m_i - i_{k,i} - g_i - D\bar{m} - E\tilde{m}_i\right)^2 \\
+ \eta m_i - \frac{1}{2}\phi m_i^2 - \theta_1 \bar{m} - \theta_2 \tilde{m}_i.&
\end{aligned}$$

Define net income $Y_i(t) := A_i(t)k_i(t) + B_i(t)m_i(t) - D\bar{m}(t) - E\tilde{m}_i(t)$. Then:

$$c_i(t) = Y_i(t) - i_{k,i}(t) - g_i(t).$$

This formulation enables us to write the Hamiltonian for the control problem in terms of $i_k$ and $g$, denoting $V_k = \partial_k V_i, V_m = \partial_m V_i$:

$$\begin{aligned}
\mathcal{H}_i(t,k,m; i_k, g; V_k, V_m) = a(Y_i - i_k - g) - \frac{b}{2}(Y_i - i_k - g)^2 + \eta m - \frac{1}{2}\phi m^2 \\
+ V_k(i_k - \delta^k k) + V_m(g - \delta^m m).
\end{aligned}$$

The agent chooses controls $i_k$, $g$ to maximize $\mathcal{H}_i$, taking all other quantities as given.

### 3.2 First-Order Conditions

The first-order conditions for interior optima are:

$$\begin{aligned}
\frac{\partial \mathcal{H}_i}{\partial i_k} &= -a + b(Y_i - i_k - g) + V_k = 0, \\
\frac{\partial \mathcal{H}_i}{\partial g} &= -a + b(Y_i - i_k - g) + V_m = 0.
\end{aligned}$$

Hence, we obtain the consistency condition:

$$V_k = V_m,$$

and summing the equations:

$$i_k + g = Y_i - \frac{1}{b}(a - V_k).$$

This equation defines the total investment demand $i_k + g$ as a function of shadow value $V_k$, agent income $Y_i$, and preference parameters.

### 3.3 HJB Equation

The HJB equation for the value function $V_i(t, k, m)$ becomes:

$$\begin{aligned}\rho V_i(t,k,m) = \sup_{i_k, g} \Bigg\{ & a(Y_i - i_k - g) - \frac{b}{2}(Y_i - i_k - g)^2 + \eta m - \frac{1}{2}\phi m^2 \\ & + V_k(i_k - \delta^k k) + V_m(g - \delta^m m) + \frac{1}{2}\sigma_k^2 V_{kk} + \frac{1}{2}\sigma_m^2 V_{mm} \Bigg\}.\end{aligned}$$

Given the quadratic nature of the objective and linearity of dynamics, we postulate that the value function admits a quadratic form:

$$V_i(t,k,m) = \alpha_0(t) + \alpha_1(t)k + \alpha_2(t)m + \frac{1}{2}\beta_1(t)k^2 + \beta_2(t)km + \frac{1}{2}\beta_3(t)m^2.$$

Substituting this ansatz into the HJB equation, we derive a system of Riccati-type ODEs governing $\alpha_j(t), \beta_j(t)$, which will be solved backward in time (or in the stationary case, solved algebraically). The inclusion of $B_i(t)$ introduces heterogeneity across agents in their income sensitivity to military stock, which alters their optimal arms investment behavior and affects their likelihood of becoming dominant actors.

### 3.4 Fokker–Planck Dynamics

Let $p(t, k, m)$ be the density of agents across states $(k, m)$. Its evolution satisfies the Fokker–Planck–Kolmogorov (FPK) equation:

$$\frac{\partial p}{\partial t} + \frac{\partial}{\partial k}(\mu_k p) + \frac{\partial}{\partial m}(\mu_m p) = \frac{\sigma_k^2}{2}\frac{\partial^2 p}{\partial k^2} + \frac{\sigma_m^2}{2}\frac{\partial^2 p}{\partial m^2}.$$

where:

$$\begin{aligned}\mu_k &= i_k(t,k,m) - \delta^k k \\ \mu_m &= g(t,k,m) - \delta^m m\end{aligned}$$

are the optimally chosen drift terms derived from the control solution above.

In equilibrium, the aggregate militarization $\bar{m}(t) = \int mp(t,k,m)dk\,dm$ and the personalized threat exposure $\tilde{m}_i(t)$ must be consistent with the population distribution. Thus, the solution requires finding a fixed point where:

$$\bar{m}(t) = \int m \cdot p(t,k,m)\,dk\,dm,$$

and the policy functions $i_k(t,k,m), g(t,k,m)$ solve the HJB equation given the distribution $p(t,k,m)$, which itself evolves under these same policies.

This interdependence between individual behavior and aggregate distribution lies at the heart of the mean field game. The heterogeneity introduced by $B_i(t)$ implies that some agents will more rapidly accumulate military stocks, and—depending on the dynamics introduced in later sections—may evolve into dominant protective actors or proto-states.

# 4 Mean Field Equilibrium and Consistency Conditions

In this section, we formally define the notion of equilibrium in the decentralized system of agents described above. Each agent solves a dynamic optimization problem, taking as given the time paths of the average militarization $\bar{m}(t)$ and their personalized threat exposure $\tilde{m}_i(t)$. However, since all agents are drawn from a continuum, these aggregate objects are themselves determined endogenously by the distribution of individual states. A Mean Field Equilibrium (MFE) arises when these individual decisions and aggregate quantities are mutually consistent.

## 4.1 Individual Optimality

From the previous section, we have shown that each agent i solves a stochastic control problem that results in:

1. A quadratic value function $V_i(t,k,m)$ capturing expected discounted utility.

2. Linear feedback controls $i_{k,i}(t,k,m)$ and $g_i(t,k,m)$, derived from the first-order conditions of the HJB.

3. Optimal consumption given by:

$$c_i(t) = A_i(t)k_i(t) + B_i(t)m_i(t) - D\bar{m}(t) - E\tilde{m}_i(t) - i_{k,i}(t) - g_i(t),$$

which ensures intertemporal budget feasibility.

These controls are optimal given the expectations about $\bar{m}(t)$ and $\tilde{m}_i(t)$, and are influenced by both private capital and military stocks and the environmental threat.

## 4.2 Evolution of State Distributions

Let $p(t,k,m)$ denote the joint density of capital and military stock in the agent population at time $t$. Its evolution satisfies the Fokker–Planck–Kolmogorov (FPK) equation:

$$\frac{\partial p}{\partial t} = -\frac{\partial}{\partial k}[(i_k(t,k,m)-\delta^k k)p] - \frac{\partial}{\partial m}[(g(t,k,m)-\delta^m m)p] + \frac{1}{2}\sigma_k^2\frac{\partial^2 p}{\partial k^2} + \frac{1}{2}\sigma_m^2\frac{\partial^2 p}{\partial m^2},$$

where $i_k$ and $g$ are the optimal feedback policies derived from the HJB equation, which are functions of current states $(k,m)$ and time $t$. This equation governs the probability flow over the population. The distribution $p(t,k,m)$ evolves based on how agents invest in capital and arms and how stochastic shocks perturb their state trajectories.

## 4.3 Aggregation and Mean Field Quantities

The mean field quantities of interest are defined as:

$$\bar{m}(t) := \int m \cdot p(t,k,m)\, dk\, dm, \quad \bar{k}(t) := \int k \cdot p(t,k,m)\, dk\, dm.$$

Each agent's threat exposure from others is decomposed into:
- A global externality $\bar{m}(t)$, common to all agents.
- A personalized threat exposure:

$$\tilde{m}_i(t) := \int m_j(t) \cdot \phi_{ij}(t)\, dj,$$

where $\phi_{ij}(t)$ captures how close agent $j$'s arms are to agent $i$ (spatially or strategically).

In equilibrium, these objects must be consistent with the time-evolving density $p(t,k,m)$. Hence, we require a fixed point condition: the mean field $\bar{m}(t)$ and personalized threats $\tilde{m}_i(t)$ used in individual optimization must coincide with those generated by the resulting optimal policy profile and state distribution.

## 4.4 Definition of Mean Field Equilibrium

A Mean Field Equilibrium (MFE) in this model is defined by a tuple:

$$\{V_i(t,k,m),\, i_{k,i}(t,k,m),\, g_i(t,k,m),\, p(t,k,m)\}$$

such that:

1. $V_i(t,k,m)$ satisfies the Hamilton–Jacobi–Bellman (HJB) equation given $\bar{m}(t), \tilde{m}_i(t)$.
2. $i_k(t,k,m), g(t,k,m)$ solve the agent's optimization problem.
3. $p(t,k,m)$ solves the Fokker–Planck (FPK) equation given the policy feedbacks.

4. The aggregate variables $\bar{m}(t), \bar{k}(t)$, and $\tilde{m}_i(t)$ are self-consistent:

$$\bar{m}(t) = \int m \cdot p(t,k,m)\, dk\, dm, \quad \tilde{m}_i(t) = \int m \cdot \phi_{ij}(t) \cdot p(t,k,m)\, dk\, dm.$$

This system defines a rational expectations equilibrium: agents' beliefs about the environment are confirmed by the realized aggregate dynamics.

### 4.5 Existence and Uniqueness

In our linear-quadratic-Gaussian (LQG) framework, existence and uniqueness of MFE follow from standard fixed-point arguments:

• The feedback maps $(i_k, g)$ are continuous and Lipschitz in $(k, m)$, given the value function is quadratic.

• The FPK equation generates a unique density flow $p(t,k,m)$ under well-posed initial conditions.

• The aggregation operators (integrals) are continuous in $p$, ensuring the map $p \mapsto \bar{m}(t)$ is continuous.

Under these conditions, Schauder's fixed-point theorem guarantees the existence of a fixed point $(\bar{m}(t), \tilde{m}_i(t))$, and hence an MFE. Uniqueness follows if the marginal utility of arming is concave (via penalties $\phi, \theta_1, \theta_2$) and the feedback policies are monotonic.

## 5 State Emergence and Divergence in the Distribution of Power

The equilibrium structure developed in Section 4 reveals a decentralized world of self-protective agents facing threat externalities. However, when embedded in a stochastic dynamic framework with agent heterogeneity and multiplicative reinforcement in capital and arms, this setting generates endogenous asymmetries over time. In this section, we show how path-dependent divergence in capital and military stock leads to the emergence of dominant actors—proto-states—that evolve from a uniform population of equals.

### 5.1 Stochastic Divergence and the Breakdown of Symmetry

Let us begin with a population of initially identical agents, each with $k_i(0) = k_0$, $m_i(0) = m_0$, and common productivity parameters $A_i(t) = A, B_i(t) = B$. Despite initial symmetry, the stochastic dynamics

$$dk_i(t) = \left(i_{k,i}(t) - \delta^k k_i(t)\right) dt + \sigma_k\, dW_i^k(t),$$
$$dm_i(t) = \left(g_i(t) - \delta^m m_i(t)\right) dt + \sigma_m\, dW_i^m(t),$$

ensure that individual trajectories diverge due to idiosyncratic Brownian motion. Small differences in shocks lead to increasing heterogeneity in $k_i(t)$ and

$m_i(t)$. In the absence of redistribution or institutional limits, these gaps can persist and widen.

### 5.2 Feedback Amplification through Militarization

The divergence becomes self-reinforcing once we account for the strategic logic of arming. Agents with higher military stocks face reduced effective threat (since others are less likely to attack them), while simultaneously extracting more deterrent value from arms due to concavity in the utility function:

$$u_i(t) = ac_i(t) - \frac{1}{2}bc_i(t)^2 + \eta m_i(t) - \frac{1}{2}\phi m_i(t)^2 - \theta_1 \bar{m}(t) - \theta_2 \tilde{m}_i(t).$$

Agents with higher $m_i(t)$ thus experience a dual advantage:

1. They are less burdened by external threats due to their own coercive credibility.

2. They extract greater utility per unit of arms due to increased deterrence effectiveness.

This yields an endogenous feedback loop. Higher arms $\rightarrow$ greater stability $\rightarrow$ higher capital accumulation $\rightarrow$ greater resources for arms. Formally, this feedback appears in the HJB via the cross-derivatives of the value function:

$$\frac{\partial^2 V_i}{\partial k \partial m} > 0.$$

This complementarity implies increasing returns to capital and arms jointly, which drives divergence in trajectories.

### 5.3 Threshold for Dominance

Let us define a dominance threshold $\bar{\mu}(t)$ for military stock:

$$\bar{\mu}(t) := \bar{m}(t) + \epsilon,$$

for some small $\epsilon > 0$. If an agent i satisfies $m_i(t) > \bar{\mu}(t)$ and simultaneously has a capital ratio $k_i(t)/\bar{k}(t) \gg 1$, then that agent becomes de facto hegemonic. Others will no longer attempt to challenge it militarily, and will begin to condition their own investment and defense strategies on its behavior.

The dominant agent then shifts from a mutual threat actor to a protective authority—an entity whose superior power ensures stability and discipline within the population.

### 5.4 Emergence of a Monopolist Protective Agency

Once an agent's capital and arms cross this threshold, a qualitative shift occurs:

- The deterrent function of its military becomes non-rivalrous: all others benefit from its presence.

• Its arms reduce global militarization by obviating the need for others to arm individually.

• Its capital base grows through taxation (in later sections) or voluntary payments for protection.

This reflects the birth of a monopolist protective agency or a state. Mathematically, this transition can be characterized as a bifurcation in the equilibrium structure of the FPK distribution $p(t, k, m)$:

$$k_i(t)/\bar{k}(t) \to \infty \text{ and } m_i(t)/\bar{m}(t) \to \infty \text{ as } t \to T.$$

This divergence is not accidental but endogenous, driven by recursive feedback loops in the optimal control structure.

### 5.5 Endogenous Stratification and Political Interpretation

The result is a stratified society: a dominant agent enforces order, while others optimize in its shadow. This reproduces the political logic of Hobbes's Leviathan, and anticipates Nozick's vision of a minimal state emerging from self-interested protective contracts.

In formal terms, the mean field game now admits non-symmetric equilibria, where the dominant agent occupies a singular region of the state space $(k, m)$, while all others cluster below. The equilibrium distribution $p(t, k, m)$ develops a heavy tail, signifying concentrated capital and arms in the hands of the emerging state.

## 6 War, Redistribution, and Recursive Consolidation

This section introduces warfare into the mean field framework as a dynamic redistribution mechanism that reallocates capital and coercive resources (arms) among agents. Following the insight of Charles Tilly that "war made the state," we model war not merely as a destructive shock, but as an endogenous event triggered by military asymmetries and resolved through the seizure of productive and coercive assets. Crucially, we maintain consistency with each agent's dynamic budget constraint and optimization problem, ensuring that any post-war redistribution enters only through the state variables (capital and military stock), while investment and consumption remain subject to control-based constraints.

Let time be continuous, but suppose that at a discrete sequence of epochs $\{t_n\}n \in \mathbb{N}$, a war event occurs. At each war event, we determine the military winner as the agent possessing the highest military stock in the population. Formally, the winner at time $t_n$ is defined as

$$i^*(t_n) = \arg\max_{i \in [0,1]} m_i(t_n).$$

All other agents $j \neq i^*(t_n)$ are defined as losers. We assume that the winner seizes a fixed fraction $\rho_k \in (0,1)$ of the capital stock and $\rho_m \in (0,1)$ of the military stock of each loser at time $t_n$. These represent the economic and strategic resources acquired through conquest—corresponding historically to looting, annexation of productive land, and absorption of manpower into the winner's military forces.

The redistribution process operates directly on the state variables, not on the control variables. Specifically, for each loser $j \neq i^*$, we define:

$$\Delta k_{i^*}^{(j)}(t_n) = \rho_k \cdot k_j(t_n^-),$$
$$\Delta m_{i^*}^{(j)}(t_n) = \rho_m \cdot m_j(t_n^-),$$

which are transferred instantaneously to the winner's capital and military stock:

$$k_{i^*}(t_n^+) = k_{i^*}(t_n^-) + \sum_{j \neq i^*} \Delta k_{i^*}^{(j)}(t_n),$$
$$m_{i^*}(t_n^+) = m_{i^*}(t_n^-) + \sum_{j \neq i^*} \Delta m_{i^*}^{(j)}(t_n).$$

Correspondingly, each losing agent $j \neq i^*$ experiences a reduction in state variables:

$$k_j(t_n^+) = (1 - \rho_k) \cdot k_j(t_n^-),$$
$$m_j(t_n^+) = (1 - \rho_m) \cdot m_j(t_n^-).$$

Note that this redistribution does not affect control variables such as consumption $c_i(t)$, investment $i_{k,i}(t)$, or arms spending $g_i(t)$ directly. Instead, the impact of war flows through a changed state base: after each war, the affected agent's optimal policy is updated based on their new levels of capital and arms.

For the war winner $i$, the post-war dynamics of capital and military stock resume under the standard equations:

$$dk_{i^*}(t) = \big(i_{k,i^*}(t) - \delta^k \cdot k_{i^*}(t)\big)\, dt + \sigma_k \, dW_{i^*}^k(t),$$
$$dm_{i^*}(t) = (g_{i^*}(t) - \delta^m \cdot m_{i^*}(t))\, dt + \sigma_m \, dW_{i^*}^m(t),$$

with optimal investment and military expenditure determined by the usual budget constraint:

$$c_{i^*}(t) + i_{k,i^*}(t) + g_{i^*}(t) = A_{i^*}(t) \cdot k_{i^*}(t) + B_{i^*}(t) \cdot m_{i^*}(t) - D \cdot \bar{m}(t) - E \cdot \tilde{m}_{i^*}(t).$$

For each loser $j \neq i^*$, the post-war dynamics take the same structural form but with proportionally reduced $k_j(t)$ and $m_j(t)$. This guarantees that agents do not behave irrationally after war: their control behavior adjusts rationally to the new, lower state base.

The key insight here is that war reallocates state variables rather than affecting flows directly. The agent who wins accumulates larger base levels of capital and coercive power. This allows higher consumption, investment, and military accumulation in future periods. Because the winner has now increased their military dominance, they are more likely to win again in future war rounds. This generates a recursive loop of divergence in the state space:

$$\text{Victory } t \Rightarrow \text{Higher State Base at } t^{+} \Rightarrow \text{Higher Flows } \Rightarrow \text{Greater Probability of Victory } t{+}1,$$

which under mild conditions leads to exponential divergence between the winner's capital and military stocks and the rest of the population. Mathematically, if the same agent $i^*$ continues to win, then

$$\lim_{t\to\infty} \frac{k_{i^*}(t)}{\bar{k}(t)} \to \infty, \qquad \lim_{t\to\infty} \frac{m_{i^*}(t)}{\bar{m}(t)} \to \infty.$$

This signals the endogenous emergence of the state: a monopolistic protective agency with overwhelming dominance in both production and coercive capacity.

To incorporate this into the population dynamics, we update the Fokker–Planck-Kolmogorov equation with a jump operator that redistributes probability mass at each war date $t_n$. The modified Fokker–Planck–Kolmogorov (FPK) equation is given by:

$$\frac{\partial p}{\partial t} + \nabla \cdot (\mu(t,k,m)p) = \delta(t - t_n) \cdot \mathcal{W}[p] + \frac{1}{2}\nabla^2(\Sigma^2 p),$$

where $\mu(t,k,m)$ is the drift induced by optimal investment and arms controls, and $\mathcal{W}[p]$ is the war redistribution operator, redistributing mass from losing regions $(k_j, m_j)$ to the winner's region $(k_{i^*}, m_{i^*})$. This operator preserves total probability mass but shifts its distribution, creating increasingly heavy right tails in both capital and military stock distributions. Over time, the result is a power-law–like divergence consistent with empirical war-driven hegemonies.

This mechanism captures the economic and political structure behind Tilly's hypothesis. Wars are not simply exogenous shocks; they are events caused by endogenous asymmetries and result in redistribution that accelerates those very asymmetries. In doing so, they generate a path-dependent trajectory in which early military advantage compounds into long-run sovereign dominance. This completes the formal argument that war, by altering the stochastic evolution of state variables, generates a centralized state as an emergent equilibrium of strategic conflict.

# 7 Taxation and the Fiscal Expansion of the State

The emergence of a dominant agent through repeated military victories does not merely result in accumulation of coercive superiority—it also establishes the institutional conditions for the extraction of economic resources. Once an agent accumulates sufficient military stock such that rebellion is irrational and deterrence is fully credible, it can impose taxation on other agents. In this section, we incorporate a fiscal mechanism into our mean field framework by modeling taxation as a share of productive output, funneled from the rest of the population to the monopolist protective agency. This allows us to capture the endogenous expansion of the state via fiscal means, alongside its coercive foundation established through warfare.

Let $i^* \in [0, 1]$ denote the dominant agent, whose military and capital stocks $(k_{i^*}(t), m_{i^*}(t))$ have diverged significantly from the mean field due to the recursive process described in Section 6. Suppose that after a critical threshold of coercive dominance is achieved—say,

$$\frac{m_{i^*}(t)}{\bar{m}(t)} \gg 1 \quad \text{and} \quad \frac{k_{i^*}(t)}{\bar{k}(t)} \gg 1,$$

all other agents recognize that resistance is futile and accept taxation as a rational tradeoff for security. The dominant agent $i^*$ becomes a monopolist provider of protection, and its coercive authority is regularized into a system of taxation.

## 7.1 Income Taxation on Output

We assume the state levies a proportional income tax $\tau \in (0, 1)$ on each agent's gross output. The production function for each agent $i \neq i^*$ is:

$$y_i(t) = A_i(t) \cdot k_i(t) + B_i(t) \cdot m_i(t) - D \cdot \bar{m}(t) - E \cdot \tilde{m}_i(t).$$

Under taxation, the effective income for the agent is reduced:

$$(1 - \tau) \cdot y_i(t),$$

while the state collects:

$$T_i(t) = \tau \cdot y_i(t).$$

Thus, the budget constraint for each minor agent becomes:

$$c_i(t)+i_{k,i}(t)+g_i(t) = (1-\tau)\cdot\left(A_i(t) \cdot k_i(t) + B_i(t) \cdot m_i(t) - D \cdot \bar{m}(t) - E \cdot \tilde{m}_i(t)\right).$$

The dominant agent $i^*$, in contrast, does not pay taxes but instead receives transfers:

$$R(t) = \int_0^1 \tau \cdot (A_j(t) \cdot k_j(t) + B_j(t) \cdot m_j(t) - D \cdot \bar{m}(t) - E \cdot \tilde{m}_j(t))\, dj.$$

Hence, its updated budget constraint is:

$$c_{i^*}(t)+i_{k,i^*}(t)+g_{i^*}(t) = A_{i^*}(t)\cdot k_{i^*}(t)+B_{i^*}(t)\cdot m_{i^*}(t)-D\cdot\bar{m}(t)-E\cdot\tilde{m}_{i^*}(t)+R(t).$$

### 7.2 Dynamic Evolution with Tax Receipts

The evolution equations for the dominant agent now reflect the productive use of taxes. We continue to model capital and military stock as:

$$dk_{i^*}(t) = \left(i_{k,i^*}(t) - \delta^k \cdot k_{i^*}(t)\right) dt + \sigma_k \, dW_{i^*}^k(t),$$
$$dm_{i^*}(t) = \left(g_{i^*}(t) - \delta^m \cdot m_{i^*}(t)\right) dt + \sigma_m \, dW_{i^*}^m(t),$$

but with investment flows $i_{k,i^*}(t)$ and $g_{i^*}(t)$ now indirectly enhanced by tax inflows. This leads to self-financing expansion: taxation increases productive capacity, which increases fiscal base, which increases taxation, and so forth.

The long-run consequence of this feedback is accelerated divergence between the state and the rest of the population. Even absent further war, the state's capital and arms accumulate exponentially faster due to its monopoly on both coercion and taxation. Mathematically, this is reflected in the updated Fokker–Planck-Kolmogorov system, in which the state agent $i^*$ acquires an effective drift term boosted by the population-wide production:

$$\mu_{i^*}^{\text{drift}}(t, k, m) = f(k, m) + \tau \cdot \mathbb{E}_j[y_j(t)],$$

whereas the rest of the population faces the depressed net productivity:

$$\mu_i^{\text{drift}}(t, k, m) = (1 - \tau) \cdot f(k, m), \quad i \neq i^*.$$

This differential implies that the state's share of total output and military stock expands over time, even without direct military conquest.

### 7.3 Interpretative Implications

The model shows that taxation emerges as a second-order effect of coercive dominance. It is not exogenous; rather, it arises endogenously once the marginal costs of rebellion exceed the marginal burden of tax compliance. In this sense, taxation is the regularized expression of the war victory. Historically, this corresponds to the transformation from conquest states (e.g., early Qin, Ottoman, Prussian) into bureaucratic fiscal-military states. Moreover, the taxes are not simply extractions—they are used by the state to enhance both productive and coercive capacities, allowing for even more tax extraction.

Thus, taxation and war are complementary channels of state consolidation. While war initiates the divergence, taxation sustains and amplifies it. The state thus emerges not only as a war-making machine but also as a revenue-maximizing rational agent, reinforcing the twin foundations of coercion and extraction identified by Tilly, Schumpeter, and modern fiscal sociology.

## 8 Recursive War Cycles and Long-Run State Formation

In this section, we present a complete dynamic system that models how a sequence of recursive war cycles, reinforced by fiscal extraction, generates the endogenous emergence of the state. This process begins in a Hobbesian world of symmetric agents armed for self-defense and production, but ultimately results in a stable hierarchy with a dominant protective agency that commands both coercive and productive capacities far exceeding those of all others. Our model formalizes Charles Tilly's famous dictum—"war made the state, and the state made war"—as a stochastic recursive dynamic within a mean field game (MFG) system.

## 8.1 Recursive Feedback via Military Superiority

Suppose that at war times $\{t_n\}_{n\in\mathbb{N}}$, military conflict redistributes capital and arms according to the war mechanism defined in Section 6. After each war, the winner $i^*(t_n)$ captures a fraction $\rho_k$ and $\rho_m$ of the losing agents' capital and arms. The new post-war capital and military stocks for the winner become:

$$\begin{aligned} ki^*(t_n^+) &= k_{i^*}(t_n^-) + \sum_{j\neq i^*} \rho_k \cdot k_j(t_n^-), \\ m_{i^*}(t_n^+) &= m_{i^*}(t_n^-) + \sum_{j\neq i^*} \rho_m \cdot m_j(t_n^-). \end{aligned}$$

These sudden jumps produce a higher base for the winner's subsequent capital and arms accumulation, governed by:

$$\begin{aligned} dk_{i^*}(t) &= \left(i_{k,i^*}(t) - \delta^k \cdot k_{i^*}(t)\right) dt + \sigma_k \, dW_{i^*}^k(t), \\ dm_{i^*}(t) &= \left(g_{i^*}(t) - \delta^m \cdot m_{i^*}(t)\right) dt + \sigma_m \, dW_{i^*}^m(t), \end{aligned}$$

where control variables $i_{k,i^*}(t), g_{i^*}(t)$ are now chosen from a higher resource base, augmented by tax revenues after fiscal institutionalization (Section 7).

The critical point is that each war redistributes resources in a way that makes future victory more likely. Because the winner not only keeps but uses seized capital and arms productively, the probability of winning again increases. Let $\mathbb{P}_i(t_{n+1})$ denote the probability that agent $i$ wins the next war at $t_{n+1}$. Then:

$$\mathbb{P}_{i^*}(t_{n+1}) > \mathbb{P}_{i^*}(t_n) \quad \text{if victory at } t_n.$$

This recursive probability growth eventually leads to almost certain dominance by one agent, as the population distribution $p(t,k,m)$ becomes increasingly concentrated around the dominant $(k_{i^*}(t), m_{i^*}(t))$.

## 8.2 Dynamic Divergence and Long-Run Dominance

We can now trace the full trajectory of this recursive process. Suppose the winner $i^*$ continues to dominate over many cycles. Then the ratio of its capital and military stock relative to the population mean satisfies:

$$\lim_{t\to\infty} \frac{k_{i^*}(t)}{\bar{k}(t)} \to \infty, \qquad \lim_{t\to\infty} \frac{m_{i^*}(t)}{\bar{m}(t)} \to \infty.$$

This implies that the winner increasingly resembles a monopolist provider of security—the state—while all others become effectively powerless or tributary. In equilibrium, the population mean remains bounded because losers are repeatedly depleted:

$$\bar{k}(t) = \int_0^1 k_i(t)\, di \leq K < \infty, \qquad \bar{m}(t) = \int_0^1 m_i(t)\, di \leq M < \infty,$$

whereas $k_{i^*}(t), m_{i^*}(t)$ grow at compounding rates due to seizures and tax flows.

### 8.3 Fokker–Planck Dynamics with War and Tax Feedback

The full population dynamics are governed by a Fokker–Planck–Kolmogorov (FPK) equation:

$$\frac{\partial p}{\partial t} + \nabla \cdot (\mu(t,k,m)p) = \mathcal{W}[p] + \mathcal{T}[p] + \frac{1}{2}\nabla^2(\Sigma^2 p),$$

where:

- $\mu(t,k,m)$: endogenous drift from optimal control,
- $\mathcal{W}[p]$: redistribution operator from war (jumps at $t_n$),
- $\mathcal{T}[p]$: tax-based resource extraction (continuous drift to $i^*$),
- $\Sigma^2$: variance operator from Brownian motions.

This system gradually evolves toward a singular mass distribution in which almost all population density is pushed to low-wealth, low-arms states, while a dominant state agent accumulates the upper tail.

### 8.4 Political and Historical Interpretation

Mathematically, the emergence of the state arises from recursive divergence in a stochastic control model, but politically, it reflects the pattern seen throughout human history: from Mesopotamian city-states to the Qin unification, to early modern Europe, the same structure recurs. Initially symmetric agents engage in conflict. Winners grow. War becomes institutionalized. Taxation follows. Over time, these mechanisms produce a centralizing force—what we now call the state.

Our model thus provides a rigorous microfoundation for the Tillyian mechanism: war as both capital destruction and rent transfer, institutionalized through tax and bureaucracy, leading to hierarchical consolidation. The recursive structure also explains why timing matters: early winners enjoy path dependence. The historical state is not just a social contract—it is the equilibrium of repeated violence, appropriation, and fiscal stabilization.

## 9 Conclusion and Future Directions

This paper has developed a dynamic mean field game framework to model the endogenous emergence of the state from a Hobbesian environment, where each agent begins as their own producer and defender. Drawing from the political philosophy of Thomas Hobbes and Robert Nozick, and building on Charles Tilly's historical thesis that "war made the state, and the state made war," we constructed a model in which production, self-defense, conflict, and taxation are jointly determined by decentralized optimization and strategic interaction under uncertainty.

In the base model, each agent optimizes over consumption, capital investment, and military accumulation, facing threat externalities from the average and directed military stock of others. When war is introduced as a mechanism for redistributing capital and arms, the recursive advantage of winning leads to a divergence process: each victory increases future probability of success by augmenting coercive and productive capacities. Once a dominant agent emerges, taxation becomes a rational response to deterrent asymmetry. The monopolist protective agent uses tax revenue to further accumulate resources, reinforcing their superiority and establishing a fiscal-military state. Over time, this recursive loop creates a singular equilibrium structure with one dominant protective agency and a passive population of taxed agents.

This modeling framework offers several powerful implications and directions for future research:

1. Quantitative Calibration to Historical State Formation: Our model can be calibrated to study real episodes of state consolidation, such as the unification of China during the Warring States period, early modern European state-building, or the formation of modern Leviathans in post-colonial settings.

2. Coalitional Dynamics and Failed States: The model could be extended to include coalition formation among weaker agents, enabling resistance or secession dynamics. This would allow us to study the conditions under which states fail, fragment, or are contested.

3. Public Goods and Endogenous Legitimacy: Taxation in our model is coercive and extractive. However, public goods provision—roads, defense, education—may change the utility structure. Incorporating endogenous legitimacy would permit an analysis of when states are supported rather than tolerated.

4. Multiple Competing States and Borders: In reality, multiple protective agencies (states) co-exist. Introducing multiple dominant agents would allow modeling of inter-state warfare, diplomacy, and the endogenous emergence of borders and empires.

5. Stochastic Institutional Shocks: Agents could transition not only between productivity regimes but also institutional types (autocratic, feudal, democratic), with implications for warfare and taxation. This would allow us to model how institutional evolution interacts with coercive asymmetries.

6. Endogenous Technology and Capital-Military Substitution: The current model treats capital and arms as separate inputs. One can introduce substitutability between them, or technological innovation in weaponry, enabling richer tradeoffs between economic growth and militarization.

7. War Cycles and Collapse: Finally, our framework can be extended to model the rise and fall of states by allowing war to cause overextension, capital depletion, and rebellion.

In summary, we have built a coherent and rigorous dynamic model of state formation as an endogenous equilibrium outcome of decentralized conflict and coercion. War, taxes, and political authority emerge not as assumptions but as the byproducts of strategic asymmetries and rational individual behavior. The state is thus neither a social contract nor a benevolent planner, but the stochastic limit of recursively reinforced dominance. The tools we developed here—stochastic control, mean field games, Fokker–Planck-Kolmogorov dynamics, and fiscal redistribution—offer a robust foundation for future explorations of political and economic evolution in an uncertain and contested world.

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