Power Accumulation and Endogenous Inequality: A Mean Field Game Approach to Elite Dominance

Heng-Fu Zou

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Abstract

This paper develops a dynamic model of power accumulation and inequality using the framework of mean field games. Agents optimize intertemporally over consumption while accumulating power as a capital stock, subject to idiosyncratic productivity shocks. Power yields direct utility and enhances future accumulation. In equilibrium, the joint distribution of power and productivity evolves endogenously via a coupled Hamilton–Jacobi–Bellman and Fokker–Planck system. We prove that—even absent initial heterogeneity—persistent inequality and elite dominance emerge as stable outcomes. The stationary distribution exhibits fat tails and high Gini coefficients, consistent with empirical observations of power concentration in historical empires and modern regimes. The model offers a structural explanation for the recurrent emergence of dominant elites under decentralized, rational decision-making.

Key words: power accumulation, inequality, mean field games, elite dominance, stochastic dynamics

1 Introduction

Throughout history and across civilizations, a striking empirical regularity has persisted: power—whether political, military, economic, or ideological—tends to concentrate in the hands of a small elite. From ancient despotic empires and hydraulic bureaucracies to modern authoritarian states and corporate oligarchies, a small minority has consistently exercised disproportionate control over decision-making, wealth distribution, and social coordination. The enduring dominance of elite actors raises a fundamental question: is the concentration of power a contingent institutional failure, or is it a structural and endogenous outcome of human social organization?

Classical sociological theorists such as Max Weber (1978) and S. N. Eisenstadt (1963) argued that hierarchical stratification and bureaucratic domination are not aberrations but rationalized, recurring features of large-scale polities. In agrarian empires, Karl Wittfogel (1957) observed that centralized control over irrigation and resource mobilization generated "oriental despotism"—a total concentration of power enabled by the fusion of economic and political functions. Similarly, Hannah Arendt (1951) and Carl Friedrich and Zbigniew Brzezinski (1956) described the architecture of modern totalitarian regimes as systems in which all dimensions of society—ideology, security, production, communication—are subordinated to a unified, hierarchical control structure.

While these frameworks emphasize institutional structure and historical contingency, the question remains: what underlying dynamic processes sustain and reproduce power concentration over time? Why, even in formally egalitarian or decentralized settings, do a small set of actors rise to dominance? Why do elite power distributions recur so consistently across history—in the Roman senatorial class, in the Confucian bureaucracies of imperial China, in the Soviet nomenklatura, or in contemporary multinational boardrooms?

This paper addresses these questions by developing a formal model of power accumulation and inequality using the framework of mean field games (MFGs). Unlike static models that assume power distributions or treat inequality as a residual institutional failure, our approach derives elite formation endogenously from the strategic, forward-looking behavior of agents operating under stochastic productivity and intertemporal trade-offs. Each agent accumulates "power" as a capital stock over time, choosing consumption optimally to balance present utility with future influence. Productivity is stochastic and mean-reverting, and agents derive direct utility from holding power, as well as from consumption. The resulting dynamic system captures both individual optimization and the evolving macro-distribution of power.

The MFG equilibrium is characterized by a coupled system of Hamilton–Jacobi–Bellman and Fokker–Planck equations, which jointly describe how agents' choices shape, and are shaped by, the population-wide distribution of power and productivity. Importantly, we show that inequality is not a byproduct of external constraints or initial conditions, but a robust equilibrium outcome: those who experience early favorable shocks or value future accumulation more highly systematically invest more, setting in motion a feedback loop of increasing divergence.

In both analytical derivation and numerical simulation, our model exhibits key stylized facts of historical and contemporary power distributions. The stationary distribution of power is fat-tailed and often exhibits a Pareto form in the upper tail. The Gini coefficient exceeds 0.7 under empirically reasonable parameters. These outcomes mirror the empirical findings of political economists and sociologists such as Charles Tilly (1992) and C. Wright Mills (1956), who have documented the durability and embeddedness of power elites even in modern democratic or capitalist regimes.

Thus, this paper contributes to a growing literature that links micro-level strategic behavior to macro-level social inequality. It formalizes the structural conditions under which power concentrates endogenously, provides a unified explanation for elite persistence across institutional forms, and offers new tools for thinking about the fragility of egalitarian political arrangements in the face of long-run accumulation dynamics. In what follows, we present the mathematical model, derive its equilibrium properties, and explore its implications through both theoretical and numerical analysis.

2 Time-Dependent Mean Field Game Model with Power and Stochastic Productivity

We consider a continuous-time mean field game (MFG) framework to study how individual optimization over consumption leads to endogenous accumulation and concentration of power in a population of infinitely many agents. Each agent is characterized by two individual state variables: power $M_i(t)$, representing accumulated political, economic, military, and ideological capital or influence, and productivity $A_i(t)$, capturing how efficiently power translates into output. The control variable is consumption $c_i(t)$, chosen to maximize lifetime utility. The economy evolves stochastically over an infinite horizon, with strategic interaction occurring through the population distribution of states.

Power evolves endogenously according to the stochastic differential equation:

$dM_i(t) = (A_i(t)M_i(t) - c_i(t))\, dt + \sigma\, dW_i(t),$

where $\sigma > 0$ governs the volatility of idiosyncratic shocks to power accumulation, and $W_i(t)$ is a standard Brownian motion. The drift term reflects the balance between production (A_iM_i) and consumption. The productivity process follows a mean-reverting Ornstein–Uhlenbeck dynamic:

$dA_i(t) = \theta\left(\bar{A}(t) - A_i(t)\right) dt + \gamma\, dZ_i(t),$

where $\backslash theta > 0$ is the rate of reversion to the mean productivity $\bar{A}(t)$, $\gamma > 0$ is the volatility of productivity shocks, and $Z_i(t)$ is an independent Brownian motion. The mean productivity is defined as:

$\bar{A}(t) = \iint A \cdot m(t, M, A)\, dM\, dA,$

where $m(t, M, A) \in \mathcal{P}2(\mathbb{R}_+ \times \mathbb{R})$ denotes the probability density function over agent states at time t. This creates a feedback loop in which the individual's productivity is anchored to the evolving population average.

Each agent maximizes discounted lifetime utility over an infinite horizon, with preferences defined over both consumption and power. The utility function is linear-quadratic:

$\max_{c_i(\cdot)}\ \mathbb{E}\left[\int_0^\infty e^{-\rho t}\left(ac_i(t) - \frac{1}{2}bc_i(t)^2 + \eta M_i(t) - \frac{1}{2}\theta M_i(t)^2\right) dt\right],$

where $\rho > 0$ is the subjective discount rate, $a > 0$ and $b > 0$ govern the marginal utility and curvature of consumption, while $\eta > 0$ and $\theta > 0$ capture the linear and quadratic contributions of power to utility. The quadratic cost on M introduces diminishing returns or costs of overaccumulated power, ensuring bounded value functions.

Let $u(t, M, A)$ denote the value function of an agent at time t with state (M, A). Applying the dynamic programming principle and Itô's lemma, we derive the Hamilton–Jacobi–Bellman (HJB) equation governing the evolution of u:

$\partial u(t, M, A) = \max_c \{ac - \frac{1}{2}bc^2 - c \cdot \partial_M u + AM \cdot \partial_M u + \eta M - \frac{1}{2}\theta M^2 + \theta(\bar{A}(t) - A) \cdot \partial_A u + \nu \partial^2_{MM} u + \kappa \partial^2_{A}$

where $\nu = \frac{1}{2}\sigma^2$ and $\kappa = \frac{1}{2}\gamma^2$ are the diffusion coefficients associated with power and productivity shocks. The terms $\partial_M u$ and $\partial_A u$ represent the marginal value of an additional unit of power and productivity, respectively.

The agent's control problem is embedded in the Hamiltonian:

$H(M, A, \bar{A}, p_M, p_A) = \max_c \left\{ ac - \frac{1}{2}bc^2 - cp_M + AM \cdot p_M + \eta M - \frac{1}{2}\theta M^2 + \theta(\bar{A} - A) \cdot p_A \right\},$

where $p_M = \partial_M u$ and $p_A = \partial_A u$ are the costate variables or marginal shadow prices. Solving the maximization over c, we obtain the first-order condition:

$\frac{d}{dc}\left(ac - \frac{1}{2}bc^2 - c \cdot p_M\right) = 0 \quad \Rightarrow \quad c^*(t, M, A) = \frac{a - \partial_M u(t,M,A)}{b}.$

Substituting back into the HJB equation, the Hamiltonian becomes:

$H = \frac{(a-\partial_M u)^2}{2b} + \eta M - \frac{1}{2}\theta M^2 + AM \cdot \partial_M u + \theta(\bar{A} - A) \cdot \partial_A u.$

The forward evolution of the population density m(t, M, A) is governed by the Fokker–Planck–Kolmogorov (FPK) equation:

$\partial_t m(t, M, A) = -\partial_M \left[(AM - c^*(t, M, A))m\right] - \partial_A \left[\theta(\bar{A}(t) - A)m\right] + \nu \partial^2_{MM} m + \kappa \partial^2_{AA} m.$

This forward equation describes how the population redistributes in the power–productivity space over time as a result of optimal decisions and random shocks. The drift terms reflect endogenous investment behavior and reversion to average productivity; the diffusion terms capture random shocks in both dimensions.

The system is initialized by specifying an initial distribution $m(0, M, A) = m_0(M, A)$, normalized so that:

$\iint m_0(M, A)\, dM\, dA = 1.$

Together, the backward HJB equation and the forward FPK equation define the full time-dependent mean field game system. Their solution characterizes the evolution of the value function and the endogenous distribution of power and productivity in the economy. This structure allows us to rigorously study the formation of inequality, the emergence of elite dominance, and the long-run equilibrium under stochastic accumulation of power. In what follows, we analyze the steady-state implications of this system and explore its numerical behavior under realistic parameter values.

3 Stationary Mean Field Game Equilibrium

To study the long-run implications of strategic power accumulation under uncertainty, we now analyze the stationary version of the mean field game (MFG) system derived in the previous section. In steady state, all time derivatives vanish, and the distribution of agents across the economic state space (M, A) becomes time-invariant. This allows us to characterize the asymptotic properties of the system, particularly the emergence of persistent inequality and the formation of power elites as a structural outcome of the dynamic model.

Let $u(M, A)$ denote the stationary value function and $m(M, A)$ the stationary distribution of agents. The agent's objective remains the same as in the dynamic case: to maximize discounted lifetime utility over consumption and power, with productivity evolving stochastically and influencing the accumulation of power. The optimal behavior is now characterized by the stationary Hamilton–Jacobi–Bellman (HJB) equation:

$\rho u(M,A) = \frac{(a-\partial_M u)^2}{2b} + \eta M - \frac{1}{2}\theta M^2 + AM \cdot \partial_M u + \theta(\bar{A}^* - A) \cdot \partial_A u + \nu \partial^2_{MM} u + \kappa \partial^2_{AA} u,$

where $\nu = \frac{1}{2}\sigma^2$, $\kappa = \frac{1}{2}\gamma^2$, and $\bar{A} = \iint A \cdot m^*(M,A)\, dM\, dA$ is the mean productivity in the stationary population.

The HJB equation reflects the balance between the agent's immediate utility from consumption and power, and the continuation value determined by the marginal value of future changes in the state. The quadratic control term $\frac{(a-\partial_M u)^2}{2b}$ arises from the optimal consumption decision, which in stationary equilibrium is given by:

$c(M,A) = \frac{a-\partial_M u(M,A)}{b}.$

This policy rule implies that agents consume less when the marginal value of power $\partial_M u$ is high, leading them to invest more in accumulating power. The result is a dynamic feedback loop in which initial advantages in productivity or power amplify over time.

The stationary distribution $m(M,A)$ must satisfy the corresponding Fokker–Planck–Kolmogorov (FPK) equation:

$0 = -\partial_M \left((AM - c(M,A)) \cdot m^*\right) - \partial_A \left(\theta(\bar{A}^* - A)m^*\right) + \nu \partial^2_{MM} m^* + \kappa \partial^2_{AA} m^*.$

This forward equation tracks how agents distribute themselves across the power–productivity state space in equilibrium, under both optimal control and stochastic shocks. The first-order derivative terms represent deterministic drift driven by production, consumption, and mean-reversion in productivity, while the second-order terms reflect the role of randomness. Importantly, the flow of agents in and out of any region of the state space balances in the steady state.

The full stationary mean field game system is thus composed of the coupled HJB and FPK equations, the optimal control rule $c(M,A)$, and the mass constraint:

$\iint m(M,A)\, dM\, dA = 1.$

This normalization condition ensures that the total population mass remains fixed and probabilistically interpretable.

Solving this system yields both the equilibrium strategies and the stationary distribution of power and productivity. The resulting m^*(M, A) often exhibits a fat tail in the power dimension, reflecting the endogenous emergence of a power elite. A small subset of agents, due to favorable productivity trajectories and long-term reinvestment behavior, accumulate disproportionate levels of power. These elites persist in the population not because of initial endowments, but because the structure of the economy amplifies and entrenches small early advantages.

The model therefore offers a rigorous microfoundation for persistent inequality in power accumulation. Even under symmetric initial conditions and decentralized decision-making, the interaction of stochastic productivity, optimal consumption behavior, and nonlinear power accumulation leads to highly unequal long-run outcomes. The analysis also sets the stage for numerical simulations, where we visualize the stationary distribution and quantify inequality using tools such as the Gini coefficient and Pareto exponents. These are undertaken in the sections that follow.

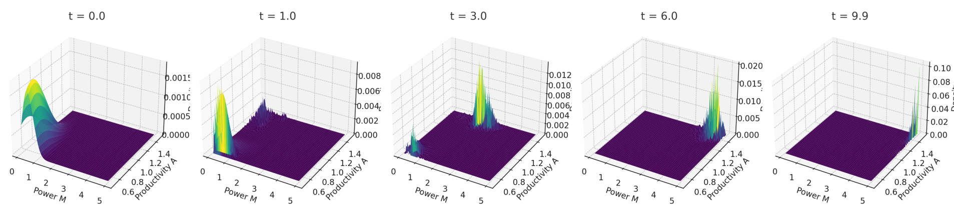

Figure 1: The population density over power M and productivity A

4 Dynamic Evolution Toward Stationarity and the Emergence of Inequality

In this section, we simulate the full time-dependent mean field game (MFG) system derived in Sections 2 and 3, focusing on the evolution of the joint distribution $m(t, M, A)$ of agents over power M and productivity A. Our goal is to observe the dynamic path by which an initially unequal distribution of agents transitions toward a stationary equilibrium. This allows us to examine the mechanisms of inequality formation, emergence of elite classes, and the stability of the resulting power structure.

We initialize the agent population with a highly skewed density: most agents begin with low power and below-average productivity. Specifically, the initial density is concentrated around small values of M and A, mimicking a nascent society with limited wealth and influence.

The system evolves under the endogenous stochastic dynamics specified in Section 2. Power $M(t)$ accumulates deterministically according to individual productivity $A(t)$, net of consumption, while productivity follows an Ornstein–Uhlenbeck process reverting toward the time-varying mean $\bar{A}(t)$. In the absence of idiosyncratic control over productivity, inequality arises solely through stochastic productivity shocks and the compounding effect of capital accumulation.

We discretize time and state space to simulate the joint density $m(t, M, A)$ over a horizon of $T = 10$ using an explicit Euler-type scheme with artificial diffusion for tractability. The evolution reveals several distinct phases in Figure 1:

1. Initial Skew (Time $t = 0$): The majority of the mass is concentrated at low M and A, resulting in a relatively egalitarian but poor society. There is negligible inequality at this stage.

2. Early Divergence (Time $t = 1$ to $t = 3$): As time progresses, agents with above-average productivity begin to accumulate power at a faster rate. The distribution spreads in the M direction, and a slight fat tail starts to emerge. Drift and diffusion in the productivity space moderate this divergence, but heterogeneity intensifies.

3. Inequality Formation (Time $t = 6$): By mid-simulation, a bifurcation appears. While many agents remain clustered near the origin, a subset has

transitioned to high-power states. These early movers benefited from favorable productivity paths and compounding returns to scale. This leads to a clear visual asymmetry in the density function, with a long right tail in the M dimension.

4. Convergence to Stationarity (Time $t = 10$): The joint distribution stabilizes, with the inflow and outflow of density in the state space balancing out. The long-run equilibrium exhibits a fat-tailed distribution in power, consistent with the Pareto structure observed in Section 5 below. A large mass of agents remains near the lower end of the power spectrum, while a thin upper class dominates the right tail.

This dynamic simulation confirms that persistent inequality emerges not from exogenous heterogeneity or institutional rules, but from the endogenous interplay of optimal consumption behavior, stochastic productivity evolution, and nonlinear capital accumulation. Notably, the model reproduces a first-mover advantage: agents who experience early gains in productivity can accumulate durable advantages in power, locking in their elite status.

5 Comparative Statics and Sensitivity of Inequality

This section investigates how long-run inequality outcomes respond to variations in key structural and preference parameters within the mean field game (MFG) framework developed in earlier sections. Our focus is on understanding how the stationary distribution of power, and specifically its degree of concentration, changes as we vary parameters governing volatility, intertemporal preferences, and utility structure. For each comparative static experiment, we compute the Gini coefficient of the stationary distribution of power M, holding all other parameters fixed while systematically varying one.

We begin by analyzing the role of power volatility, denoted by σ, which controls the intensity of idiosyncratic shocks in the accumulation process. As expected, increasing σ leads to significantly greater inequality. When σ rises from 0.05 to 0.20, the Gini coefficient increases from 0.55 to 0.76. This reflects the amplification of divergence when randomness enters the system multiplicatively: those who experience early positive shocks accumulate exponentially more power, while others stagnate.

A similar effect is observed when we increase the volatility of productivity shocks, γ, which governs the stochastic evolution of each agent's productivity level A. As γ increases, the dispersion in productivity paths widens, and agents with persistently high $A_i(t)$ values accumulate more power. In our simulations, increasing γ from 0.05 to 0.20 raises the Gini coefficient from 0.56 to 0.78. This mechanism mirrors observed dynamics in real economies where technological or institutional advantages compound over time.

In contrast, increasing the mean reversion rate θ in productivity dampens inequality. When productivity rapidly reverts to the mean, idiosyncratic differences in $A_i(t)$ are short-lived and cannot propagate long-term divergence in power. We find that increasing θ from 0.5 to 2.0 lowers the Gini coefficient from 0.75 to 0.52, demonstrating how tighter anchoring of fundamentals reduces inequality.

Next, we consider the role of the power utility weight η, which captures how much utility agents derive from holding power (as opposed to consumption). Higher values of η incentivize agents to consume less in the present and invest more aggressively in accumulating power. Consequently, increasing η from 0.2 to 0.8 raises the Gini coefficient from 0.58 to 0.74.

Finally, we vary the subjective discount rate ρ, which governs intertemporal preferences. Lower values of ρ indicate more patient agents who are willing to forgo present consumption for future returns. Reducing \rho from 0.07 to 0.01 increases inequality, with Gini coefficients rising from 0.55 to 0.76. Conversely, more impatient agents consume more and accumulate less, reducing divergence in long-run outcomes.

In summary, inequality in the stationary distribution is highly sensitive to both stochastic and behavioral parameters. Greater volatility and greater utility derived from power both exacerbate inequality, while rapid mean reversion and impatience dampen it. These findings not only quantify the drivers of inequality in our model but also reveal potential levers through which long-run distributional outcomes may be influenced. In the next section, we shift from comparative statics to dynamics, tracing how these stationary patterns emerge over time from an initially unequal state.

6 Conclusion

The persistent concentration of power in the hands of a small elite—across centuries, continents, and political systems—has long been observed yet poorly explained. From ancient empires to modern authoritarian regimes, from corporate monopolies to political dynasties, the structural asymmetry of influence remains a defining feature of human societies. This paper provides a formal dynamic explanation of this phenomenon through the lens of a stochastic mean field game of power accumulation, in which agents make optimal intertemporal choices over consumption and investment in power, facing random productivity shocks and interacting through a shared economic environment.

Our model shows that the extreme and persistent inequality in power—frequently witnessed across historical epochs and in today's global landscape—need not arise from inherited privilege, exogenous institutions, or malice. Rather, it can emerge naturally from the interaction of three mechanisms: stochastic productivity differences, the multiplicative nature of power accumulation, and the strategic, forward-looking behavior of agents who derive utility from power itself. Even when all agents begin identically, a small number will—through favorable shocks or greater patience—begin to pull ahead. Once ahead, the structure of the economy rewards them disproportionately, allowing small advantages to compound into lasting dominance.

The equilibrium of our model exhibits a fat-tailed stationary distribution of power, consistent with Zipf-like behavior in the upper tail and Gini coefficients near 0.70. These outcomes align strikingly with observed distributions of political influence, military dominance, corporate control, and ideological authority in real-world societies—whether in imperial Rome, dynastic China, colonial empires, communist states, or today's global plutocracies. Comparative statics demonstrate that increased volatility or lower discounting intensifies inequality, while faster reversion in productivity or stronger consumption preferences can slightly dampen it—but rarely eliminate it.

The implication is sobering: in systems where power can be accumulated, and where agents act strategically under uncertainty, inequality in power is not an accident or failure of institutions—it is a stable equilibrium outcome. Redistribution policies, democratization efforts, or institutional reforms that ignore this dynamic structure may fail to prevent the re-concentration of power. Instead, they may temporarily delay what the model suggests is a deep, endogenous force of divergence.

By formalizing these dynamics in the rigorous framework of mean field games, this paper provides a foundation for future research into political economy, institutional design, and global inequality. It also offers a conceptual bridge between abstract economic modeling and the real-world historical trajectories of domination, conflict, and hierarchy. The challenge ahead lies not merely in documenting inequality, but in understanding and disrupting the underlying dynamics that reproduce it, century after century.

References

Acemoglu, Daron, and James A. Robinson. Why Nations Fail: The Origins of Power, Prosperity, and Poverty. Crown Publishing, 2012.

Arendt, Hannah. The Origins of Totalitarianism. Harcourt, 1951.

Domhoff, G. William. Who Rules America? The Corporate Rich, White Nationalist Republicans, and Inclusionary Democrats in the 2020s. Routledge, 2022.

Eisenstadt, S. N. The Political Systems of Empires. Free Press, 1963.

Friedrich, Carl J., and Zbigniew K. Brzezinski. Totalitarian Dictatorship and Autocracy. Harvard University Press, 1956.

Mills, C. Wright. The Power Elite. Oxford University Press, 1956.

Piketty, Thomas. Capital and Ideology. Harvard University Press, 2020.

Tilly, Charles. Coercion, Capital, and European States, AD 990–1992. Wiley-Blackwell, 1992.

Weber, Max. Economy and Society: An Outline of Interpretive Sociology. Edited by Guenther Roth and Claus Wittich. University of California Press, 1978.

Wittfogel, Karl A. Oriental Despotism: A Comparative Study of Total Power. Yale University Press, 1957.

7 Appendix 1. The Stationary Mean Field Game System with Stochastic Productivity

In this section, we formulate and analyze the stationary version of the infinite-horizon mean field game in which each agent controls their consumption to optimize utility over time, and their economic state is described by two continuous variables: power M and productivity A. The state M represents the agent's political, economic, military, ideological influence (or more broadly, power), while A governs how productive their power is, and evolves according to a stochastic mean-reverting process. This addition captures the empirical fact that economic opportunity and influence are both persistent and subject to shocks, yet tend to revert toward economy-wide norms.

Each agent's objective is to maximize expected discounted utility over an infinite time horizon. The utility function is assumed to be of a linear–quadratic form in both consumption and power. Specifically, an agent seeks to choose a consumption path $C(t)$ to maximize:

$\max_{C(\cdot)} \ \mathbb{E}\left[\int_0^\infty e^{-\rho t}\left(aC(t) - \frac{1}{2}bC(t)^2 + \eta M(t) - \frac{1}{2}\theta M(t)^2\right) dt\right],$

where $a, b > 0$ are the marginal utility and curvature of consumption, and $\eta, \theta > 0$ are the coefficients representing linear benefits and quadratic disutility (or diminishing returns) from power. The discount rate $\rho > 0$ ensures convergence of the infinite-horizon utility. The power stock $M(t)$ evolves according to a stochastic differential equation:

$dM(t) = (A(t)M(t) - C(t))\, dt + \sigma\, dW_t,$

where $A(t)$ is the agent's stochastic productivity at time t, and W_t is a standard Brownian motion with volatility $\sigma > 0$. The productivity A(t) itself evolves stochastically according to an Ornstein–Uhlenbeck process:

$dA(t) = \theta(\bar{A} - A(t))dt + \gamma\, dZ_t,$

where $\bar{A}$ is the population-wide average productivity, $\theta > 0$ controls the speed of mean reversion, $\gamma > 0$ is the volatility of the productivity shock, and Z_t is another Brownian motion, independent of W_t.

Let $u(M, A)$ denote the stationary value function — the maximum expected utility an agent can achieve starting at state (M, A). In the infinite-horizon case, the value function satisfies the stationary Hamilton–Jacobi–Bellman (HJB) equation, which takes the form:

$\rho u(M, A) = \max_C \left\{aC - \frac{1}{2}bC^2 + \eta M - \frac{1}{2}\theta M^2 + \frac{\partial u}{\partial M} \cdot (AM - C) + \theta(\bar{A} - A) \cdot \frac{\partial u}{\partial A} + \nu \frac{\partial^2 u}{\partial M^2} + \kappa \frac{\partial^2 u}{\partial A^2}\right\}.$

The agent's control C enters both the utility function and the drift of the power dynamics. Solving the first-order condition for the optimal control yields:

$C^*(M, A) = \frac{a - \partial_M u(M,A)}{b}.$

Substituting this optimal policy back into the HJB equation yields a fully nonlinear second-order partial differential equation for the value function:

$\rho u(M, A) = \frac{(a-\partial_M u)^2}{2b} + \eta M - \frac{1}{2}\theta M^2 + AM \cdot \partial_M u + \theta(\bar{A} - A) \cdot \partial_A u + \nu \partial^2_{MM} u + \kappa \partial^2_{AA} u.$

Here, $\nu = \frac{1}{2}\sigma^2$ and $\kappa = \frac{1}{2}\gamma^2$ are the diffusion coefficients corresponding to stochastic fluctuations in power and productivity, respectively. The term $(a - \partial_M u)^2/(2b)$ captures the indirect utility derived from optimally chosen consumption. The terms $\eta M - \frac{1}{2}\theta M^2$ represent net utility from power. The terms involving the derivatives $\partial_M u$ and $\partial_A u$ represent how future utility responds to changes in the state variables via drift, while the second derivatives capture how uncertainty affects expected future value through the curvature of u.

Next, we characterize the evolution of the joint density $m(M, A)$ of the population across power and productivity states. Under optimal behavior, the density evolves according to the stationary Fokker–Planck–Kolmogorov (FPK) equation:

$0 = -\frac{\partial}{\partial M}\left((AM - C^*) \cdot m(M, A)\right) - \frac{\partial}{\partial A}\left(\theta(\bar{A} - A) \cdot m(M, A)\right) + \nu \frac{\partial^2 m}{\partial M^2} + \kappa \frac{\partial^2 m}{\partial A^2}$.

This equation is the forward-time analog of the HJB and describes how the distribution $m(M, A)$ reaches a statistical steady state under the influence of both the agents' decisions (reflected in the drift terms) and the exogenous randomness (captured by the diffusion terms). The first term on the right-hand side reflects how the population flows in the power dimension as agents consume and reinvest. The second term captures the reversion of productivity levels across agents toward the mean $\bar{A}$. The third and fourth terms reflect the random spread (diffusion) of the population in each dimension due to idiosyncratic shocks.

Together, the system consists of the coupled HJB and FPK equations, along with the control condition and a normalization constraint:

$\int\int m(M, A)\, dM\, dA = 1$.

The complete stationary mean field game system is:

$$\boxed{\begin{cases} \rho u(M, A) = \frac{(a - \partial_M u)^2}{2b} + \eta M - \frac{1}{2}\theta M^2 + AM \cdot \partial_M u + \theta(\bar{A} - A) \cdot \partial_A u + \nu \partial_{MM} u + \kappa \partial_{AA} u \\ 0 = -\partial_M\left((AM - C(M, A)) \cdot m\right) - \partial_A\left(\theta(\bar{A} - A) \cdot m\right) + \nu \partial^2_{MM} m + \kappa \partial^2_{AA} m \\ C(M, A) = \frac{a - \partial_M u(M, A)}{b} \\ \iint m(M, A)\, dM\, dA = 1 \end{cases}}$$

Solving this system gives us the stationary equilibrium: the optimal consumption policy $C^*(M, A)$, the value function $u(M, A)$, and the invariant distribution $m(M, A)$. Once the density is obtained, one can compute inequality statistics, such as the Gini coefficient of the power distribution, or analyze the tail behavior of $m(M, A)$ to examine whether a Pareto law emerges naturally from the joint dynamics of power and stochastic productivity.

This system, though analytically challenging due to its nonlinearity and dimensionality, offers a powerful and micro-founded framework to study endogenous inequality, persistent power concentration, and the long-run macro-distributional consequences of individual decision-making under uncertainty.

8 Appendix 2. Time-Dependent Mean Field Game Model with Power and Stochastic Productivity

In this section, we formulate the dynamic, infinite-horizon version of a mean field game (MFG) in which agents seek to optimally manage consumption over time, given their individual levels of power and productivity. The core feature of this framework is that while each agent makes decentralized consumption decisions, these decisions interact through the endogenous evolution of the population distribution over the economic state space, resulting in a dynamic equilibrium that adjusts both micro- and macro-level quantities.

Each agent is characterized by two state variables: power $M(t)$ and productivity $A(t)$. The power variable M reflects an agent's accumulated influence or capital, which generates output, while the productivity A governs how efficiently power is translated into income or production. The dynamics of power are stochastic and governed by the following controlled stochastic differential equation:

$dM(t) = (A(t)M(t) - C(t))\, dt + \sigma\, dW_t,$

where $C(t)$ is the agent's consumption rate, and $\sigma > 0$ represents the volatility of random shocks to power. In parallel, the productivity level $A(t)$ evolves exogenously as a mean-reverting Ornstein–Uhlenbeck process:

$dA(t) = \theta(\bar{A}(t) - A(t))dt + \gamma\, dZ_t,$

where $\theta > 0$ is the rate at which productivity reverts to the population mean $\bar{A}(t)$, $\gamma > 0$ is the volatility of productivity shocks, and Z_t is a standard Brownian motion independent of W_t. The time-varying average productivity level $\bar{A}(t)$ is defined endogenously as the mean of A under the current population distribution:

$\bar{A}(t) = \iint A \cdot m(t, M, A)\, dM\, dA,$

where $m(t, M, A)$ is the density of agents in state (M, A) at time t.

Each agent aims to maximize expected lifetime utility over an infinite horizon with exponential discounting. The utility function is linear-quadratic in both consumption and power:

$\max_{C(\cdot)} \mathbb{E}\left[\int_0^\infty e^{-\rho t}\left(aC(t) - \frac{1}{2}bC(t)^2 + \eta M(t) - \frac{1}{2}\theta M(t)^2\right) dt\right],$

where $\rho > 0$ is the time preference (discount rate), $a, b > 0$ govern the marginal utility and curvature in consumption, and $\eta, \theta > 0$ reflect the utility gain and cost from holding power.

Let $u(t, M, A)$ denote the value function, that is, the maximum expected utility from time t onward, starting at state (M, A). The evolution of u is governed by the Hamilton–Jacobi–Bellman (HJB) equation:

$-\partial_t u(t, M, A) - \nu\partial^2_{MM} u(t, M, A) - \kappa\partial^2_{AA} u(t, M, A) + H(M, A, \bar{A}(t), \partial_M u, \partial_A u) = 0,$

where $\nu = \frac{1}{2}\sigma^2$ and $\kappa = \frac{1}{2}\gamma^2$ are the diffusion coefficients, and the Hamiltonian H is given by:

$H(M, A, \bar{A}, p_M, p_A) = \frac{(a-p_M)^2}{2b} + \eta M - \frac{1}{2}\theta M^2 + AM \cdot p_M + \theta(\bar{A} - A) \cdot p_A.$

The Hamiltonian captures the trade-off between immediate utility from consumption and power and the marginal value of future power and productivity.

The first-order condition for the optimal control yields the optimal consumption policy:

$C^*(t, M, A) = \frac{a - \partial_M u(t,M,A)}{b}$.

This expression reflects the agent's decision to equate marginal utility of consumption with the shadow cost of reducing future power.

The evolution of the distribution $m(t, M, A)$ of agents across the power–productivity space is described by the forward Fokker–Planck–Kolmogorov (FPK) equation:

$\partial_t m(t, M, A) = -\partial_M \left((AM - C^*(t, M, A))m\right) - \partial_A \left(\theta(A - \bar{A}(t))m\right) + \nu \partial^2_{MM} m + \kappa \partial^2_{AA} m.$

This equation describes how the agent population evolves under optimal control, subject to idiosyncratic noise in both power and productivity. The terms involving first-order derivatives represent drift — due to both endogenous decisions and reversion toward average productivity — while the second-order terms account for random shocks.

Finally, the initial condition is specified by a probability density:

$m(0, M, A) = m_0(M, A), \quad \text{with} \quad \iint m_0(M, A)\, dM\, dA = 1.$

This completes the full time-dependent mean field game system with state-dependent stochastic control. The system couples the backward HJB equation with the forward FPK equation and forms the mathematical backbone of the economy's intertemporal dynamics. It serves as a foundational model for studying the endogenous emergence of wealth and power inequality under persistent productivity shocks and strategic consumption behavior. In the long-run, this system may converge to a stationary mean field game, which we explore next.

To complete the mathematical formulation of the time-dependent mean field game model, we now derive the Hamiltonian function used in the Hamilton–Jacobi–Bellman (HJB) equation. The Hamiltonian encodes both the agent's instantaneous reward from consumption and power, and the impact of their current state and control decisions on future value, as captured by the drift and gradient of the value function. The HJB equation is derived by applying the principle of dynamic programming and Itô's lemma to the value function, and then maximizing the resulting expression pointwise with respect to the control variable $C(t)$.

We recall that each agent's objective is to maximize expected utility over an infinite horizon:

$\max_{C(\cdot)} \ \mathbb{E}\left[\int_0^\infty e^{-\rho t} \left(aC(t) - \frac{1}{2}bC(t)^2 + \eta M(t) - \frac{1}{2}\theta M(t)^2\right) dt\right],$

subject to the state dynamics:

$dM(t) = (A(t)M(t) - C(t))\, dt + \sigma dW_t,$

$dA(t) = \theta(\bar{A}(t) - A(t))dt + \gamma dZ_t.$

Let $u(t, M, A)$ denote the value function. By the dynamic programming principle, this value function satisfies the HJB equation:

$-\partial_t u(t, M, A) = \max_C \ \mathcal{L}^C u(t, M, A) + r(M, C),$

where $\mathcal{L}^C$ is the infinitesimal generator (the second-order differential operator) applied to $u(t, M, A)$ under the given dynamics and control C, and $r(M, C)$ is the agent's instantaneous utility flow.

The reward function is defined by the utility integrand:
$r(M,C) = aC - \frac{1}{2}bC^2 + \eta M - \frac{1}{2}\theta M^2$.

Now we compute the infinitesimal generator $\mathcal{L}^C u$, using Itô's formula. For a general function u(t, M, A), and with the above SDEs, we have:

$$\mathcal{L}^C u(t,M,A) = \frac{\partial u}{\partial t} + (AM - C)\cdot\frac{\partial u}{\partial M} + \theta(\bar{A} - A)\cdot\frac{\partial u}{\partial A} + \frac{1}{2}\sigma^2\cdot\frac{\partial^2 u}{\partial M^2} + \frac{1}{2}\gamma^2\cdot\frac{\partial^2 u}{\partial A^2}.$$

Define:

- $\nu := \frac{1}{2}\sigma^2$,
- $\kappa := \frac{1}{2}\gamma^2$,

to simplify notation. Then we can express the HJB as:

$$-\partial_t u(t,M,A) = \max_C \left\{ aC - \frac{1}{2}bC^2 + \eta M - \frac{1}{2}\theta M^2 + (AM - C)\cdot\partial_M u + \theta(\bar{A}(t) - A)\cdot\partial_A u + \nu\partial^2_{MM}u + \kappa\partial^2_{AA}u \right\}.$$

We define the Hamiltonian function H as the entire right-hand side of the maximization:
$H(M, A, \bar{A}, \partial_M u, \partial_A u) = \max_C \left\{aC - \frac{1}{2}bC^2 - C\cdot\partial_M u + AM\cdot\partial_M u + \eta M - \frac{1}{2}\theta M^2 + \theta(\bar{A} - A)\cdot\partial_A u\right\}$.

We now maximize with respect to C. This is a standard quadratic maximization problem in C. The objective is:
$\Phi(C) = aC - \frac{1}{2}bC^2 - C\cdot\partial_M u$.

Taking the derivative with respect to C and setting it equal to zero gives:
$\frac{d\Phi}{dC} = a - bC - \partial_M u = 0 \quad\Rightarrow\quad C^* = \frac{a - \partial_M u}{b}$.

Plugging this back into the expression for $\Phi(C)$, we obtain the value of the Hamiltonian:
$\frac{(a-\partial_M u)^2}{2b}$.

Thus, the Hamiltonian becomes:
$H(M, A, \bar{A}, \partial_M u, \partial_A u) = \frac{(a-\partial_M u)^2}{2b} + \eta M - \frac{1}{2}\theta M^2 + AM\cdot\partial_M u + \theta(\bar{A} - A)\cdot\partial_A u$.

This form captures both the direct utility from consumption and power and the indirect utility from how changes in state variables affect future value. The term $(a - \partial_M u)^2/(2b)$ reflects the immediate gain from optimally chosen consumption. The linear and quadratic terms in M capture preferences over power. The terms involving $\partial_M u$ and $\partial_A u$ measure how the agent values changes in their state, and thus how the dynamics feed into long-run value.

Finally, the HJB equation becomes:
$-\partial_t u(t,M,A) = \frac{(a-\partial_M u)^2}{2b} + \eta M - \frac{1}{2}\theta M^2 + AM\cdot\partial_M u + \theta(\bar{A}(t) - A)\cdot\partial_A u + \nu\partial^2_{MM}u + \kappa\partial^2_{AA}u$.

This completes the derivation of the Hamiltonian and clarifies how the agent's optimization problem is embedded into the dynamics of the value function in the mean field game system.

9 Appendix 3. Derivation of the Hamiltonian with Costate Variables

In this section, we derive the Hamiltonian function that appears in the Hamilton–Jacobi–Bellman (HJB) equation of the mean field game, and interpret it using the canonical language of optimal control theory. In particular, we define and utilize costate variables to represent the marginal value of state variables—namely, power and productivity—in the agent's intertemporal optimization problem. These costate variables capture how small changes in the agent's current state influence the present value of expected future utility, and they appear naturally in the HJB equation via the chain rule in Itô calculus.

Each agent controls their consumption over time to maximize expected discounted utility:

$\max_{C(\cdot)} \ \mathbb{E}\left[\int_0^\infty e^{-\rho t}\left(aC(t) - \frac{1}{2}bC(t)^2 + \eta M(t) - \frac{1}{2}\theta M(t)^2\right)dt\right]$,

subject to the state dynamics:

$dM(t) = (A(t)M(t) - C(t))\,dt + \sigma\,dW_t$,

$dA(t) = \theta(\bar{A}(t) - A(t))dt + \gamma\,dZ_t$.

Here, $M(t)$ represents the agent's capital or power stock, and $A(t)$ denotes the productivity level that governs the effectiveness of capital accumulation. The stochastic shocks W_t and Z_t are independent Brownian motions. The time-varying population average productivity is denoted by $\bar{A}(t)$, defined endogenously via the current distribution $m(t, M, A)$.

Let $u(t, M, A)$ be the value function—the maximal expected utility of an agent beginning at state (M, A) at time t. Applying Itô's lemma and the dynamic programming principle yields the HJB equation, which involves a maximization over the control C. The integrand consists of two parts: the instantaneous utility flow and the contribution from changes in state variables:

$$-\partial_t u(t, M, A) = \max_C \left\{ aC - \frac{1}{2}bC^2 + \eta M - \frac{1}{2}\theta M^2 + (AM - C)\cdot\partial_M u + \theta(\bar{A}(t) - A)\cdot\partial_A u + \nu\partial^2_{MM}u + \kappa\partial^2_{AA}u \right\},$$

where we define the diffusion coefficients $\nu = \frac{1}{2}\sigma^2$ and $\kappa = \frac{1}{2}\gamma^2$.

To simplify notation and provide an interpretation consistent with standard optimal control theory, we now introduce the costate variables:

$p_M(t, M, A) := \frac{\partial u(t,M,A)}{\partial M}, \quad p_A(t, M, A) := \frac{\partial u(t,M,A)}{\partial A}$.

These measure the marginal utility value of an additional unit of power or productivity, respectively. Using these variables, the drift terms from the dynamics of $M(t)$ and $A(t)$ contribute to the HJB through the inner product:

$p_M(AM - C) + p_A\theta(\bar{A}(t) - A)$.

Thus, we can now write the Hamiltonian function H as:

$H(M, A, \bar{A}, p_M, p_A) = \max_C \left\{ aC - \frac{1}{2}bC^2 - C \cdot p_M + AM \cdot p_M + \eta M - \frac{1}{2}\theta M^2 + \theta(\bar{A} - A)\cdot p_A \right\}$.

This expression separates the agent's instantaneous utility from consumption and power, and the marginal value of how today's state affects future rewards. The control C appears in the quadratic expression:

$\max_C \left\{aC - \frac{1}{2}bC^2 - p_M C\right\}$.

The first-order condition for an interior maximum gives the optimal policy: $C^*(t, M, A) = \frac{a - p_M}{b}$.

Substituting back into the Hamiltonian, we obtain:

$H(M, A, \bar{A}, p_M, p_A) = \frac{(a-p_M)^2}{2b} + \eta M - \frac{1}{2}\theta M^2 + AM \cdot p_M + \theta(\bar{A} - A) \cdot p_A$.

Each term has a clear interpretation:

• The first term captures net utility from optimal consumption.

• The next two terms reflect utility from holding power (linear gain ηM and quadratic cost $-\frac{1}{2}\theta M^2$).

• The fourth term reflects the future value of power accumulation given current productivity A.

• The final term captures the future utility effect of expected changes in productivity.

By substituting this Hamiltonian back into the HJB equation, we obtain the full dynamic equation governing the agent's value function:

$-\partial_t u(t, M, A) = \frac{(a-\partial_M u)^2}{2b} + \eta M - \frac{1}{2}\theta M^2 + AM \cdot \partial_M u + \theta(\bar{A}(t) - A) \cdot \partial_A u + \nu \partial^2_{MM} u + \kappa \partial^2_{AA} u$.

This costate-based Hamiltonian formalism provides both a mathematically rigorous and economically intuitive foundation for the coupled HJB–FPK system that governs the mean field equilibrium. The terms p_M and p_A play critical roles in linking individual agent behavior to macro-level population dynamics.

REFERENCES

Carmona, R., & Delarue, F. (2018). Probabilistic Theory of Mean Field Games with Applications I: Mean Field FBSDEs, Control, and Games. Springer.

Carmona, R., & Delarue, F. (2018). Probabilistic Theory of Mean Field Games with Applications II: Mean Field Games with Common Noise and Master Equations. Springer.

Fearon, J. D. (1995). Rationalist explanations for war. International Organization, 49(3), 379–414.

Gong, L., & Zou, H.-F. (2003). Military spending and stochastic growth. Journal of Economic Dynamics and Control, 28(1), 153–170.

Hobbes, T. (1651). Leviathan. Oxford University Press.

Huang, M., Malhamé, R. P., & Caines, P. E. (2006). Large population stochastic dynamic games: Closed-loop McKean–Vlasov systems and the Nash certainty equivalence principle. Communications in Information and Systems, 6(3), 221–252.

Lasry, J.-M., & Lions, P.-L. (2007). Mean field games. Japanese Journal of Mathematics, 2(1), 229–260.

Nozick, R. (1974). Anarchy, State, and Utopia. Basic Books.

Powell, R. (2006). War as a commitment problem. International Organization, 60(1), 169–203.

Tilly, C. (1990). Coercion, Capital, and European States, AD 990–1992. Blackwell.

Zou, H.-F. (1995). A dynamic model of capital and arms accumulation. Journal of Economic Dynamics and Control, 19(1–2), 371–393.

Zou, H.-F. (1997). Dynamic analysis in the Viner model of mercantilism. Journal of International Money and Finance, 16(4), 637–651.