Militarization and Capital Accumulation: A Mean Field Game of Symmetric Nations under Strategic Externalities

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May 3, 2025

Abstract

This paper develops a dynamic mean field game model of capital accumulation and militarization among symmetric countries facing stochastic shocks. Each country optimizes consumption, productive investment, and military spending while responding to the average militarization level of others. We derive the coupled Hamilton–Jacobi–Bellman and Fokker–Planck equations governing equilibrium, and prove existence and uniqueness under strategic externalities. A linear–quadratic–Gaussian version yields closed-form feedback rules and forward dynamics. Simulations reveal that low internalization of global militarization leads to arms traps and stagnation, while stronger externality awareness supports growth and restraint. The model offers a rigorous framework to study decentralized security dilemmas and their macroeconomic consequences without assuming hierarchy or conflict.

Key words: Mean Field Games, Militarization, Strategic Externalities, Capital Accumulation, Arms Race

1 Introduction

Military investment decisions are among the most consequential and persistent features of state behavior. Even in the absence of active conflict or coercion, countries routinely allocate significant resources toward building and maintaining military capacity. This arms accumulation often occurs not because of direct rivalry or imminent threats, but because each nation fears the aggregate military posture of others. In such environments, individual optimization under strategic interdependence can generate collective outcomes that are inefficient, unstable, or self-reinforcing. This paper develops a dynamic, decentralized framework to formally analyze how militarization and capital accumulation co-evolve across a continuum of symmetric countries when each optimizes under uncertainty and strategic externalities.

We introduce a continuous-time mean field game model in which all countries are symmetric ex ante and face stochastic dynamics in capital and military stocks. Each country chooses consumption, investment in productive capital, and military expenditure to maximize intertemporal utility, subject to a resource constraint and depreciation. Crucially, countries do not directly interact with specific rivals but are instead influenced by the mean field—that is, the average level of militarization across the global population. The strategic tension arises because the mean field enters negatively into both the production function (representing resource or security inefficiencies from global tension) and the utility function (reflecting strategic anxiety). Each country optimizes by taking this aggregate militarization level as given, but in equilibrium, the mean field must coincide with the average outcome of all countries' decentralized behavior.

We analyze this model through the coupled system of a Hamilton–Jacobi–Bellman (HJB) equation (describing the optimal policy of a representative agent), a Fokker–Planck–Kolmogorov (FPK) equation (describing the evolution of the population distribution of states), and a fixed-point condition equating perceived and realized average militarization. We show that under regularity conditions, this system admits a unique equilibrium. In a simplified linear–quadratic–Gaussian (LQG) version of the model, we derive closed-form expressions for optimal investment policies and simulate the joint evolution of capital, militarization, and welfare. Our numerical results reveal that small differences in how much countries internalize the disutility of global militarization lead to dramatically different long-run outcomes: either a militarized equilibrium with stagnant growth and low welfare, or a restrained equilibrium with capital accumulation and higher well-being. These results demonstrate that systemic militarization can emerge endogenously—driven by decentralized fear and strategic symmetry—even when no country intends aggression.

The contribution of this paper is threefold. First, it offers a tractable and rigorous framework for studying the dynamic co-evolution of militarization and capital in a fully symmetric, non-hierarchical world. Second, it shows how strategic externalities, even when modest, can fundamentally alter macroeconomic outcomes and security equilibria. Third, it provides a foundation for analyzing policy interventions, such as disarmament incentives or alliance formation, that might shift equilibrium outcomes without requiring centralized enforcement. While much of the existing literature focuses on pairwise rivalry, hegemonic actors, or repeated signaling, our framework highlights the endogenous emergence of arms races under complete symmetry and decentralized rationality.

The remainder of the paper proceeds as follows. Section 2 introduces the model setup, including the stochastic dynamics, utility specification, and mean field structure. Section 3 derives the coupled equations that characterize equilibrium. Section 4 analyzes qualitative properties of the stationary distribution. Section 5 specializes the model to a linear–quadratic–Gaussian benchmark and derives closed-form solutions. Section 6 establishes existence and uniqueness of equilibrium using fixed-point arguments. Section 7 presents the explicit Fokker–Planck dynamics and demonstrates their consistency with the equilibrium feedback laws. Section 8 provides numerical simulations illustrating how different externality regimes shape capital, militarization, and welfare. Section 9 concludes and outlines promising extensions related to heterogeneity, institutional design, learning dynamics, and spatial conflict diffusion.

2 Model Setup

We consider a continuum of symmetric countries indexed by $i \in [0, 1]$, each of which simultaneously optimizes its own consumption, capital accumulation, and military investment under strategic interdependence with the rest of the population. The key state variables for each country i are its productive capital $k_i(t) \in \mathbb{R}+$ and military capital $m_i(t) \in \mathbb{R}+$, both of which evolve over continuous time $t \geq 0$. The countries are atomistic, but interact through a mean field represented by the population distribution of military stock.

2.1 State Dynamics

Each country controls its capital investment $i_{k,i}(t)$ and military investment $g_i(t)$, subject to depreciation and stochastic shocks. The capital and military dynamics follow:

$$dk_i(t) = \left(i_{k,i}(t) - \delta^k k_i(t)\right) dt + \sigma_k dW_i^k(t),$$

$$dm_i(t) = \left(g_i(t) - \delta^m m_i(t)\right) dt + \sigma_m dW_i^m(t),$$

where $\delta^k, \delta^m > 0$ are constant depreciation rates, and $W_i^k(t), W_i^m(t)$ are independent Brownian motions representing idiosyncratic shocks to capital and military dynamics, with volatilities $\sigma_k, \sigma_m > 0$.

2.2 Production and Budget Constraint

Output is produced linearly using capital and military resources, but is penalized by strategic congestion effects arising from average global militarization. The instantaneous budget constraint for country i is:

$$c_i(t) + i_{k,i}(t) + g_i(t) = Ak_i(t) + Bm_i(t) - D\bar{m}(t),$$

where:

- $c_i(t)$ is consumption,
- $A, B > 0$ represent the productivity of capital and military resources,
- $\bar{m}(t) = \int_0^1 m_j(t)\, dj$ is the mean militarization across the population,
- $D > 0$ captures the negative externality from global arms buildup on productive output (e.g., insecurity, arms races, inefficient resource use).

2.3 Utility Function

Each country maximizes expected discounted utility, which depends on consumption, military strength, and strategic externalities from average militarization. The instantaneous utility of country i is given by:

$$u_i(t) = a\, c_i(t) - \frac{1}{2} b\, c_i(t)^2 + \eta\, m_i(t) - \frac{1}{2}\phi\, m_i(t)^2 + \eta_1\, c_i(t)\, m_i(t) - \eta_2\, \bar{m}(t) - \eta_3\, c_i(t)\, \bar{m}(t),$$

where:

- $a > 0, b > 0$: utility parameters for concave consumption preferences,
- $\eta, \phi > 0$: represent the marginal benefit and cost of militarization,
- η_1: captures the consumption–security complementarity (more military allows safer consumption),

• $\eta_2, \eta_3 > 0$: represent strategic disutility from global militarization and its interaction with domestic consumption.

2.4 Objective Functional

Each country chooses control processes $\{i_{k,i}(t), g_i(t)\}_{t\geq 0}$ to maximize expected discounted utility over an infinite horizon:

$\max_{\{i_{k,i}(t),g_i(t)\}} \mathbb{E}\left[\int_0^\infty e^{-\rho t} u_i(t)\, dt\right]$,

subject to the state dynamics and the budget constraint, where $\rho > 0$ is the constant discount rate. All countries are ex ante identical and solve the same optimization problem, but interact through the evolving distribution $m(t, x)$ of military stocks across the population.

2.5 Population Distribution and Mean Field Interaction

Let $m(t, x)$ denote the density of countries at time t with state $x = (k, m) \in \mathbb{R}^2$. Then:

$\bar{m}(t) = \int_{\mathbb{R}^2} m \cdot m(t, k, m)\, dk\, dm$

feeds into the budget constraint and utility function of each player. This gives rise to a fully coupled forward–backward system, where each agent's optimal strategy affects, and is affected by, the population distribution.

In the next section, we derive the coupled Hamilton–Jacobi–Bellman (HJB) and Fokker–Planck (FPK) system that characterizes equilibrium in this symmetric mean field game.

3 The HJB–FPK System and Equilibrium Characterization

We now turn to the formal characterization of equilibrium in the symmetric mean field game introduced in Section 2. Each country seeks to optimally allocate resources between consumption, capital accumulation, and military investment, accounting for the strategic influence of the average militarization level across the population. The resulting system of equations features a backward Hamilton–Jacobi–Bellman (HJB) partial differential equation representing individual optimality and a forward Fokker–Planck–Kolmogorov (FPK) equation describing the evolution of the distribution of agents. These equations are coupled through the population-level mean field, and jointly characterize the decentralized rational expectations equilibrium.

3.1. The Individual Optimization Problem and the HJB Equation

Let $u(t, k, m)$ denote the value function of a representative country at time $t \geq 0$ and state $x = (k, m) \in \mathbb{R}^2_+$, where k and m represent capital and military stock, respectively. Given a path for the average military level $\bar{m}(t)$, the value function solves the dynamic programming equation:

$$\partial_t u(t,k,m) + \rho u(t,k,m) = \sup_{i_k, g} \Big\{ ac(k,m,i_k,g;\bar{m}) - \frac{1}{2} bc(k,m,i_k,g;\bar{m})^2 + \eta m - \frac{1}{2}\phi m^2$$
$$+ \eta_1 c(k,m,i_k,g;\bar{m}) m - \eta_2 \bar{m}(t) - \eta_3 c(k,m,i_k,g;\bar{m})\bar{m}(t)$$
$$+ \partial_k u(t,k,m)(i_k - \delta^k k) + \partial_m u(t,k,m)(g - \delta^m m)$$
$$+ \frac{1}{2}\sigma_k^2 \partial_{kk}^2 u(t,k,m) + \frac{1}{2}\sigma_m^2 \partial_{mm}^2 u(t,k,m) \Big\},$$

where consumption is determined residually by the instantaneous resource constraint:

$c(k,m,i_k,g;\bar{m}) = Ak + Bm - D\bar{m}(t) - i_k - g.$

The control variables i_k and g represent the investment rates in capital and military stocks. The agent maximizes utility from consumption and military security, net of both internal and strategic externalities arising from aggregate militarization.

3.2. Optimal Investment Policy and Feedback Form

Assuming interior solutions, the first-order conditions with respect to i_k and g yield the optimal feedback controls:

$$i_k(t,k,m) = \frac{1}{b + \eta_3 \bar{m}(t)} \left[a + \eta_1 m - b(Ak + Bm - D\bar{m}(t)) - \partial_k u(t,k,m) \right],$$

$$g(t,k,m) = \frac{1}{b + \eta_3 \bar{m}(t)} \left[a + \eta_1 m - b(Ak + Bm - D\bar{m}(t)) - \partial_m u(t,k,m) \right].$$

These policies are state-dependent and reflect the marginal value of investment relative to its opportunity cost and strategic incentives. The feedback form highlights how local investment is distorted by global arms levels $\bar{m}(t)$ through the utility interaction terms.

3.3. The Population Law and the Fokker–Planck Equation

Let $\mu(t,k,m)$ denote the joint density of countries over the state space $\mathbb{R}_+^2$ at time t. Given the optimal policy functions derived above, the evolution of the distribution of countries follows the Fokker–Planck–Kolmogorov equation:

$$\partial_t \mu(t,k,m) = -\,\partial_k \left[(i_k(t,k,m) - \delta^k k)\mu(t,k,m) \right] - \partial_m \left[(g(t,k,m) - \delta^m m)\mu(t,k,m) \right]$$
$$+ \frac{1}{2}\sigma_k^2 \partial_{kk}^2 \mu(t,k,m) + \frac{1}{2}\sigma_m^2 \partial_{mm}^2 \mu(t,k,m),$$

subject to an initial condition $\mu(0,k,m) = \mu_0(k,m) \in \mathcal{P}2(\mathbb{R}+^2)$. The first two terms describe deterministic drifts in state variables driven by optimal policy, while the last two represent diffusion due to uncertainty.

The average militarization level is defined consistently by:

$\bar{m}(t) = \int_{\mathbb{R}_+^2} m \cdot \mu(t,k,m)\, dk\, dm.$

This creates the key fixed-point interaction between micro-level optimization and macro-level distributional dynamics.

3.4. Definition of Mean Field Equilibrium

We now formally define equilibrium in the symmetric mean field game:

Definition 3.1 (Mean Field Equilibrium). A mean field equilibrium is a triple $(u(t,k,m), \mu(t,k,m), \bar{m}(t))$ such that:

1. $u(t,k,m)$ solves the HJB equation given $\bar{m}(t)$,

2. $\mu(t,k,m)$ solves the Fokker–Planck equation given optimal feedback controls (i_k, g),

3. The consistency condition $\bar{m}(t) = \int m\,\mu(t,k,m)\,dk\,dm$ holds for all $t \geq 0$.

This equilibrium concept captures a decentralized rational expectations environment in which all countries behave optimally, taking the mean field as given, while the mean field is itself generated by the aggregation of their decisions.

4 Stationary Equilibrium and Qualitative Structure

To gain deeper insight into the long-run implications of decentralized arms and capital investment, we now examine the structure of stationary equilibria. We focus on time-invariant solutions to the coupled HJB–FPK system, corresponding to the case in which all agents face a constant mean field $\bar{m}$, adopt time-invariant feedback controls, and the population distribution converges to a stationary probability measure over the state space $\mathbb{R}_+^2$. The goal of this section is twofold: (1) to reduce the complexity of the full dynamic system by eliminating time dependence, and (2) to extract qualitative properties of global militarization and economic development under symmetric strategic interaction.

4.1. Stationary HJB Equation

In a stationary equilibrium, the value function $u(k,m)$ no longer depends on time, and satisfies the time-independent HJB equation:

$$\begin{aligned}\rho u(k,m) = \sup_{i_k,g} \Big\{ & ac(k,m,i_k,g;\bar{m}) - \frac{1}{2}bc(k,m,i_k,g;\bar{m})^2 + \eta m - \frac{1}{2}\phi m^2 \\ & + \eta_1 c(k,m,i_k,g;\bar{m})m - \eta_2\bar{m} - \eta_3 c(k,m,i_k,g;\bar{m})\bar{m} \\ & + \partial_k u(k,m)(i_k - \delta^k k) + \partial_m u(k,m)(g - \delta^m m) \\ & + \frac{1}{2}\sigma_k^2\partial_{kk}^2 u(k,m) + \frac{1}{2}\sigma_m^2\partial_{mm}^2 u(k,m)\Big\}.\end{aligned}$$

The stationary consumption rule is given by:

$c(k,m,i_k,g;\bar{m}) = Ak + Bm - D\bar{m} - i_k - g$,

where $\bar{m} \in \mathbb{R}_+$ is the endogenous average military level that will be determined in equilibrium.

4.2. Stationary Fokker–Planck Equation

The associated stationary distribution $\mu^*(k,m)$ over capital and military stocks satisfies the stationary Fokker–Planck equation:

$$\begin{aligned}0 = & -\partial_k\left[(i_k(k,m) - \delta^k k)\mu(k,m)\right] - \partial_m\left[(g(k,m) - \delta^m m)\mu(k,m)\right] \\ & + \frac{1}{2}\sigma_k^2\partial_{kk}^2\mu(k,m) + \frac{1}{2}\sigma_m^2\partial_{mm}^2\mu(k,m),\end{aligned}$$

subject to the normalization condition:

$\int_{\mathbb{R}_+^2} \mu^*(k,m)\,dk\,dm = 1$.

The equilibrium mean field is:

$\bar{m} = \int_{\mathbb{R}_+^2} m\,\mu^*(k,m)\,dk\,dm$.

This equation determines the fixed point of the distribution given the individual responses to the mean field $\bar{m}$.

4.3. Qualitative Structure of Stationary Solutions

From the structure of the HJB equation and the FPK equation, several qualitative insights can be derived:

• Strategic amplification: Higher values of η_2 and η_3—i.e., stronger disutility from global militarization—reduce optimal arms m^* and induce lower $\bar{m}$ in equilibrium. This feedback is stabilizing.

• Consumption–military complementarity: When $\eta_1 > 0$, military investment enhances the marginal value of consumption, encouraging higher arms in states with abundant capital—potentially destabilizing when combined with low depreciation or low variance.

• Diffusion stabilizes dispersion: Larger values of σ_k and σ_m (i.e., higher uncertainty in capital or arms dynamics) smooth out the distribution $\mu^*(k, m)$ and reduce concentration at high-militarization equilibria.

• Multiple equilibria: If the utility function is sufficiently nonlinear in m and interactions $\bar{m}$, the equilibrium fixed point may not be unique. In particular, strategic complementarities (large η_1, small η_2) may yield both high- and low-arms steady states, separated by an unstable threshold.

These features suggest that in a purely symmetric world, even without a dominant country, endogenous militarization may arise from decentralized strategic anxiety—especially under low costs of arms and strong strategic spillovers.

5 Linear–Quadratic–Gaussian (LQG) Benchmark Model

To obtain explicit closed-form solutions and facilitate quantitative analysis, we specialize the general model developed in Sections 2–4 to a linear–quadratic–Gaussian (LQG) framework. This allows us to reduce the full coupled HJB–FPK system to a pair of solvable linear equations, while preserving the core economic and strategic features: consumption–military trade-offs, decentralized investment behavior, and mean field externalities through average arms accumulation.

5.1. Simplified Structure of the LQG Model

We impose the following assumptions to transform the general framework into the LQG setting:

1. State dynamics are deterministic in mean with additive Gaussian noise:

$dk_t = \left(i_k(t) - \delta^k k_t\right) dt + \sigma_k dW_t^k$,

$dm_t = \left(g(t) - \delta^m m_t\right) dt + \sigma_m dW_t^m$,

with constant volatilities σ_k, σ_m and depreciation rates $\delta^k, \delta^m > 0$.

2. Production is linear, and consumption is defined residually:

$c(t) = Ak_t + Bm_t - D\bar{m}(t) - i_k(t) - g(t)$,

where $A, B, D > 0$, and $\bar{m}(t) = \mathbb{E}[m_t]$ in symmetric equilibrium.

3. Instantaneous utility is quadratic:

$$u(t) = ac(t) - \frac{1}{2}bc(t)^2 + \eta m_t - \frac{1}{2}\phi m_t^2 + \eta_1 c(t)m_t - \eta_2\bar{m}(t) - \eta_3 c(t)\bar{m}(t),$$

with all parameters positive, capturing diminishing marginal utility and strategic disutility.

5.2. Quadratic Value Function Ansatz

We conjecture that the value function takes a quadratic form:

$u(t, k, m) = -\frac{1}{2}\gamma_k(t)k^2 - \frac{1}{2}\gamma_m(t)m^2 - \gamma_{km}(t)km + \theta_k(t)k + \theta_m(t)m + \xi(t),$

where $\gamma_k(t), \gamma_m(t), \gamma_{km}(t) \geq 0$, and $\theta_k(t), \theta_m(t), \xi(t)$ are functions to be determined. This allows closed-form derivation of feedback controls and evolution equations.

5.3. Optimal Controls in Feedback Form

Using the HJB equation and first-order conditions, the optimal investment policies are:

$i_k(t, k, m) = \partial_k u(t, k, m) = -\gamma_k(t)k - \gamma_{km}(t)m + \theta_k(t),$

$g(t, k, m) = \partial_m u(t, k, m) = -\gamma_m(t)m - \gamma_{km}(t)k + \theta_m(t).$

Substituting into the budget constraint yields a linear expression for consumption:

$c(t, k, m; \bar{m}) = Ak + Bm - D\bar{m}(t) - i_k(t, k, m) - g(t, k, m).$

This form makes the HJB equation linear in k, m, and the diffusion terms constant due to additive Brownian motion, enabling explicit solution of the Riccati equations for $\gamma(t)$.

5.4. Mean Field and Closed-Form Consistency

Let $\bar{k}(t) = \mathbb{E}[k_t], \bar{m}(t) = \mathbb{E}[m_t]$. Taking expectations of the SDEs:

$\frac{d}{dt}\bar{k}(t) = \mathbb{E}[i_k(t, k_t, m_t)] - \delta^k \bar{k}(t),$

$\frac{d}{dt}\bar{m}(t) = \mathbb{E}[g(t, k_t, m_t)] - \delta^m \bar{m}(t).$

Using the linear feedback policies and the fact that the system is jointly Gaussian (under LQG assumptions), we compute:

$\frac{d}{dt}\bar{k}(t) = -\gamma_k(t)\bar{k}(t) - \gamma_{km}(t)\bar{m}(t) + \theta_k(t) - \delta^k \bar{k}(t),$

$\frac{d}{dt}\bar{m}(t) = -\gamma_m(t)\bar{m}(t) - \gamma_{km}(t)\bar{k}(t) + \theta_m(t) - \delta^m \bar{m}(t).$

This closed system can be solved forward in time for $(\bar{k}(t), \bar{m}(t))$, once the coefficients are solved backward from the Riccati ODEs derived from the HJB.

5.5. Equilibrium Characterization in LQG

The LQG equilibrium consists of:

1. A value function $u(t, k, m)$ quadratic in the state variables;
2. Linear feedback policies $i_k(t, k, m), g(t, k, m)$;
3. A forward system for $(\bar{k}(t), \bar{m}(t))$ consistent with the law of motion;
4. A fixed point condition: $\bar{m}(t) = \mathbb{E}[m_t]$, closing the mean field loop.

These elements characterize the symmetric decentralized equilibrium with strategic military externalities under rational expectations and Gaussian uncertainty.

6 Existence of LQG Mean Field Equilibrium

In this section, we formally establish the existence of a symmetric decentralized equilibrium in the linear–quadratic–Gaussian (LQG) benchmark model. We consider the setting developed in Section 5, in which each agent faces linear dynamics, quadratic utility, and Gaussian uncertainty. The key to existence lies in demonstrating that a fixed point exists between the backward optimization (the HJB problem) and the forward population law of motion (the Fokker–Planck equation). Under LQG structure, these problems can be reduced to a backward Riccati system and a forward linear ordinary differential equation (ODE) system, both of which admit unique solutions under standard conditions.

6.1. Summary of the LQG Structure

Recall from Section 5 that the LQG equilibrium consists of the following elements:

1. A quadratic value function $u(t,k,m)$ of the form:

$u(t,k,m) = -\frac{1}{2}\gamma_k(t)k^2 - \frac{1}{2}\gamma_m(t)m^2 - \gamma_{km}(t)km + \theta_k(t)k + \theta_m(t)m + \xi(t)$,

where $\gamma_k(t), \gamma_m(t), \gamma_{km}(t), \theta_k(t), \theta_m(t), \xi(t)$ are continuous, differentiable functions of time.

2. Linear feedback policies:

$i_k(t,k,m) = -\gamma_k(t)k - \gamma_{km}(t)m + \theta_k(t), \quad g(t,k,m) = -\gamma_m(t)m - \gamma_{km}(t)k + \theta_m(t)$.

3. A forward linear ODE system for the mean field states:

$\dot{\bar{k}}(t) = i_k(t,\bar{k}(t),\bar{m}(t)) - \delta^k\bar{k}(t)$,

$\dot{\bar{m}}(t) = g(t,\bar{k}(t),\bar{m}(t)) - \delta^m\bar{m}(t)$,

with given initial values $\bar{k}(0), \bar{m}(0) \in \mathbb{R}_+$.

4. A fixed-point condition:

$\bar{m}(t) = \mathbb{E}[m_t]$,

ensuring that the average military stock in the population equals the expected value under optimal decentralized behavior.

We now prove that this system admits a unique solution.

6.2. Backward Riccati System: Existence and Uniqueness

Given the LQG structure, the HJB equation reduces to a matrix Riccati differential equation. The coefficients $\gamma_k(t), \gamma_m(t), \gamma_{km}(t)$ satisfy a coupled system of backward ODEs. Under standard conditions (e.g., positive definiteness of quadratic cost coefficients, boundedness of drift terms), the following classical result applies (see Yong & Zhou, 1999; Bardi & Capuzzo-Dolcetta, 1997):

Lemma 6.1 (Riccati Well-Posedness). The Riccati system derived from the LQG HJB equation with quadratic terminal condition admits a unique global solution $(\gamma_k(t), \gamma_m(t), \gamma_{km}(t)) \in C^1([0,T])$ for any finite horizon $T > 0$, or in the limit $T \to \infty$ under stabilization conditions.

Once the Riccati terms are determined, the linear terms $\theta_k(t), \theta_m(t), \xi(t)$ can be solved using linear backward ODEs with coefficients depending on the known mean field $\bar{m}(t)$, which we treat provisionally as fixed.

6.3. Forward Mean Field ODE: Existence and Uniqueness

For any given set of feedback coefficients, the optimal controls (i_k^, g^) are linear in (\bar{k}(t), \bar{m}(t)). Therefore, the system:

$\dot{\bar{k}}(t) = -\gamma_k(t)\bar{k}(t) - \gamma_{km}(t)\bar{m}(t) + \theta_k(t) - \delta^k\bar{k}(t),$

$\dot{\bar{m}}(t) = -\gamma_m(t)\bar{m}(t) - \gamma_{km}(t)\bar{k}(t) + \theta_m(t) - \delta^m\bar{m}(t),$

is a linear system of ODEs in $\mathbb{R}^2$, driven by time-dependent coefficients. Classical theory guarantees:

Lemma 6.2 (Linear ODE Well-Posedness). For any initial condition $(\bar{k}(0), \bar{m}(0)) \in \mathbb{R}^2$, and continuous coefficient functions $\gamma_\cdot(t), \theta_\cdot(t)$, there exists a unique global solution $(\bar{k}(t), \bar{m}(t)) \in C^1([0,T])$.

This establishes the forward component of the MFG system for any given mean field path $\bar{m}(t)$.

6.4. Fixed-Point Argument and Continuity

To complete the proof, we must show that the map:

$\bar{m}(t) \mapsto \gamma_\cdot(t), \theta_\cdot(t) \mapsto (\bar{k}(t), \bar{m}(t))$

has a fixed point, i.e., a self-consistent $\bar{m}(t)$ that is both used in the optimization and generated by the resulting dynamics. This follows by the Schauder fixed-point theorem:

Proposition 6.3 (Existence of LQG MFG Equilibrium). Let $\mathcal{M} \subset C([0,T])$ be the set of continuous paths $\bar{m}(t)$ with uniform boundedness. Then the operator:

$\mathcal{T} : \bar{m}(\cdot) \mapsto \mathbb{E}[m_t]$ under optimal controls is continuous and maps a convex compact set into itself. Therefore, $\mathcal{T}$ has a fixed point.

Hence, there exists a path $\bar{m}(t) \in C([0,T])$ consistent with the rational expectations equilibrium under optimal decentralized behavior.

6.5. Conclusion of the Proof

We conclude that the coupled backward–forward LQG system admits a unique solution:

- A value function $u(t,k,m)$ quadratic in states;
- Linear optimal feedback policies $i_k(t,k,m), g(t,k,m)$;
- A consistent forward solution $(\bar{k}(t), \bar{m}(t))$;
- A fixed point $\bar{m}(t) = \mathbb{E}[m_t]$.

This establishes the existence of a symmetric decentralized mean field equilibrium under rational expectations in the presence of strategic military spillovers and Gaussian uncertainty.

7 Explicit FPK Equations and Fixed Point Structure under Simplified Utility

In this section, we derived the explicit forward Fokker–Planck–Kolmogorov (FPK) equation under a strategically simplified utility function. We now turn to the mathematical structure of the fixed point problem implied by the mean field equilibrium. In particular, we show that the system—comprising the backward HJB equation, the forward FPK equation, and the population mean field closure—admits a well-defined fixed point. Under suitable regularity and monotonicity assumptions, we also show that the equilibrium is unique.

7.1. Recap: Population Distribution and Fixed Point Problem

Given optimal controls derived from the HJB equation and a conjectured path $\bar{m}(t) \in C([0,T],\mathbb{R})$, the FPK equation uniquely determines the time-evolving joint density $\mu(t,k,m)$ of capital k and military stock m. The consistency condition in mean field games requires:

$\bar{m}(t) = \mathbb{E}[m_t] = \int_{\mathbb{R}_+^2} m \cdot \mu(t,k,m)\, dk\, dm.$

Thus, the equilibrium problem is equivalent to finding a fixed point of the mapping:

$\mathcal{T} : \bar{m}(\cdot) \mapsto \int_{\mathbb{R}_+^2} m \cdot \mu^{\bar{m}}(t,k,m)\, dk\, dm,$

where $\mu^{\bar{m}}(t,k,m)$ is the solution to the FPK equation corresponding to the drift coefficients generated by $\bar{m}(t)$.

7.2. Functional Form of the FPK Drift

Let us recall the FPK system with the linear drift derived from the optimal controls. Denote $x = (k,m)$, and define the drift vector:

$b^{\bar{m}}(t,x) = \begin{pmatrix} i_k(t,k,m) - \delta^k k \\ g(t,k,m) - \delta^m m \end{pmatrix} = -B(t)x + \theta(t),$

where:

• $B(t) \in \mathbb{R}^{2\times 2}$ is a matrix of feedback coefficients with components $\gamma_k(t) + \delta^k, \gamma_m(t) + \delta^m$, and $\gamma_{km}(t)$,

• $\theta(t) \in \mathbb{R}^2$ is a vector of affine terms $(\theta_k(t), \theta_m(t))$.

The FPK equation thus becomes:

$\partial_t \mu(t,x) = -\nabla \cdot (b^{\bar{m}}(t,x)\mu(t,x)) + \frac{1}{2}\nabla \cdot (\Sigma \nabla \mu(t,x)),$

with $\Sigma = \mathrm{diag}(\sigma_k^2, \sigma_m^2)$.

7.3. Existence of a Fixed Point

Let us now state and prove the formal existence result.

Theorem 7.1 (Existence of Mean Field Fixed Point).

Let $\bar{m}(t) \in \mathcal{M} \subset C([0,T],\, \mathbb{R})$ be a bounded set of continuous candidate paths. Then the map:

$\mathcal{T} : \bar{m}(\cdot) \mapsto \int_{\mathbb{R}_+^2} m \cdot \mu^{\bar{m}}(t,k,m)\, dk\, dm$

is continuous and maps $\mathcal{M}$ into itself. Hence, $\mathcal{T}$ admits a fixed point $\bar{m}^*(t)$ by Schauder's fixed point theorem.

Proof.

1. Well-posedness of the FPK equation:

For every bounded and continuous $\bar{m}(t)$, the drift term $b^{\bar{m}}(t,x)$ is Lipschitz continuous in x, uniformly in t, because the coefficients $\gamma_k(t), \gamma_m(t), \gamma_{km}(t), \theta_k(t), \theta_m(t)$ are derived from the backward Riccati equation (well-posed under LQG structure). The diffusion term Σ is constant and strictly positive definite. Therefore, standard parabolic PDE theory (see Bogachev et al., 2016) guarantees existence and uniqueness of $\mu^{\bar{m}}(t,x) \in C([0,T],\, \mathcal{P}_2(\mathbb{R}^2))$.

2. Continuity of $\mathcal{T}$:

Suppose $\bar{m}_n(t) \to \bar{m}(t)$ uniformly in t. Then the corresponding drift terms $b^{\bar{m}_n}(t,x) \to b^{\bar{m}}(t,x)$ uniformly on compact subsets. The solutions $\mu^{\bar{m}_n}(t,x)$

to the FPK equation depend continuously on the drift by stability of linear parabolic PDEs (Bogachev et al., Theorem 6.1). Hence,

$\bar{m}_n\prime(t) = \int m\mu^{\bar{m}_n}(t,k,m)\,dk\,dm \longrightarrow \int m\mu^{\bar{m}}(t,k,m)\,dk\,dm = \mathcal{T}\bar{m}$,

establishing continuity.

3. Compactness and invariance:

The set $\mathcal{M} = \{\bar{m} \in C([0,T]) : \|\bar{m}\|_\infty \leq M\}$ is convex and compact under the sup norm. For bounded initial conditions and bounded drift terms, the moment bound $\bar{m}(t) \leq C$ holds for all t, ensuring $\mathcal{T}[\bar{m}] \in \mathcal{M}$.

By Schauder's theorem, the continuous, self-mapping operator $\mathcal{T}$ admits a fixed point $\bar{m}^*(t) \in \mathcal{M}$.

7.4. Uniqueness of Equilibrium Path

Uniqueness of the mean field equilibrium requires an additional contraction property or monotonicity condition.

Proposition 7.2 (Uniqueness under Strategic Concavity).

If $\eta_2 > 0, D > 0$, and the feedback coefficients $\gamma_k(t), \gamma_m(t)$ are monotone in $\bar{m}(t)$, then $\mathcal{T}$ is a contraction in a weighted norm, and the fixed point $\bar{m}^*(t)$ is unique.

Sketch of Proof.

Let $\bar{m}_1(t), \bar{m}_2(t) \in \mathcal{M}$. Let $\mu^1(t,k,m), \mu^2(t,k,m)$ be the corresponding FPK solutions. Since $b^{\bar{m}_1} - b^{\bar{m}_2} = -\delta b(t,x)$ is Lipschitz in $\bar{m}$, stability results for linear FPK equations (via Gronwall inequality inL^1 or Wasserstein norms) imply:

$\|\mu^1(t,\cdot) - \mu^2(t,\cdot)\|_{TV} \leq C\|\bar{m}_1 - \bar{m}2\|\infty$.

Therefore,

$|\mathcal{T}\bar{m}_1 - \mathcal{T}\bar{m}_2| \leq \int |m| \cdot |\mu^1 - \mu^2| \leq C\|\bar{m}_1 - \bar{m}2\|\infty$.

If $C < 1$, $\mathcal{T}$ is a contraction. Thus uniqueness follows.

The existence and uniqueness results confirm that even under uncertainty and continuous heterogeneity, decentralized countries acting in response to a strategic mean field of militarization will coordinate around a unique, time-consistent path $\bar{m}(t)$. This result lays the foundation for simulation, sensitivity analysis, and welfare policy in the symmetric version of the model.

8 Simulated Equilibrium Dynamics under Varying Externalities

In this section, we compute and analyze the equilibrium time paths of average capital $\bar{k}(t)$, average military stock $\bar{m}(t)$, and expected utility flow $\bar{u}(t)$ in the symmetric mean field game framework, focusing particularly on the role of the strategic disutility parameter η_2. This parameter governs the extent to which each country internalizes the external cost of global militarization. We compare the dynamics under two representative regimes: one where η_2 is small, indicating weak strategic externality awareness, and another where η_2 is large, representing heightened sensitivity to systemic over-militarization.

We work with the simplified utility function $u(t) = ac(t) - \frac{1}{2}bc(t)^2 + \eta m(t) - \frac{1}{2}\phi m(t)^2 - \eta_2\bar{m}(t)$, along with the instantaneous resource constraint $c(t) =$

$Ak(t) + Bm(t) - D\bar{m}(t) - i_k(t) - g(t)$. Under the linear–quadratic–Gaussian assumptions of the model, the optimal policies take linear feedback form: $i_k(t, k, m) = -\gamma_k k - \gamma_{km} m + \theta_k$, and $g(t, k, m) = -\gamma_m m - \gamma_{km} k + \theta_m$, where all coefficients are held constant over time. Substituting these policies into the law of motion for expected capital and military stocks, we obtain a forward linear system of ordinary differential equations:

$\frac{d}{dt}\bar{k}(t) = -(\gamma_k + \delta^k)\bar{k}(t) - \gamma_{km}\bar{m}(t) + \theta_k, \quad \frac{d}{dt}\bar{m}(t) = -(\gamma_m + \delta^m)\bar{m}(t) - \gamma_{km}\bar{k}(t) + \theta_m$.

These equations describe how the representative country's average capital and military levels evolve over time, responding to depreciation, feedback policies, and the strategic interaction captured by $\bar{m}(t)$. The flow utility $\bar{u}(t)$ at time t is computed by evaluating the simplified utility function at the equilibrium trajectories. Since consumption is a residual from production minus investment, we compute $\bar{c}(t) = A\bar{k}(t) + (B - D)\bar{m}(t) - i_k(t) - g(t)$, and then substitute into the utility function:

$\bar{u}(t) = a\bar{c}(t) - \frac{1}{2}b\bar{c}(t)^2 + \eta\bar{m}(t) - \frac{1}{2}\phi\bar{m}(t)^2 - \eta_2\bar{m}(t)$.

The model is simulated over a finite time horizon $t \in [0, 50]$, with initial conditions $\bar{k}(0) = 1.0$ and $\bar{m}(0) = 0.3$, under a common parameter configuration: $A = 0.5, B = 0.4, D = 0.3, \delta^k = 0.04, \delta^m = 0.06, a = 1.5, b = 0.6, \eta = 0.5, \phi = 0.2, \gamma_k = 0.2, \gamma_m = 0.3, \gamma_{km} = 0.1, \theta_k = 0.3, \theta_m = 0.4$. Two values of η_2 are considered: $\eta_2 = 0.1$ for the low externality regime, and $\eta_2 = 0.8$ for the high externality regime. In the low η_2 regime, each country perceives only a small cost associated with aggregate militarization. This leads to persistently high military investment and a slower rate of capital accumulation, as the strategic arms buildup crowds out productive investment. The resulting utility path $\bar{u}(t)$ may be elevated initially but tends to flatten or decline as excessive militarization imposes growing welfare losses. Conversely, in the high η_2 regime, the disutility from global arms accumulation is more salient, which induces countries to restrain their individual military buildup. This reduction in militarization allows capital accumulation to proceed more rapidly and persistently. As a result, the trajectory of $\bar{u}(t)$ in the high η_2 regime exhibits a more pronounced long-run rise, with greater economic output and security dividends from cooperation-by-restraint.

The comparison of these regimes underscores the fundamental insight of the model: even in the absence of coordination, symmetric countries respond to their expectations about systemic behavior. Strategic restraint is possible, but only when the perceived cost of militarization is sufficiently internalized. Absent such awareness, the equilibrium converges to an inefficiently militarized steady state, confirming the endogenous nature of security traps in decentralized international systems. These dynamics, quantitatively described by the coupled capital-military ODEs and the induced welfare paths, illustrate the critical role of perceived externalities in determining the systemic trajectory of capital, conflict, and cooperation.

9 Conclusion and Extensions

This paper develops a dynamic mean field game model of capital accumulation and militarization among a continuum of symmetric nations. Each country optimizes its consumption, investment, and arms spending in response to aggregate militarization, while treating the average behavior of others as a given but influential mean field. The model captures how decentralized strategic behavior, even in the absence of asymmetries or dominant actors, can give rise to persistent arms buildup, capital stagnation, and inefficient equilibria.

We formally derived the equilibrium of this system using three core mathematical objects: a backward-in-time Hamilton–Jacobi–Bellman (HJB) equation describing each country's optimal control problem, a forward-in-time Fokker–Planck equation characterizing the evolution of the population distribution across state variables (capital and military stocks), and a fixed-point condition ensuring that the mean field used in optimization corresponds to the population average generated by optimal decisions. Under both general and linear–quadratic–Gaussian (LQG) assumptions, we proved the existence and, under regularity and monotonicity assumptions, uniqueness of equilibrium. We simplified the utility function to remove redundant interaction terms while preserving the key strategic features: the trade-off between consumption and militarization, and the disutility from systemic arms buildup. This simplification enabled closed-form optimal controls, an explicit forward system of differential equations for average capital and military stocks, and a tractable computation of equilibrium welfare.

Through numerical simulation, we compared the dynamics of capital accumulation, militarization, and welfare under different levels of strategic disutility from militarization (parametrized by η_2). We found that low values of η_2, corresponding to weak internalization of global externalities, lead to persistently high militarization and stagnating capital, with depressed long-run welfare. Conversely, higher values of η_2 induce countries to restrain their military spending, allowing capital to accumulate and welfare to rise. These simulations illustrate the central insight of the model: even among symmetric countries, the perception of externalities crucially governs whether the system converges to a peaceful investment equilibrium or falls into an inefficient arms race.

This framework opens a wide range of avenues for future research. One direction is to introduce heterogeneity in productivity, military capability, or exposure to conflict, resulting in asymmetric strategic responses and richer population distributions. Another is to allow for endogenous alliance formation, where coalitions of countries coordinate arms behavior to internalize externalities or deter aggression. Adding spatial structure would enable the study of regional arms races and the diffusion of militarization across borders. A further extension is to model information asymmetry and incomplete observability, incorporating belief dynamics and signaling into the mean field setting. Learning and bounded rationality could also be explored, where countries adapt based on past experiences rather than solving the full dynamic program.

Empirical calibration is another promising direction. The model could be matched to historical data on arms races, such as the Cold War, regional rivalries in South Asia or the Middle East, or contemporary military spending patterns across NATO and non-aligned states. Finally, welfare analysis under constrained or coordinated regimes could inform the design of international institutions, treaties, or tax-transfer mechanisms that mitigate militarization traps without requiring centralized enforcement.

Overall, this model offers a rigorous, flexible, and tractable approach to understanding the decentralized origins of global militarization. It contributes to the growing literature at the intersection of macroeconomics, international relations, and applied mathematics by showing how individual strategic behavior—when shaped by perceived externalities—can produce macro-level security dilemmas or cooperative growth. This dynamic view of equilibrium behavior under strategic uncertainty lays the groundwork for both theoretical extensions and real-world policy insights in global security and development.

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