The Stability and Instability of Total Power: A Nonlinear Dynamic Analysis of Power Accumulation and Collapse

Wei Liang and Heng-Fu Zou

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Abstract

This paper models the cyclical rise and fall of total power—an aggregate of political, economic, military, and ideological strength—through the interplay of power accumulation, consumption, and endogenous nonlinear feedback effects such as corruption and inefficiency. By framing total power as a dynamic system, we derive an optimal logistic decision rule for power accumulation from an infinite-horizon optimization problem where the state maximizes long-term utility from power and consumption. A key finding is that the state's time preference (discount factor β) intrinsically determines the logistic map's growth parameter (A). Analyzing this logistic rule using bifurcation diagrams, Lyapunov exponents, and the Feigenbaum constant, we demonstrate how decreasing patience (lower β, thus higher A) drives transitions from stable equilibria through period-doubling cascades (limit cycles) into chaotic regimes, leading to collapse. Finally, we simulate historical power collapses, including those of the Roman Empire, the Soviet Union, and the Ming Dynasty, showing that all state collapses follow the same universal mathematical path—from order to chaos—driven by shifts in effective growth parameters.

Keywords: Nonlinear Dynamics, Chaos Theory, Power Cycles, State Collapse, Dynamic Optimization

JEL Classification: C61, D72, D74, H11, N40

1 Introduction

The cyclical rise and fall of empires, states, and regimes is a recurring theme throughout human history. From ancient empires such as Rome and the Han Dynasty to modern powers like the Soviet Union, dominant states have experienced predictable patterns of growth, stagnation, and collapse. Historians often attribute these patterns to a confluence of factors: external conflicts, such as invasions or protracted wars that drain resources and manpower (e.g., the barbarian pressures on Rome or the Mongol conquests); economic crises, including currency debasement, unsustainable debt, trade disruptions, or resource depletion (seen in various forms from late Rome to pre-revolutionary France); or leadership failures, encompassing incompetent or corrupt rulers, succession crises, and paralyzing internal power struggles that divert attention from effective governance. While these factors undoubtedly play significant roles as proximate causes and catalysts for decline, they often mask or are themselves symptoms of a more fundamental process. However, beneath these surface-level causes lies a deeper, universal law governing the life cycle of power accumulation, an internal dynamic that predisposes states to eventual fragility regardless of specific external triggers. This paper aims to uncover those universal dynamics by modeling the accumulation and collapse of total power using tools from dynamic optimization and nonlinear dynamics.

In this paper, we define total power as an aggregate of political, economic, military, and ideological power. These components are interdependent: economic power, for instance, funds the military and administrative apparatus; military strength can secure trade routes, expand territory (thus economic base), and enforce political will; political control shapes ideology, resource allocation, and legal frameworks; and ideology can foster social cohesion, legitimize the state, and mobilize popular support for its objectives. This interconnectedness means that growth in one area can fuel growth in others, leading to periods of rapid power accumulation. Yet, as states accumulate power, they introduce self-limiting feedback effects—corruption, inefficiency, and internal strife—that ultimately lead to decline. For example, unchecked political consolidation can lead to autocracy, reduced accountability, and the stifling of innovation, fostering corruption as elites exploit their positions. Concentrated economic power can lead to extreme inequality, social unrest, and rent-seeking behaviors that drain productive capacity, contributing to inefficiency and internal strife. Military overextension, often a product of past successes, can strain resources, neglect domestic needs, and provoke both internal dissent and external coalitions. Even the solidification of ideological power can lead to dogmatism and an inability to adapt to changing circumstances, further breeding inefficiency and discontent. These internal feedback mechanisms, inherent in the very process of power accumulation, create vulnerabilities that can be exploited by the aforementioned external conflicts or economic shocks, but their origins are fundamentally endogenous.

This paper models these dynamics by framing total power as a dynamic system. We derive an optimal decision rule for power accumulation from an infinite-horizon optimization problem, where a state seeks to maximize long-term utility. This approach reveals how rational, optimizing behavior, when interacting with the inherent nonlinearities of power, can lead to complex, cyclical, and even chaotic outcomes. The resulting model demonstrates transitions from stable equilibria to limit cycles and eventually to chaotic regimes, governed by a logistic-like dynamic. We show that this path—from order to chaos—is a universal feature of power accumulation and collapse, applicable across diverse historical contexts.

The rest of the paper is organized as follows: Section 2 introduces the general mathematical form of the power dynamic and provides foundational definitions of dynamical systems, including fixed points, periodic points, and the concept of chaos via the Li-Yorke Theorem. Section 3 discusses how chaotic behavior can arise within an optimization framework. Section 4 details the model setup, formulating the state's dynamic maximization problem, including the objective function, utility function, production function, and power accumulation dynamics, leading to the Bellman equation. Section 5 focuses on the derivation of the optimal decision rule, showing how it collapses to a logistic decision rule. Section 6 examines the special case of the logistic decision rule $f(P) = AP(1-P)$ and its properties. Sections 7, 8, and 9 analyze the model's behavior, exploring the stability of total power, the transition from stability to oscillations (limit cycles), and the emergence of instability and chaotic regimes at high growth parameters. Section 10 presents a combined bifurcation diagram overlaid with Lyapunov exponents to measure chaos intensity. Finally, Section 11 concludes by summarizing the universal law of power collapse derived from the model.

2 The General Form

The saying "power tends to corrupt, and absolute power corrupts absolutely" highlights the psychological and social impacts of power. Psychologically, individuals with power often develop a sense of superiority and entitlement, which can lead to ethical lapses and a disregard for the consequences of their actions on others. Socially, absolute power often lacks checks and balances, enabling leaders to make decisions without accountability. This lack of oversight can foster an environment where corruption becomes systemic, as leaders manipulate laws and policies to benefit themselves or their supporters at the expense of the broader public good. Historical examples show that when leaders operate without constraints, they are more likely to engage in corrupt practices, suppress dissent, and prioritize personal gain over public welfare.

Economically, power concentration can distort market mechanisms, leading to inefficiencies and corruption. Politically, it undermines democratic institutions and erodes civic trust. When power is concentrated, decisions are often made to benefit the few rather than the many, leading to policies that can stifle economic innovation and concentrate wealth, which in turn reduces competition and economic growth. This setup also limits political pluralism, reducing political competition, and thereby decreasing governmental accountability and responsiveness to citizen needs. In both cases, the long-term stability and health of a society can be compromised.

In totalitarian and authoritarian regimes, the concentration of power tends to be more pronounced and systemic, leading to more severe economic and political consequences. Economically, these regimes often control major aspects of production and distribution, stifling entrepreneurship and innovation due to heavy regulations and state control. Politically, the suppression of dissent and limitation of freedoms under such regimes leads to a lack of accountability and transparency. This can result in widespread corruption, as the lack of checks and balances allows leaders and their close associates to exploit state resources for personal gain. These dynamics inhibit economic development and degrade the quality of governance, often resulting in significant public dissatisfaction and unrest.

In this paper, we explore the dynamics of power corruption using the framework of chaotic cycles. We define P_t to represent the level of power at period t, and $P_{t+1} = f(P_t)$ to represent the degree of development of a country, which becomes the level of power at period $t + 1$.

The function f exhibits an inverted U-shape, reflecting the dual impact of power on development. When the power level is very low, the country is too fragmented to progress effectively. This fragmentation arises because a low concentration of power results in weak central authority, leading to disorganization, lack of coordination, and inefficient resource allocation. Local entities may act independently without a unified strategy, resulting in conflicts and inefficiencies that hinder overall development.

Conversely, when the power level is excessively high, it leads to totalitarianism. In such scenarios, power is concentrated in the hands of a few or a single entity, leading to an authoritarian regime. This centralization stifles innovation and growth because it often comes with suppression of dissent, lack of political freedom, and rigid control over economic activities. The concentration of power discourages creative thinking and entrepreneurship, as individuals and businesses may fear retribution for actions that do not align with the ruling authority's preferences. Additionally, the lack of checks and balances can lead to corruption and mismanagement of resources, further harming the country's development.

The equation $P_{t+1} = f(P_t)$ highlights how power, when not balanced, can oscillate between extremes, resulting in cycles of growth and decline, mirroring the chaotic trajectories in economic growth models. These chaotic cycles demonstrate how small variations in power can lead to unpredictable and complex outcomes, emphasizing the importance of maintaining a balanced distribution of power to foster sustainable development. Generally, we assume that f is a continuous function, and the initial state is $P_0 = P$.

2.1 Definitions of a Dynamical System

First come some definitions of a dynamical system.

Definition 1. A dynamical system is described by a pair (X, f), where X is called the *state space* that is a nonempty metric space (with metric denoted by d), and $f : X \to X$ is called the *law of motion.* Therefore, we have

$$x_{t+1} = f(x_t). \tag{1}$$

Definition 2. Denote f^j as the j-th *iterate* of function f, i.e. $f^0(x) = x$, $f^1(x) = f(x)$, and for every positive integer $j \geq 1$,

$$f^{j+1}(x) = f(f^j(x)). \tag{2}$$

Definition 3. The *trajectory* generated by $f : x \to f(x)$ is the sequence $\tau(x) = \{f^j(x)_{j=0}^{\infty}\}$. The *orbit* from x is the set $\gamma(x) = \{y : y = f^j(x)$ for some $j \geq 0\}$.

Definition 4. A point $x \in X$ is a *fixed point* or a steady state if $x = f(x)$.

Definition 5. A point $x \in X$ is a *period point* of period $k \geq 2$, if $f^k(x) = x$ and $f^j(x) \neq x$ for $1 \leq j < k$.

Therefore, we can consider a fixed point as a period point of period 1. The following proposition is a useful result on the existence of fixed points of f.

Proposition 1. Let $X = \mathbb{R}$, and f be continuous. If there is a (nondegenerate) closed interval $I = [a, b]$ such that

$$\text{(i) } f(I) \subset I \text{ or (ii) } f(I) \supset I, \tag{3}$$

then there is a fixed point of f in I.

Proof. (i) If $f(I) \subset I$, then $f(a) \geq a$, and $f(b) \leq b$. If $f(a) = a$ or $f(b) = b$, the conclusion is immediate. Otherwise, $f(a) > a$ and $f(b) < b$. This means that the function $g(x) = f(x) - x$ is positive at a and negative at b. Using the intermediate value theorem, $g(x^*) = 0$ for some x^* in (a, b). Then $f(x^*) = x^*$. (ii) If $f(I) \supset I$, by the Weierestrass' theorem, there are points x_m and x_M in I such that $f(x_m) \leq f(x) \leq f(x_M)$ for all x in I. Write $f(x_m) = m$ and $f(x_M) = M$. Again, by the intermediate value theorem, $f(I) = [m, M]$. Since $f(I)$ is assumed to contain I, $m \leq a \leq b \leq M$. In other words,

$$f(x_m) = m \leq a \leq x_m, \tag{4}$$

and

$$f(x_M) = M \geq b \geq x_M. \tag{5}$$

The proof can now be completed by an argument similar to that in case (i).

Definition 6. A fixed point x^* of f is *locally attracting* if there is an open set U containing x^*such that for all $x \in U$, the trajectory $\tau(x)$ converges to x^*.

Definition 7. A fixed point x^* is *repelling* if there is an open set U containing x^*such that for all $x \in U, x \neq x^*$, there is some $k \geq 1$, $f^k(x) \notin U$.

Here is a simple proposition to distinguishing between attracting and repelling fixed points of nonlinear functions.

Definition 8. A function $f : X \to X$ is a contraction, if there is some $\beta \in (0, 1)$ such that for all $x, y \in X, x \neq y$, one has

$$d(f(x), f(y)) < \beta d(x, y). \tag{6}$$

Proposition 2. Let (X, d) be a (nonempty) complete metrix space, and $f : X \to X$ a contraction. Then f has a unique fixed point $x^* \in X$. Moreover, for any x in X, the trajectory $\tau(x) = \{f^j(x)_{j=0}^{\infty}\}$ converges to x^*.

Proof. Choose an arbitrary $x \in X$. Consider the trajectory $\tau(x) = (x_t)$ from x, where $x_{t+1} = f(x_t)$. Note that $d(x_2, x_1) = d(f(x_1), f(x)) < \beta d(x_1, x)$. Therefore, for any $t \geq 1$,

$$d(x_{t+1}, x_t) < \beta^t d(x_1, x). \tag{7}$$

Then,

$$\begin{aligned} d(x_{t+2}, x_t) &\leq d(x_{t+2}, x_{t+1}) + d(x_{t+1}, x_t) \\ &< \beta^{t+1} d(x_1, x) + \beta^t d(x_1, x) \\ &= \beta^t (1+\beta) d(x_1, x). \end{aligned} \tag{8}$$

It follows that, for any integer $k \geq 1$,

$$d(x_{t+k}, x_t) < \frac{\beta^t}{1-\beta} d(x_1, x), \tag{9}$$

and this implies that $\tau(x) = (x_t)$ is a Cauchy sequence. Since X is assumed to be complete, $\lim_{t\to\infty} x_t = x^*$ exists. By continuity of f and $x_{t+1} = f(x_t)$, we have $f(x^*) = x^*$. If there are two distinct fixed points x^* and x^{**} of f, we see that there is a contradiction:

$$0 < d(x^*, x^{**}) = d(f(x^*), f(x^{**})) < \beta d(x^*, x^{**}), \tag{10}$$

where $\beta \in (0, 1)$.

Proposition 3. Let $X = [a, b]$ and f be continuous on $[a, b]$ and continuously differentiable on (a, b).

(a) If $x^* \in (a, b)$ is a fixed point of f and $|f'(x^*)| < 1$, then x^* is locally attracting.

(b) If $x^* \in (a, b)$ is a fixed point of f with $|f'(x^*)| > 1$, then x^* is repelling.

Proof. (a) Assume there is some $\mu > 0$ such that $|f'(x)| < \beta < 1$ for all x in $I = [x^* - \mu, x^* + \mu]$. By the mean value theorem,

$$|x - x^*| = |f(x) - f(x^*)| \leq \beta |x - x^*| < \beta\mu < \mu. \tag{11}$$

Hence, f maps I into I and, by the mean value theorem is a contraction on I. The result follows from Proposition 1. Similarly, we can define locally attracting and repelling periodic points accordingly.

2.2 Chaotic Dynamical System

Then we come to the definition of a chaotic dynamical system, which is introduced well in Devaney (1986).

Definition 8. Suppose X is a set and Y is a subset of X. We say that Y is *dense* in X if, for any point $x \in X$, there is a point y in the subset Y arbitrarily close to x.

Definition 9. A dynamical system is *topologically transitive* if for any pair of nonempty open sets U and V, there exists $k \geq 1$ such that $f^k(U) \cap V \neq \varnothing$.

In other words, a transitive dynamical system has the property that, given any two points, we can find an orbit that comes arbitrarily close to both.

Definition 10. A dynamical system has *sensitive dependence on initial condition* if there is $\epsilon > 0$ such that, for any $x \in X$ and any neighborhood N of x there exists y and an integer $j \geq 0$ with the property $\left|f^j(x) - f^j(y)\right| > \epsilon$.

Devaney argues that if a dynamical system "has sensitive dependence on initial condition, then the dynamics is unsuitable for numerical computation for all practical purposes". Small errors in the calculations brought by rounding may be magnified in the iterations. As a consequence, we may be looking at an orbit that eventually diverges from the true orbit we seek. The results of numerical calculations of orbits, no matter how precise, are unlikely to have any resemblance to the actual orbit. Under Devaney's view, we have the definition of a chaotic dynamical system:

Definition 11. A dynamical system (X, f) is *chaotic* if

(i) (X, f) is topologically transitive.

(ii) (X, f) has sensitive dependence on initial condition.

(iii) The set of periodic points $\mathcal{P}(X)$ of X is dense in X.

After the definition of a chaotic dynamical system, it is natural to ask that under what conditions chaotic behaviour may occur in a specific economic model. We should refer to the Li-Yorke Theorem:

Theorem 1 (Li-Yorke Theorem). Let I be an interval and $f : I \to I$ be continuous. Assume that there is some point a in I for which there are points $b = f(a)$, $c = f(b) = f^2(a)$ and $d = f(c) = f^3(a)$ satisfying:

$$d \le a < b < c \text{ (or, } d \ge a > b > c). \tag{12}$$

Then,

A. For every positive integer $k = 1, 2, ...$, there is a periodic point of period k in I.

B. Let $\mathcal{P}(X)$ denote the set of periodic points of X, $\mathcal{N}(X)$ denote the set of non-periodic points of X. Then

B1.There is an uncountable set $S \subset \mathcal{N}(X)$ such that for all $p, q \in S$, $p \neq q$,

$$\limsup_{n\to\infty} |f^n(p) - f^n(q)| > 0; \tag{13}$$

$$\liminf_{n\to\infty} |f^n(p) - f^n(q)| = 0. \tag{14}$$

B2. If $p \in S$ and $q \in \mathcal{P}(X)$,

$$\limsup_{n\to\infty} |f^n(p) - f^n(q)| > 0. \tag{15}$$

We will say that the dynamical system (X, f) exhibits *topological chaos* if conditions [A] and [B] of Theorem 1 are satisfied. The Li-Yorke Theorem is a easily verified sufficient condition for topological chaos.

Proof of [A]. Step 1. Let G be a real-valued continuous function on an interval I. For any compact subinterval I_1 of $G(I)$ there is a compact subinterval Q of I such that $G(Q) = I_1$.

Proof of Step 1. One can figure out the subinterval Q directly as follows: Let $I_1 = [G(x), G(y)]$ where x, y are in I. Assume that $x < y$. Let r be the last point of $[x, y]$ such that $G(r) = G(x)$; let s be the first point after r such that $G(s) = G(y)$. Then $Q = [r, s]$ is mapped onto I_1 under G. The case $x > y$ is similar.

Step 2. Let I be an interval and $f : I \to I$ be continuous. Suppose that $(I_n)_{n=0}^{\infty}$ is a sequence of compact subintervals of I and, for all n,

$$I_{n+1} \subset f(I_n). \tag{16}$$

Then there is a sequence of compact subintervals (Q_n) of I such that, for all n,

$$Q_{n+1} \subset Q_n \subset Q_0 = I_0 \tag{17}$$

and

$$f^n(Q_n) = I_n. \tag{18}$$

Hence, there is

$$x \in \bigcap_n Q_n \text{ such that } f^n(x) \in I_n \text{ for all } n. \tag{19}$$

Proof of Step 2. The construction of the sequence Q_n proceeds "inductively" as follows: Define $Q_0 = I_0$. Recall that f^0 is defined as the identity mapping, so $f^0(Q_0) = I_0$ and $I_1 \subset f(I_0)$. If Q_{n-1} is defined as a compact subinterval such that $f^{n-1}(Q_{n-1}) = I_{n-1}$ then $I_n \subset f(I_{n-1}) = f^n(Q_{n-1})$. Use Step 1, with $G = f^n$ on Q_{n-1} in order to get a compact subinterval Q_n of Q_{n-1} such that $f^n(Q_n) = I_n$. This completes the induction argument (establishing (17) and (18)). Compactness of Q_n leads to (19).

Now we prove [A]. Assume that $d \le a < b < c$ (the other case $d \ge a > b > c$ is treated similarly). Write $K = [a, b]$ and $L = [b, c]$. Let k be any positive integer. For $k > 1$, define a sequence of intervals (I_n) as follows: $I_n = L$ for $n = 0, 1, 2, ..., k-2$; $I_{k-1} = K$, and $I_{n+k} = I_n$ for $n = 0, 1, 2, ...$ For $k = 1$, let $I_n = L$ for all n. Let Q_n be the intervals in Step 2. Notice that $Q_k \subset Q_0 = I_0$ and $f^k(Q_k) = I_k = I_0$. Hence, Proposition 1 applied to f^k gives us a fixed point p_k of f^k in Q_k. Now, p_k cannot have period less than k; otherwise, we need to have $f^{k-1}(p_k) = b$, contrary to$f^{k+1}(p_k) \in L$.

Proof of B. Let $\mathcal{M}$ be the set of sequences $M = \{M_n\}_{n=1}^{\infty}$ of intervals with

A1: $M_n = K$ or $M_n \subset L$, and $f(M_n) \supset M_{n+1}$.

if $M_n = K$ then

A2: n is the square of an integer and $M_{n+1}, M_{n+2} \subset L$.

For $M \in \mathcal{M}$ let $P(M, n)$ denote the number of i's in $\{1, ..., n\}$ for which $M_i = K$. For each $r \in (3/4, 1)$ choose $M^r = \{M_n^r\}_{n=1}^{\infty}$ to be a sequence in $\mathcal{M}$ such that

A3: $\lim_{n\to\infty} P(M^r, n^2)/n = r$.

Let $\mathcal{M}_0 = \{M^r : r \in (3/4, 1)\} \subset \mathcal{M}$. Then $\mathcal{M}_0$ is uncountable since $M^{r_1} \ne M^{r_2}$ for $r_1 \ne r_2$. For each $M^r \in \mathcal{M}_0$, by Step 2, there exists a point x_r, with $f^n(x_r) \in M_n^r$ for all n. Let $S = \{x_r : r \in (3/4, 1)\}$. Then S is also uncountable. For $x \in S$, let $P(x, n)$ denote the number of i's in $1, ..., n$ for which $f^i(x) \in K$. We can never have $f^k(x_r) = b$, because then x_r would eventually have period 3, contrary to (A2). Consequently $P(x_r, n) = P(M^r, n)$ for all n, and so

$$\rho(x_r) = \lim_{n\to\infty} P(X_r, n^2) = r \tag{20}$$

for all r. We claim that

A4: for $p, q \in S$, $p \neq q$, there exist finitely many n's such that $f^n(p) \in K$ and $f^n(q) \in L$ or vice versa.

We may assume $\rho(p) > \rho(q)$. Then $P(p,n) - P(q,n) \to x$, and so there must be infinitely many n's such that $f^n(p) \in K$ and $f^n(q) \in L$. Since $f^2(b) = d \leq a$ and f^2 is continuous, there exists $\delta > 0$ such that $f^2(x) < (b+d)/2$ for all $x \in [b-\delta, b] \subset K$. If $p \in S$ and $f^n(p) \in K$, then (A2) implies $f^{n+1}(p) \in L$ and $f^{n+2}(p) \in L$. Therefore $f^n(p) < b - \delta$. If $f^n(q) \in L$, then $f^n(q) \geq b$ so

$$|f^n(p) - f^n(q)| > \delta. \tag{21}$$

By claim (A4), for any $p, q \in S$, $p \neq q$, it follows

$$\limsup_{n\to\infty} |f^n(p) - f^n(q)| \geq \delta > 0. \tag{22}$$

Hence (13) is proved. This technique may be similarly used to prove that [B2] is satisfied.

Proof of (14). Since $f(b) = c$, $f(c) = d$, $d \leq a$, we may choose intervals $[b^n, c^n], n = 0, 1, 2, ...$ such that

(a) $[b, c] = [b^0, c^0] \supset [b^1, c^1] \supset ... \supset [b^n, c^n] \supset ...,$

(b) $f(x) \in (b^n, c^n)$ for all $x \in (b^{n+1}, c^{n+1})$,

(c) $f(b^{n+1}) = c^n, f(c^{n+1}) = b^n$.

Let $A = \bigcap_{n=0}^{\infty}[b^n, c^n]$, $b^* = \inf A$ and $c^* = \sup A$, then $f(b^*) = c^*$ and $f(c^*) = b^*$, because of (c). In order to prove (14) we must be more specific in our choice of the sequences M^r. In addition to our previous requirements on $M \in \mathcal{M}$, we assume that if $M_k = K$ for both $k = n^2$ and $(n+1)^2$ then $M_k = [b^{2n-(2j-1)}, b^*]$ for $k = n^2 + (2j-1)$, $M_k = [c^*, c^{2n-2j}]$ for $k = n^2 + 2j$ where $j = 1, ..., n$. For the remaining k's which are not squares of integers, we assume $M_k = L$.

It is easy to check that these requirements are consistent with (A1) and (A2), and that we can still choose M^r so as to satisfy (A3). From the fact that $\rho(x)$ may be thought of as the limit of the fraction of n's for which $f^{n^2}(x) \in K$, it follows that for any r^*, $r \in (3/4, 1)$ there exist infinitely many n such that $M_k^r = M_k^{r^*} = K$ for both $k = n^2$ and $(n+1)^2$. To show (14), let $x_r \in S$ and $x_{r^*} \in S$. Since $b^n \to b^*, c^n \to c^*$ as $n \to \infty$, for any $\epsilon > 0$ there exists N with $|b^n - b^*| < \epsilon/2, |c^n - c^*| < \epsilon/2$ for all $n > N$. Then, for any n with$n > N$ and $M_k^r = M_k^{r^*} = K$ for both $k = n^2$ and $(n+1)^2$, we have

$$f^{n^2+1}(x_r) \in M_k^r = [b^{2n-1}, b^*], \tag{23}$$

with $k = (n+1)^2$ and $f^{n^2+1}(x_r)$ and $f^{n^2+1}(x_{r^*})$ both belong to $[b^{2n-1}, b^*]$. Therefore,

$$\left| f^{n^2+1}(x_r) - f^{n^2+1}(x_{r^*}) \right| \leq \epsilon. \tag{24}$$

Since there are infinitely many n with this property,

$$\liminf_{n\to\infty} |f^n(x_r) - f^n(x_{r^*})| = 0. \tag{25}$$

Li-Yorke Theorem is used for general form of dynamical system. Here we consider a more specific case in which the law of motion $f : X \to X$ takes on a "single-humped" form with $f(0) = 0$ and $f(x)$ increasing with $f'(x) > 1$ for sufficiently small positive x, for example, $f(x) = Ax(1-x)$. Therefore, we can find a unique maximization point $x^* \in X$ satisfies $x^* = \arg\max_{x\in X} f(x)$. Also, we can define a maximum attainable level x^m generated by x^*:

$$x^m = f(x^*) = \max_{x\in X} f(x) > 0. \tag{26}$$

Because of the "single-humped" property of f, x^* may have two preimages. We define the smaller one as x^c, which means that $f(x^c) = x^*$. If $f(x^m) > 0$, then we can find a interval $I = [0, x^m]$ so that it satisfies the conditions in Li-Yorke Theorem. Therefore, the sufficient condition (12) now becomes

$$0 < f(x^m) \leq x^c < x^* < x^m. \tag{27}$$

The remaining thing we need to do is to find proper parameters such that x^m exists, then we can confine the parameters by using condition (27).

3 Chaotic Behaviour Under the Optimization Framework

Next we consider if we can extend the model to an aggregative model such that the solution can exhibit optimal behavior which is chaotic (in both the "topological" and "ergodic" senses). The model specified by a *production function* $f : \mathbb{R}_+ \to \mathbb{R}_+$, a *welfare function* $w : \mathbb{R}_+^2 \to \mathbb{R}_+$, and a *discount factor* $\delta \in (0, 1)$.

The production function f follows the assumptions:

(F.1) f is continuous on $\mathbb{R}_+$ with $f(0) = 0$.

(F.2) f is non-decreasing and concave on $\mathbb{R}_+$.

(F.3) There exists some $K > 0$ such that $f(x) > x$ when $0 < x < K$ and $f(x) < x$ when $x > K$.

Next, we define the transition possibility space $\Omega \subset \mathbb{R}_+^2$ as:

$$\Omega = \{(x, z) \in \mathbb{R}_+^2 : z \leq f(x)\}. \tag{28}$$

which follows the assumptions:

(Ω.1) $(0, 0) \in \Omega$. $(0, z) \in \Omega$ implies that $z = 0$.

(Ω.2) Ω is a closed, convex.subset of $\mathbb{R}_+^2$.

(Ω.3) If $(x, z) \in \Omega$ and $x' \geq x$, $0 \leq z' \leq z$, then $(x', z') \in \Omega$.

(Ω.4) There exists some $K > 0$ such that if $(x, z) \in \Omega$ and $x > K$, then $z < x$.

The welfare function w follows the assumptions:

(W.1) $w(x,c)$ is continuous on $\mathbb{R}^2_+$.

(W.2) $w(x,c)$ is non-decreasing in x given c, and non-decreasing in c given x on $\mathbb{R}^2_+$.

(W.3) $w(x,c)$ is concave on $\mathbb{R}^2_+$. Furthermore, if $x > 0$, $w(x,c)$ is strictly concave in c on Ω.

Also, we can define the utility function $u : \Omega \to \mathbb{R}_+$ as:

$$u(x,z) = w(x, f(x) - z). \tag{29}$$

which follows the assumptions:

(U.1) $u(x,z)$ is continuous on Ω.

(U.2) If $(x,z) \in \Omega$ and $x' \geq x$, $0 \leq z' \leq z$, then $u(x',z') \geq u(x,z)$.

(U.3) $u(x,z)$ is concave on Ω and given $x > 0$, $u(x,z)$ is strictly concave in z on Ω.

There are some more definitions of the aggregative model:

Definition 12. A *program* from $\mathbf{x} \geq 0$ is a sequence $\{x_t\}_0^\infty$ satisfying

$$x_0 = \mathbf{x},\ 0 \leq x_{t+1} \leq f(x_t) \text{ for } t \geq 0. \tag{30}$$

Definition 13. A consumption sequence $\{c_{t+1}\}_0^\infty$ is given by

$$c_{t+1} = f(x_t) - x_{t+1} \text{ for } t \geq 0. \tag{31}$$

Definition 14. A program $\{\hat{x}_t\}_0^\infty$ from $\mathbf{x} \geq 0$ is *optimal* if for every program $\{x_t\}_0^\infty$ from $\mathbf{x}$, we have

$$\sum_{t=0}^{\infty} \delta^t u(\hat{x}_t, \hat{x}_{t+1}) \geq \sum_{t=0}^{\infty} \delta^t u(x_t, x_{t+1}). \tag{32}$$

Definition 15. A program $\{x_t\}_0^\infty$ from $\mathbf{x}$ is *stationary* if $x_t = \mathbf{x}$ for $t \geq 0$. It is a *stationary optimal program* if it is also an optimal program from $\mathbf{x}$.

Definition 16. A *value function* $V : \mathbb{R}_+ \to \mathbb{R}$ is defined by

$$V(x) = \sum_{t=0}^{\infty} \delta^t u(\hat{x}_t, \hat{x}_{t+1}), \tag{33}$$

and the optimal policy function $h : \mathbb{R}_+ \to \mathbb{R}_+$ is defined by

$$h(x) = \hat{x}_1, \tag{34}$$

where $\{\hat{x}_t\}$ is the optimal program from $\mathbf{x} \geq 0$.

Given a reduced form model (Ω, u, δ), satisfying $(\Omega.1)$-$(\Omega.4)$, (U.1)-(U.3), the following theorem shows that there is an optimal program from every $\mathbf{x} \geq 0$.

Theorem 2.

- (i) The value function, V, defined in (33) is the unique continuous real valued function on $[0, K]$, satisfying the functional equation of dynamic programming:
$$V(x) = \max_{(x,z)\in\Omega} [u(x, z) + \delta V(z)]. \tag{35}$$
Further, V is concave and non-decreasing on $\mathbb{R}_+$.

- (ii) The policy function h, defined in (34) satisfies the following property: for each $x \in \mathbb{R}_+$, $h(x)$ solves uniquely the constrained maximization problem:
$$\begin{array}{rl} \text{Max} & [u(x, z) + \delta V(z)] \\ \text{Subject to} & (x, z) \in \Omega \end{array} \tag{36}$$
Further, h is continuous on $\mathbb{R}_+$.

4 Model Setup: Dynamic Maximization Problem

To model the rise and fall of total power, we begin by formulating an infinite-horizon dynamic optimization problem, where the state seeks to maximize total discounted utility from power and consumption. The state derives utility from two sources: power, which provides security and control, and consumption, which satisfies immediate needs. However, there is a fundamental trade-off: investing resources to accumulate power reduces consumption, while consuming excessively weakens future power growth.

The utility function captures these preferences with diminishing returns, expressed as a quadratic function of power and consumption. In this formulation, power provides utility by securing dominance and control, while consumption offers immediate satisfaction. The quadratic terms ensure that excessive power accumulation and consumption are penalized due to increasing costs or inefficiencies. Meanwhile, the production function models how total power generates output, incorporating diminishing returns through quadratic term. This nonlinearity captures the reality that accumulating power becomes increasingly difficult as internal inefficiencies, corruption, and resistance emerge.

Power accumulation follows a state equation where next-period power depends on current production, consumption, and depreciation. Production increases power, but consumption reduces it, representing the allocation of resources between state-building and immediate needs. Additionally, depreciation captures the natural erosion of power through internal decay, corruption, or loss of influence. To solve this problem, we use dynamic programming and express the state's objective through the Bellman equation, which relates the value of power today to future value. The Bellman equation serves as the foundation for deriving the state's optimal policy and understanding its long-term behavior.

4.1 Objective Function (State's Problem)

We model a state that seeks to maximize total discounted utility from total power and consumption over an infinite horizon. The state's problem is to choose an optimal path of consumption c_t for all future periods $t = 0, 1, 2, ...$ to achieve this. The objective function, also known as the value function $V(P_t)$, given the current power level P_t, is formally expressed as:

$$\begin{aligned} V(P_t) &= \max_{c_t} \sum_{t=0}^{\infty} \frac{1}{(1+\rho)^t} u(c_t, P_t) \\ &= \max_{c_t} \sum_{t=0}^{\infty} \beta^t u(c_t, P_t). \end{aligned} \tag{37}$$

where $V(P_t)$ represents the value function: this is the maximum total discounted utility the state can achieve starting from an initial power level P_t at time t (or P_0 at time 0 if we consider the problem from the outset) and following an optimal policy thereafter. It encapsulates the long-term well-being or objective fulfillment of the state. $u(c_t, P_t)$ is the instantaneous utility function: this function captures the state's preferences and satisfaction derived in any single period t. This utility depends on the level of consumption c_t undertaken in that period and the prevailing level of total power P_t held by the state during that period. The summation $\sum_{t=0}^{\infty}$ indicates that the state considers utility over an infinite time horizon. This reflects a long-term perspective, where the state is concerned not just with immediate outcomes but with the entire future trajectory of its utility.

β is the per-period discount factor, which is central to intertemporal decision-making. It is defined as $\beta = 1/(1 + \rho)$, where $\rho > 0$ is the time discount rate. The condition $\rho > 0$ ensures that $0 < \beta < 1$. This discount factor β quantifies the state's patience or time preference. A higher β (corresponding to a lower ρ) signifies a more patient state, one that values future utility more highly relative to present utility. Conversely, a lower β (higher ρ) indicates a more impatient state, prioritizing immediate gratification or benefits more heavily.

The term β^t is the discount weight applied to utility received t periods into the future. Because $0 < \beta < 1$, this weight decreases as t increases. This means that utility streams further in the future are progressively devalued from the perspective of the present (time 0). This discounting mechanism is standard in dynamic optimization and is crucial for ensuring the convergence of the infinite sum, as well as reflecting the common economic assumption that future utility is generally valued less than present utility, either due to pure time preference, uncertainty about the future, or expected growth.

The maximization in Equation (37) is performed by choosing the sequence of consumption levels $\{c_t\}_{t=0}^{\infty}$ to optimize this discounted sum of utilities. This setup frames the state as a rational, forward-looking agent aiming to make the best possible intertemporal trade-offs to maximize its overall, long-term objective. The initial level of power P_t (or P_0) serves as the state variable at the beginning of the planning horizon.

4.2 Utility Function

To make the model tractable and capture key economic intuitions, we assume a specific functional form for the instantaneous utility $u(c_t, P_t)$. We adopt a quadratic utility function, which is widely used in economic modeling for its analytical convenience and its ability to represent diminishing marginal utility. The specific form is given by:

$$u(c_t, P_t) = c_t P_t - \frac{1}{2} c_t^2 + \frac{\beta}{2(1-\beta)} c_t. \tag{38}$$

Here, power utility shows positive returns. While Equation (38) doesn't have a term with P_t alone, the influence of power on utility is channeled through the interaction term $c_t P_t$. The "positive" aspect comes from this interaction where higher P_t can lead to higher utility for a given c_t.

For the consumption utility, it has positive but diminishing returns. This is more directly evident. The linear component (via $c_t P_t$) implies consumption is desirable. The net effect of the quadratic terms will determine the concavity. For diminishing marginal utility of consumption to hold, the second derivative $\partial^2 u / \partial c_t^2$ must be negative. Calculating this:

$$\frac{\partial^2 u}{\partial c_t^2} = \frac{2\beta - 1}{1 - \beta}. \tag{39}$$

For diminishing marginal utility, we need $(2\beta - 1)/(1 - \beta) < 0$, or $\beta < 1/2$, which will be restricted in the future sections.

4.3 Production Function

The mechanism by which a state's current level of total power, P_t, generates new output or potential for further power is described by the production function, $F(P_t)$. This function is central to the model's dynamics, as its output directly influences the state's capacity for future power accumulation and consumption. In this framework, power itself is not merely a static attribute but a productive asset capable of generating further capacity. The specific form adopted for this production function

$$F(P_t) = \left(\frac{1-\beta}{\beta} + \delta \right) P_t - \frac{1-\beta}{\beta} P_t^2 \tag{40}$$

is designed to capture both the initial growth impetus provided by power and the inherent limitations that eventually arise. Here, power, P_t, represents an aggregate measure of the state's control over resources (natural, human, and financial), its military strength (for security and projection), and its capacity to organize and direct economic production.

The production function incorporates two key components reflecting the dual nature of power's impact on output. The linear term, $((1 - \beta)/\beta + \delta)P_t$, signifies that power generates growth, particularly when existing power levels are relatively low. The coefficient $(1-\beta)/\beta+\delta$ determines the initial marginal product of power, where δ can be seen as a baseline productivity and the term $(1-\beta)/\beta$ links this productivity to the state's intertemporal preferences encapsulated by the discount factor β. This component embodies the fundamental ability of power to mobilize resources and facilitate economic activity, leading to a proportional increase in output at lower levels of power. However, this growth is not unbounded. The quadratic term, $-((1-\beta)/\beta)P_t^2$, introduces the critical concept of diminishing marginal returns and negative feedback effects. As total power P_t escalates, this negative term becomes increasingly influential, ensuring that each additional unit of power contributes progressively less to new output. This captures a range of real-world phenomena, such as rising inefficiencies in large bureaucracies, increased opportunities for corruption at high concentrations of power, internal resistance to excessive control, or the over-exploitation of resources. The concavity introduced by this quadratic term ensures that the production of new output from power eventually faces constraints, a crucial feature for modeling the cyclical dynamics of power.

4.4 Power Accumulation Dynamics and The Bellman Equation

The evolution of total power over time is governed by the state constraint, which describes how power P_t in one period transitions to power P_{t+1} in the next. This power accumulation dynamic is crucial as it defines the feasible paths of power growth and decline available to the state. The specific law of motion for power is given by:

$$P_{t+1}=F(P_t)-c_t+(1-\delta)P_t. \tag{41}$$

where next period's power P_{t+1} is determined by several factors. Firstly, power grows from production, represented by $F(P_t)$, which is the output generated from the current stock of power as detailed in Section 4.3. Secondly, power declines from consumption c_t, as resources allocated to immediate consumption are not available for reinvestment into power. Finally, the term $(1-\delta)P_t$ reflects the portion of current power that remains after accounting for internal decay and depreciation, where δ is the depreciation rate of power. This depreciation, δP_t, captures the natural erosion of power that can occur through various channels such as institutional decay, the obsolescence of existing military or economic structures, the corrosive effects of unchecked corruption, or the gradual loss of ideological influence if not actively maintained and reinforced. Thus, the net change in power from one period to the next is the result of new productive output, less what is consumed by the state and what is naturally lost to these forces of decay. This equation serves as the fundamental constraint within which the state must optimize its choices.

Given this law of motion for total power, the state's dynamic optimization problem, originally stated as maximizing the sum of discounted utilities over an infinite horizon (Equation 37), can be reformulated using the powerful tool of dynamic programming. This approach breaks the complex infinite-horizon problem into a sequence of simpler, recursive decisions, leading to the Bellman equation. The Bellman equation expresses the recursive nature of the state's problem as follows:

$$V(P_t) = \max_{c_t} \left[u(c_t, P_t) + \beta \cdot V \left(F(P_t) - c_t + (1 - \delta) P_t \right) \right]. \tag{42}$$

This equation is the cornerstone of dynamic optimization. It states that the maximum value $V(P_t)$ achievable from the current state of power P_t is found by choosing the current consumption level c_t to maximize the sum of two components: the immediate utility $u(c_t, P_t)$ derived in the current period, and the discounted expected value $\beta \cdot V(P_{t+1})$ of being in the subsequent state P_{t+1}. Critically, the argument of the future value function, $F(P_t) - c_t + (1 - \delta)P_t$, is precisely the expression for P_{t+1} from the power accumulation dynamics (Equation 41). The Bellman equation thus elegantly links the present to the future, forcing the state to consider how its current consumption choice c_t not only yields immediate utility but also determines the power level, and therefore the maximum achievable utility, in all subsequent periods. Solving this functional equation yields the optimal policy for consumption c_t as a function of P_t and the true value function $V(P_t)$.

5 Derivation of the Optimal Decision Rule (Logistic Function)

To derive the optimal decision rule, we solve the Bellman equation using dynamic programming. First, we take the first-order conditions with respect to consumption, which produce the Euler equation—a fundamental condition governing the intertemporal trade-off between consumption and power accumulation. Simultaneously, we apply the envelope theorem, which relates changes in the value function to changes in power. Solving these equations together allows us to eliminate the value function derivative and derive the state's optimal policy for power accumulation.

The result is both elegant and profound: the optimal decision rule collapses to a logistic function. This logistic function describes how power evolves over time, governed by an intrinsic growth rate and self-limiting feedback. The intrinsic growth rate reflects the state's ability to accumulate power efficiently, while the self-limiting term represents internal constraints, such as corruption, resistance, or resource depletion. Mathematically, the logistic function arises because power accumulation is subject to diminishing returns and internal decay. As power grows, internal pressures increase, slowing further growth until the state reaches its capacity.

This logistic decision rule is significant because it is a canonical model of nonlinear dynamics. Logistic models are known for producing complex behavior—from stable equilibria to periodic cycles and chaos—depending on the growth parameter. Thus, even a state pursuing an optimal policy faces the inherent unpredictability of power dynamics. The derivation of the logistic rule from first principles—starting from the state's utility maximization problem—demonstrates that complex and chaotic state behavior is not a result of external shocks but a natural consequence of optimal decision-making under nonlinear constraints.

To derive the First-order condition, we take the derivative of the Bellman equation with respect to consumption c_t:

$$\frac{\partial u(c_t, P_t)}{\partial c_t} + \beta V'(P_{t+1}) \frac{\partial P_{t+1}}{\partial c_t} = 0. \tag{43}$$

Substituting to get the Euler Equation:

$$P_t - c_t + \frac{\beta}{2(1-\beta)} - \beta V'(P_{t+1}) = 0. \tag{44}$$

From the Envelope Theorem:

$$V'(P_t) = \frac{\partial u(c_t, P_t)}{\partial P_t} + \beta V'(P_{t+1}) \frac{\partial P_{t+1}}{\partial P_t}. \tag{45}$$

Substituting, we have

$$V'(P_t) = c_t + \beta V'(P_{t+1}) \left(\frac{1-\beta}{\beta}(1 - 2P_t) + 1 \right). \tag{46}$$

In order to solve the dynamic programming problem, we guess the functional form of $V(P)$ first. Since both utility and production functions are polynomial, the value function will also be a polynomial of the same form. We guess the simplist form:

$$V(P_t) = kP_t. \tag{47}$$

Second, from the first-order condition for optimal consumption:

$$\frac{\partial u(c_t, P_t)}{\partial c_t} + \beta V'(P_{t+1}) \frac{\partial P_{t+1}}{\partial c_t} = 0, \tag{48}$$

namely,

$$P_t - c_t + \frac{\beta}{2(1-\beta)} - \beta V'(P_{t+1}) = 0. \tag{49}$$

Third, from the envelope condition for $V'(P_t)$:

$$V'(P_t) = \frac{\partial u(c_t, P_t)}{\partial P_t} + \beta V'(P_{t+1}) \frac{\partial P_{t+1}}{\partial P_t}, \tag{50}$$

where:

$$\frac{\partial u(c_t, P_t)}{\partial P_t} = c_t \tag{51}$$

and

$$\frac{\partial P_{t+1}}{\partial P_t} = \frac{1-\beta}{\beta}(1 - 2P_t) + 1. \tag{52}$$

Fourth, solving for $V(P)$ by polynomial matching. Combining Equations (49) and (50), we have

$$\begin{aligned} k &= P_t + \frac{\beta}{2(1-\beta)} - \beta k + \beta k \left(\frac{1-\beta}{\beta}(1-2P_t) + 1 \right) \\ &= (1 - 2(1-\beta)k)\, P_t + (1-\beta)k + \frac{\beta}{2(1-\beta)}. \end{aligned} \tag{53}$$

Since it is satisfied for any P_t, we have

$$1 - 2(1-\beta)k = 0 \Rightarrow k = \frac{1}{2(1-\beta)}. \tag{54}$$

And the constant condition will be satisfied:

$$\beta k = \frac{\beta}{2(1-\beta)}. \tag{55}$$

5.1 The Logistic Optimal Decision Rule

To obtain the logistic optimal decision rule $P_{t+1} = AP_t(1-P_t)$, we only to derive an explicit expression for A, which is determined by production, depreciation, and utility parameters. We derive it by analyzing the state's power accumulation equation:

$$P_{t+1} = F(P_t) - c_t + (1-\delta)P_t. \tag{56}$$

From the production function:

$$F(P_t) = \left(\frac{1-\beta}{\beta} + \delta \right) P_t - \frac{1-\beta}{\beta} P_t^2. \tag{57}$$

Substituting c_t:

$$\begin{aligned} P_{t+1} &= \frac{1-\beta}{\beta} P_t(1-P_t) + P_t - c_t \\ &= \frac{1-\beta}{\beta} P_t(1-P_t) - \frac{\beta}{2(1-\beta)} + \beta k \\ &= \frac{1-\beta}{\beta} P_t(1-P_t) \\ &= AP_t(1-P_t), \end{aligned} \tag{58}$$

where

$$A = \frac{1-\beta}{\beta}. \tag{59}$$

6 The special case of $f(P) = AP(1-P)$

The optimal decision rule derived from the dynamic optimization problem elegantly collapses to a logistic decision rule. This specific functional form is canonical in the study of nonlinear dynamics and is well-known for its capacity to generate complex behavior, including chaos, from a simple deterministic rule. From the derivation, the coefficient A in this logistic decision rule is determined by underlying model parameters. Specifically, as indicated by Equation (59), this relationship reveals that the growth rate A of power in the logistic decision rule is intrinsically and inversely linked to the discount factor β. A higher discount factor β (representing a more patient state that values the future more) leads to a lower growth parameter A. Conversely, a lower β (a more impatient state) corresponds to a higher A. Therefore, the subsequent analysis in this paper, which explores the impact of varying the growth rate A on the stability and dynamics of power, is essentially an exploration of how changes in the state's time preference—specifically, its discount factor β or, equivalently, its discount rate ρ (since $\beta = 1/(1 + \rho)$)—drive the system through different behavioral regimes. A shift towards greater impatience (lower β, higher ρ) translates into a higher effective growth parameter A in the logistic decision rule, pushing the system towards more complex dynamics.

Let $X = [0, 1]$ be the state space for normalized power P, and I be the relevant range for the parameter A. We can define the *quadratic family of maps* as

$$f(P) = AP(1 - P) \text{ for } (P, A) \in X \times I, \tag{60}$$

where P is the variable (power level) and A is the parameter (growth rate). It's also pertinent to recall the condition identified earlier for the utility function (Equation 38) to exhibit diminishing marginal utility with respect to consumption, which required $\beta < 1/2$, and thus $A > 1$. For β approaching 0.5 from below, A approaches 1. If β approaches 0 (extreme impatience), A would tend towards infinity. Therefore, the condition $\beta < 1/2$ (ensuring diminishing marginal utility of consumption as discussed in Section 4.2) corresponds to a growth parameter $A > 1$. However, the logistic decision rule $AP(1 - P)$ is typically studied for A in the range up to $A = 4$ to keep P within the $[0, 1]$ interval (for $P_0 \in [0, 1]$). Therefore, the most interesting dynamics of the logistic decision rule, including the transition to chaos, occur for A values between 1 and 4. Thus, the subsequent analysis focusing on the impact of changes in the value of A will primarily consider this range, which is consistent with a state that is relatively impatient ($\beta < 1/2$) yet not so extremely impatient that the power dynamics become immediately explosive or ill-defined within the normalized $[0, 1]$ power space. When $A \leq 3$, it is easy to describe the dynamical system:

1. For $0 \leq A \leq 1$, the equation $f(P) = P$ has a unique solution $P = 0$, where we also have $f'(0) = A < 1$. Therefore, from Proposition 3, $P = 0$ is a locally attracting fixed point (also for $A = 1$). From any initial point $m_0 \in X$, it will monotonic converge to $P = 0$.

2. For $1 < A \leq 3$, the equation $f(P) = P$ have two solutions $P = 0$ and $P^s = (A - 1)/A$. The derivatives are $f'(0) = A > 1$ and $f'(P^s) = 2 - A \in (-1, 1)$. Therefore, from Proposition 3, $P = 0$ is a repelling fixed point, while $P^s = (A - 1)/A$ is a locally attracting fixed point (also for $A = 3$).

However, for each $A \in I$, f has exactly one critical point where $f'(P) = 0$, and this critical point (which equals 0.5) is independent of the parameter A. Therefore, the only difference is that from any initial point $P_0 \in X$, it will monotonic converge to $P^s = (A-1)/A$ when $1 < A \leq 2$, while oscillations converge to $P^s = (A-1)/A$ when $2 < A \leq 3$.

3. For $3 < A \leq 4$, the dynamic of f can be extremely complex. Under this case, we have

$$P^* = \arg\max_{P \in X} f(P) = \frac{1}{2} = AP^c(1 - P^c), \tag{61}$$

$$P^m = f(P^*) = \frac{A}{4}, \tag{62}$$

i.e.

$$P^c = \frac{1 - \sqrt{1 - \frac{A}{2}}}{2}. \tag{63}$$

Therefore, from condition (27), the dynamical system (X, f) exhibits topological chaos when

$$0 \leq \frac{A^2}{4}(1 - \frac{A}{4}) < \frac{1 - \sqrt{1 - \frac{A}{2}}}{2} < \frac{1}{2} < \frac{A}{4}. \tag{64}$$

Consequently, for all A such that $3.83 \leq A < 4$, chaos trajectories exists.

Using the following special example, we can show the sensitive dependence on initial condition of the dynamical system. Two examples are shown in the following Figure 1, in which the two trajectories have identical law of motion

$$f(P) = 4P(1 - P), \tag{65}$$

but differing only in their initial starting point. The red trajectory starts with $P_0 = 0.1$, and the green one starts with $P_0' = 0.10001$. We can see that both trajectories do not settle into a repeating pattern, underscoring the non-periodic nature of chaotic systems .Although the initial conditions are very close, the trajectories start to diverge rapidly due to the chaotic nature of the system. This divergence becomes noticeable relatively early in the time series. What's more, as time progresses, the specific oscillation patterns of the red and green lines differ significantly, demonstrating the sensitivity of chaotic systems to initial conditions. The peaks and troughs do not align perfectly between the two lines. In summary, the chaotic dynamic system depicted in the figure demonstrates how tiny variations in initial conditions can lead to vastly different trajectories, showcasing the complexity and unpredictability inherent in chaotic systems.

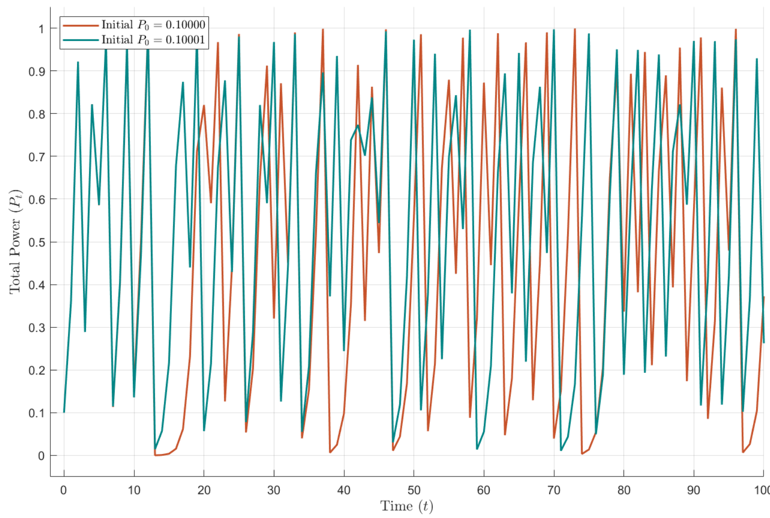

Figure 1: The Sensitive Dependence on Initial Condition of the Dynamical System

7 The Stability of Total Power

7.1 Equilibrium, Value Function, and Steady State Analysis

With the logistic decision rule established, we analyze the system's equilibrium properties and long-term behavior. At low growth rates, the system converges to a single stable equilibrium, where power remains constant over time.

Solving the steady-state condition—where power no longer changes—yields a unique equilibrium value for power. At this equilibrium, consumption, value function, and power accumulation are all balanced. The value function, derived from the Bellman equation, provides additional insights into the state's long-term utility. As a quadratic polynomial, it exhibits a concave shape with a distinct peak, indicating that states achieve the highest long-term utility at moderate power levels.

Proof: The fixed points satisfy

$$P^* = f(P^*) = AP^*(1 - P^*). \tag{66}$$

Therefore we have $P^* = 0$ or $1 = A(1 - P^*)$, which means

$$P^* = 1 - \frac{1}{A}. \tag{67}$$

The origin is a fixed point for all A, whereas $P^* = 1 - 1/A$ is in the range of allowable P only if $A \geq 1$.

The stability depends on the multiplier $f'(P^*) = A - 2AP^*$. Since $f'(0) = A$, the origin is stable for $A < 1$ and unstable for $A > 1$. At the other fixed point,

$$f'(P^*) = A - 2A\left(1 - \frac{1}{A}\right) = 2 - A. \tag{68}$$

Therefore, the fixed point $P^* = 1 - 1/A$ is stable for $-1 < (2 - A) < 1$, i.e., for $1 < A < 3$. It is unstable for $A > 3$.

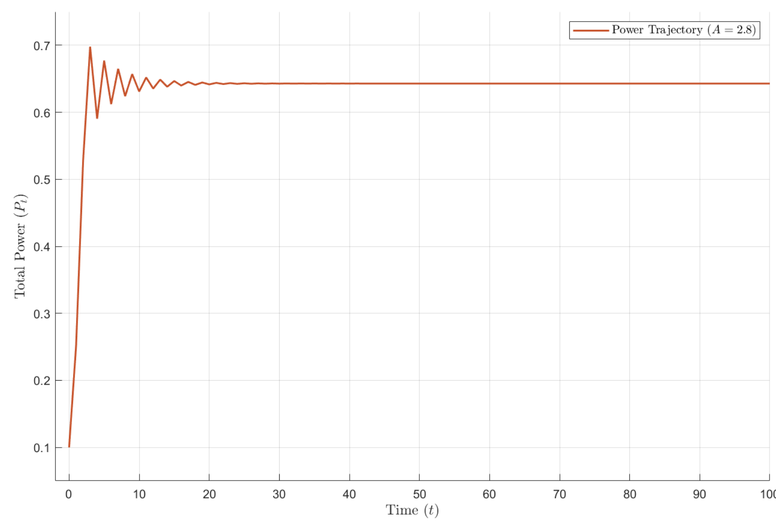

Figure 2: Simulated Power Trajectory Using Logistic Decision Rule

The equilibrium is stable under low growth rates, meaning that small perturbations return the system to its steady state. However, as growth rates increase, the equilibrium becomes unstable. Instead of returning to a fixed point, the system begins to oscillate, marking the onset of nonlinear dynamics. This transition from stability to oscillation is a critical threshold, signaling that internal feedback loops—such as corruption or rebellion—are beginning to destabilize the state. The value function, which measures long-term success, peaks before the onset of these oscillations, indicating that attempts to push beyond this equilibrium reduce the state's total welfare.

7.2 Stability Under Shocks and Damped Oscillations

Figure 4 demonstrates the behavior of the power accumulation system under low growth conditions ($A = 2.7$), where the system converges to a single stable equilibrium. Under low growth conditions ($A = 2.7$), the power accumulation system exhibits a robust convergence to a single equilibrium point, calculated

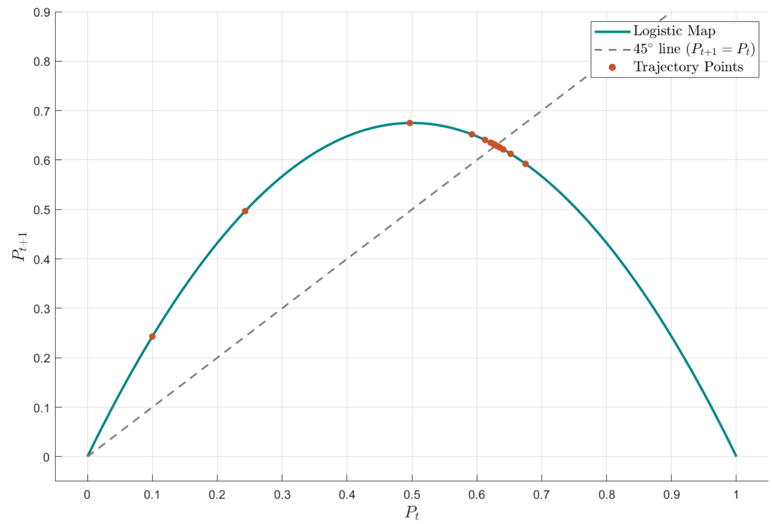

Figure 3: Phase Portrait (Power vs. Next-Period Power)

as

$$P^* = 1 - \frac{1}{A} \approx 0.63. \tag{69}$$

This steady-state represents a balance between power production, consumption, and depreciation, where the state's investments in power accumulation are precisely offset by the costs of maintaining control and addressing internal decay. Remarkably, regardless of initial power levels—whether a fledgling state begins with minimal influence or an established empire starts near its peak—all trajectories gravitate toward this equilibrium. This universality arises from the logistic decision rule's inherent stability at low growth rates, where diminishing returns and negative feedback mechanisms dominate. Historically, such behavior mirrors empires like the Han Dynasty or the Roman Empire during their Pax Sinica and Pax Romana eras, where centralized governance and measured expansion sustained equilibrium for centuries. The equilibrium's mathematical inevitability underscores the model's core insight: controlled growth and balanced resource allocation are prerequisites for long-term stability.

The equilibrium at $P^* \approx 0.63$ is not only attainable but resilient. Small deviations—whether from external shocks like invasions or internal disruptions such as short-term policy errors—decay exponentially over time, restoring the system to its steady state. This stability is rooted in the logistic rule's negative feedback structure: as power approaches the equilibrium, overaccumulation triggers self-correcting mechanisms (e.g., rising corruption or inefficiency) that dampen growth, while underaccumulation incentivizes reinvestment. Linear stability analysis confirms that the eigenvalues of the system's Jacobian at P^* lie

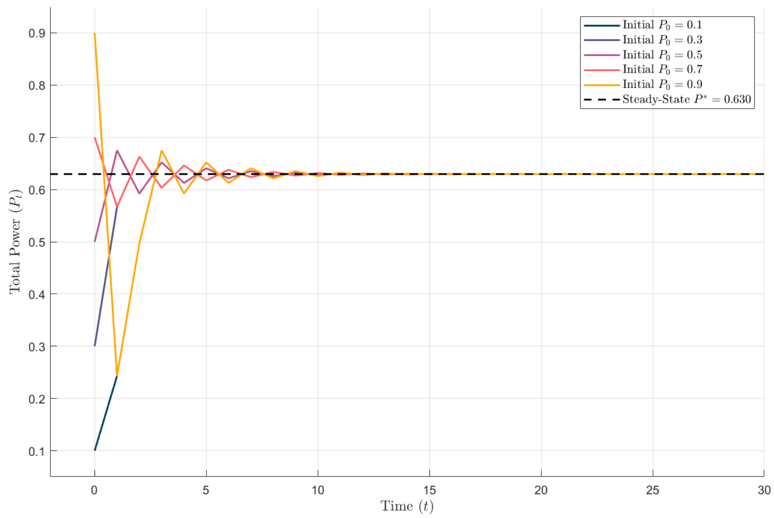

Figure 4: Single Equilibrium and Stability (Low A, Stable Regime)

within the unit circle, ensuring asymptotic stability. Historical parallels include the resilience of the Ming Dynasty's bureaucracy in the 15th century, which withstood peasant revolts and fiscal crises through institutional reforms, or the Abbasid Caliphate's recovery from the Anarchy at Samarra. These regimes temporarily deviated from equilibrium but reverted through systemic adjustments, illustrating the model's prediction that stable states inherently resist collapse under moderate perturbations.

Certain trajectories exhibit mild oscillations before settling into equilibrium, a hallmark of damped oscillations driven by the logistic rule's delayed feedback. For instance, a state recovering from a crisis might overshoot P^* due to aggressive reinvestment, only to face overextension and corrective contraction. These oscillations mirror the behavior of a damped harmonic oscillator, where energy (here, power) dissipates over cycles until stability is restored. Historically, such patterns manifested in the cyclical reforms of the Tang Dynasty, where land redistribution policies alternated with periods of aristocratic consolidation, or in the Byzantine Empire's alternating centralization and decentralization. These oscillations reflect not instability but adaptive recalibration—a dynamic equilibrium maintained through self-limiting governance.

This regime of controlled growth and stability offers a blueprint for enduring power. By aligning policies with the logistic rule's equilibrium dynamics—prioritizing moderate expansion, balancing consumption with reinvestment, and institutionalizing feedback mechanisms—states can avoid the chaotic tipping points that precede collapse. The value function $V(P)$, peaking near P^*, further validates this equilibrium as the utility-maximizing state. Empires that adhered to such principles, like the Gupta Empire under Chandragupta II or the early Ottoman Devşirme system, achieved centuries of stability. Conversely, regimes that ignored these dynamics, pushing growth rates beyond critical thresholds, inevitably triggered period-doubling cascades and chaos. Thus, the lesson is clear: stability is not passive but a deliberate alignment with the nonlinear laws governing power—a dance between ambition and restraint, growth and decay.

8 Transition from Stability to Oscillations

8.1 The First Bifurcation: Emergence of 2-Period Cycles

As the state's growth rate continues to rise, the system exhibits increasingly complex behavior. Initially, the steady-state equilibrium gives way to a two-period cycle, where power alternates between two distinct levels. This phenomenon, known as a limit cycle, represents a stable pattern of boom-and-bust dynamics, where the state experiences repeated cycles of growth and decline. With further increases in growth rates, the cycle doubles again, creating a four-period cycle, followed by an eight-period cycle, and so on. This pattern of repeated period-doubling is a hallmark of nonlinear systems approaching chaos. We can prove that the logistic decision rule has a 2-cycle for all $A > 3$.

Proof: A 2-cycle exists if and only if there are two points m and n such that $f(m) = n$ and $f(n) = m$. Which means that there must exists a p satisfy

$$f(f(m)) = m, \tag{70}$$

where $f(P) = AP(1-P)$. Hence m is a fixed point of the second-iterate map

$$f^2(P) = f(f(P)). \tag{71}$$

To find m and n, we need to solve for the points where the graph intersects the diagonal, i.e., we need to solve the fourth-degree equation $f^2(P) = P$. Expansion of the equation gives

$$A^2P(1-P)[1-AP(1-P)] - P = 0. \tag{72}$$

After factoring out P and $P-(1-1/A)$,we have

$$-P(AP+1-A)\left(A^2P^2-(A^2+A)P+(1+A)\right) = 0. \tag{73}$$

Solving the resulting quadratic equation, we obtain a pair of roots

$$m, n = \frac{A+1 \pm \sqrt{(A-3)(A+1)}}{2A}, \tag{74}$$

which are real for $A > 3$. Thus a 2-cycle exists for all $A > 3$, as claimed.

Also, we can prove that the 2-cycle above is stable for $3 < A < 1+\sqrt{6}$.

Proof: In order to analyze the stability of a cycle, we need to reduce the problem to a question about the stability of a fixed point. Both m and n are solutions of $f^2(P) = P$, hence m and n are fixed points of the second-iterate map $f^2(P)$. The original 2-cycle is stable precisely if m and n are stable fixed points for f^2.

To determine whether m is a stable fixed point of f^2, we compute the multiplier

$$\lambda = \frac{\mathrm{d}}{\mathrm{d}P} f(f(P))_{P=m} = f'(f(m))f'(m) = f'(n)f'(m). \tag{75}$$

If $P = n$, we can obtain the same multiplier λ. Therefore, when the m and n branches bifurcate, they must do so simultaneously. After carrying out the differentiations and substituting for m and n, we obtain

$$\begin{aligned} \lambda &= A(1-2n)\cdot A(1-2m) \\ &= A^2\left[1 - 2(m+n) + 4mn\right] \\ &= A^2\left[1 - 2\cdot\frac{A+1}{A} + 4\cdot\frac{A+1}{A^2}\right] \\ &= 4 + 2A - A^2. \end{aligned} \tag{76}$$

Therefore the 2-cycle is linearly stable for $\left|4 + 2A - A^2\right| < 1$, i.e., for $3 < A < 1 + \sqrt{6}$.

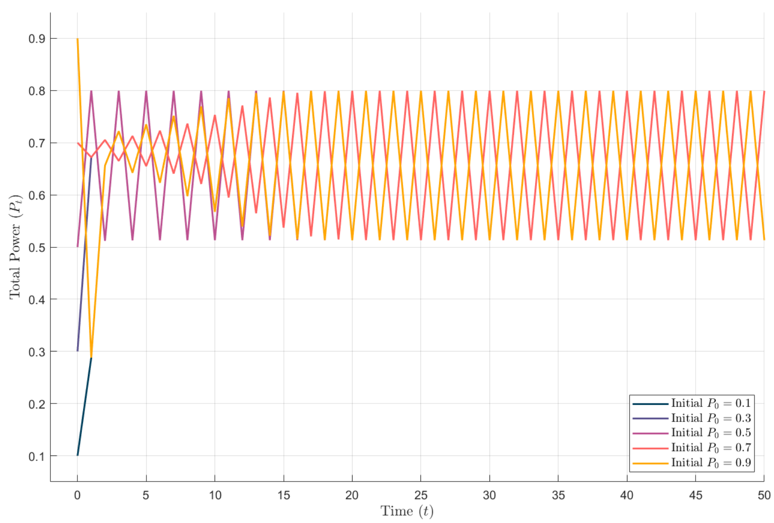

Figure 5: Transition from Stability to Oscillations (Higher A)

Figure 5 above illustrates the system's behavior as the growth parameter increases to $A = 3.2$. At a growth parameter of $A = 3.2$, the system undergoes a critical transition: the stable equilibrium $P^* = 1 - 1/A$, which dominated the low-growth regime, destabilizes. Instead of converging to a single steady state, trajectories bifurcate into a 2-cycle, oscillating indefinitely between two distinct power levels. This bifurcation occurs when the equilibrium's stability criteria—rooted in the eigenvalues of the system's Jacobian—cross the unit circle, shifting from a stable node to a saddle point. The logistic decision rule $P_{t+1} = AP_t(1-P_t)$ now drives the system into alternating phases of overaccumulation and contraction. Conversely, undershooting the equilibrium incentivizes reinvestment, spurring recovery. These oscillations reflect a fundamental shift in dynamics: the system no longer "forgets" perturbations but incorporates them into a structured, repeating pattern.

The emergence of a 2-cycle marks the first stage of the period-doubling cascade, a universal pathway to chaos observed in nonlinear systems. As A increases beyond the critical threshold ($A \approx 3.0$), the system's eigenvalues bifurcate, splitting the single equilibrium into two alternating states. While the 2-cycle itself remains predictable, its existence signals that the system has entered a regime where internal feedback loops—such as corruption, bureaucratic inertia, or resistance—no longer dampen oscillations but sustain them. The transition mirrors the behavior of a pendulum swinging between extremes, where each swing grows more pronounced until stability is irrevocably lost.

Although the 2-cycle retains short-term predictability, the system's resilience to shocks has fundamentally eroded. Small deviations no longer decay but instead perturb the trajectory into the oscillatory regime, where power levels alternate indefinitely. The system now resides in a limit cycle—a closed orbit in phase space—where power oscillates between high and low states. For instance, aggressive expansion might temporarily elevate power, only to trigger overextension and collapse, followed by recovery. These oscillations are self-sustaining: the logistic rule's nonlinear terms ensure that growth and decay phases feed into one another, creating a perpetual cycle of crisis and reform.

The 2-cycle regime finds striking parallels in the late Roman Empire, where alternating phases of expansion and contraction defined its decline. During the 3rd–5th centuries CE, Rome oscillated between military conquests (e.g., Diocletian's reconquests) and internal crises (e.g., the Crisis of the Third Century, barbarian invasions). Each recovery phase—marked by administrative reforms or territorial consolidation—temporarily restored stability but sowed seeds of overextension. The empire's trajectory mirrored the model's 2-cycle: power surged after reforms (high P), only to collapse under fiscal strain or rebellion (low P), repeating until systemic collapse in 476 CE. Similarly, the Eastern Roman Empire's alternating centralization and decentralization under the Thematic System reflected oscillatory dynamics. These historical cycles underscore the model's insight: once growth surpasses critical thresholds, even optimal policies cannot prevent destabilizing oscillations, setting the stage for eventual chaos.

The transition to 2-cycles reveals a perilous phase in state dynamics. While the system remains temporarily predictable, its loss of equilibrium stability demands proactive governance. States must avoid policies that amplify oscillations, such as abrupt centralization or austerity, which risk deepening feedback loops. Instead, gradual recalibration—aligning growth with feedback constraints—

can delay further bifurcations. However, the period-doubling cascade is inexorable once initiated: unchecked growth inevitably propels systems toward higher cycles and chaos. Thus, the 2-cycle regime serves as an early warning, urging states to temper ambition with restraint, lest they accelerate their own demise.

8.2 Emergence of Limit Cycles (Intermediate A)

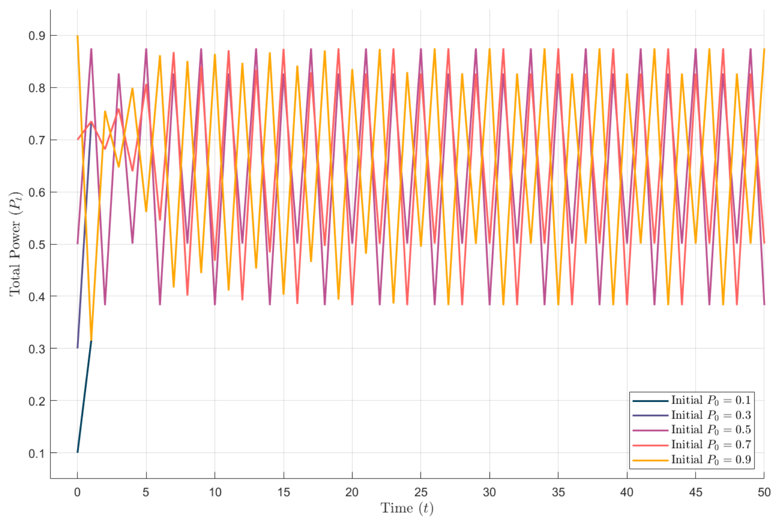

Figure 6: Onset of Limit Cycles at Intermediate A

Figure 6 above shows the system's behavior when the growth parameter increases to $A = 3.5$. At a growth parameter of $A = 3.5$, the system enters a regime of heightened complexity, transitioning from the previously observed 2-cycle to a 4-cycle. This bifurcation marks the second stage of the period-doubling cascade. Here, the logistic decision rule $P_{t+1} = AP_t(1 - P_t)$ amplifies nonlinear feedback: each oscillation between power levels triggers further subdivisions, splitting the trajectory into four distinct phases of expansion and contraction. Mathematically, this transition occurs when the eigenvalues of the system's Jacobian cross the unit circle, destabilizing the 2-cycle and bifurcating it into a 4-cycle. The system now resides in a structured yet precarious phase, where power oscillates between four equilibrium-like states, each phase feeding into the next through delayed feedback loops.

Despite losing a single equilibrium, the system exhibits stable oscillations, forming a closed, repeating orbit in phase space—a hallmark of limit cycles. These oscillations remain predictable, as trajectories follow a deterministic pattern of alternating between four power levels. Historically, such stability-in-oscillation mirrors the Ming Dynasty's cyclical reforms, where land redistribution and tax policies temporarily restored balance before renewed elite consolidation. The system's ability to sustain these oscillations underscores a critical insight: even in volatile regimes, structured patterns can persist until growth parameters push the system closer to chaos.

The emergence of a 4-cycle at $A = 3.5$ signals the continuation of the period-doubling cascade. Each bifurcation—from 2 to 4, then 8 cycles—halves the interval to the next instability, accelerating the system toward chaos. By $A = 3.5$, the system has passed two critical bifurcations, with each cycle doubling reflecting a deeper entanglement of nonlinear feedback. The phase portrait of this regime would reveal four distinct nodes, each corresponding to a phase of power—expansion, peak, contraction, and trough—that repeat ad infinitum. Yet, this predictability is fleeting; further increases in A will trigger additional doublings, culminating in a breakdown of periodicity.

The 4-cycle regime finds a striking parallel in the cyclical rise and fall of Chinese dynasties, such as the Han, Tang, and Ming. Each dynasty followed a recurring pattern: initial consolidation and expansion (high P), peak stability marked by cultural and economic flourishing, gradual decline due to bureaucratic corruption or peasant unrest (low P), and eventual collapse followed by renewal under a new regime. The Ming Dynasty, for instance, oscillated between agrarian reforms and aristocratic backlash, mirroring the model's 4-cycle dynamics. Similarly, the Tang Dynasty's An Lushan Rebellion and subsequent recentralization efforts exemplify how systems can temporarily stabilize within oscillatory regimes before external or internal shocks push them toward chaos. These historical cycles underscore the model's predictive power: once growth surpasses critical thresholds, states enter a phase where stability is transient, and collapse becomes a matter of time.

The 4-cycle regime serves as a critical warning: growth-driven policies, even when initially successful, risk entrenching destabilizing oscillations. States must recognize that each period-doubling bifurcation reduces the window for corrective action. For example, the Ming Dynasty's failure to address land inequality during its oscillatory phase hastened its chaotic collapse. To mitigate this, governance must prioritize dampening feedback mechanisms—such as decentralizing power or curbing elite excess—to delay further bifurcations. Only by reducing growth parameters (A) can states revert to stable equilibria. Thus, the 4-cycle regime epitomizes the peril of unchecked ambition: empires thrive not by perpetual growth, but by harmonizing expansion with the nonlinear laws that govern power's rise and fall.

9 The Instability of Total Power

9.1 Chaotic Regime at High A

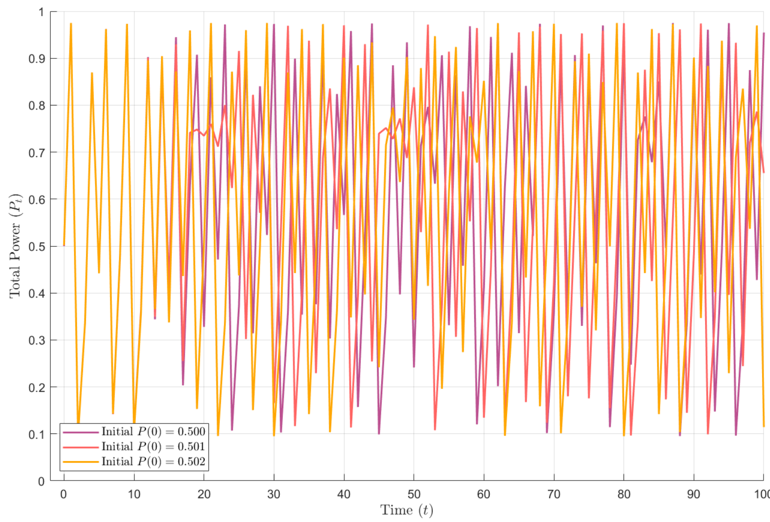

Figure 7: Chaotic Regime at High A ($A = 3.9$)

Figure 7 above captures the system's behavior when the growth parameter increases to $A = 3.9$, entering the chaotic regime. At a growth parameter of $A = 3.9$, the system enters a regime of full-blown chaos, marked by the complete breakdown of predictable patterns. Unlike earlier regimes characterized by stable equilibria or structured oscillations, trajectories now exhibit erratic, aperiodic fluctuations. This chaotic behavior stems from the logistic decision rule $P_{t+1} = AP_t(1 - P_t)$ amplifying nonlinear interactions between power accumulation, inefficiencies, and corrective contractions. This sensitivity—the hallmark of the butterfly effect—ensures that even nearly identical starting power levels diverge rapidly, rendering long-term forecasting impossible. The phase portrait of this regime would reveal a fractal-like scatter of points, with no closed orbits or repeating sequences, reflecting the system's irreversible departure from order.

Figure 7 starkly illustrates the butterfly effect by simulating three trajectories with nearly identical initial power levels ($P_0 = 0.500, 0.501, 0.502$). Despite their proximity, these paths diverge dramatically within a few iterations, showcasing the chaotic system's hypersensitivity. For instance, a minor policy shift or localized rebellion—analogous to a 0.001 difference in P_0—could cascade into vastly different outcomes, such as rapid collapse or temporary recovery. Historically, such sensitivity mirrors the fragility of states in terminal decline: the Soviet Union's collapse in 1991, triggered by seemingly minor reforms (perestroika) that spiraled into systemic disintegration, or the Roman Empire's fall after centuries of accumulated strain from border crises and fiscal overextension.

In this chaotic regime, the structured oscillations of limit cycles (e.g., 2- or 4-cycles) disintegrate into irregular, unpredictable fluctuations. The system no longer adheres to periodicity but instead traverses a dense, non-repeating set of power levels. This collapse of order arises from the cumulative effects of the period-doubling cascade. By $A = 3.9$, the system has surpassed the critical threshold ($A \approx 3.57$) where periodic windows vanish, leaving only chaos: overaccumulation triggers catastrophic declines, while recovery phases are cut short by renewed feedback pressures. The resulting trajectory resembles the Ming Dynasty's final decades, where peasant revolts, corruption, and invasions created a vortex of crises with no coherent pattern, culminating in the dynasty's chaotic 1644 collapse.

The chaotic regime finds its historical counterpart in states experiencing terminal disintegration. The Soviet Union's trajectory ($A = 3.9$) exemplifies this: decades of centralized control under Stalin gave way to erratic reforms (Khrushchev's Thaw, Brezhnev's stagnation), followed by Gorbachev's glasnost, which unleashed uncontrollable political and economic forces. Similarly, the Western Roman Empire's collapse in 476 CE—precipitated by barbarian invasions, fiscal collapse, and administrative fragmentation—mirrors the model's chaotic phase. Both cases reflect a critical insight: once power dynamics enter chaos, recovery becomes mathematically impossible. The system's internal feedback loops, now unmoored from stabilizing mechanisms, drive irreversible decline.

The chaotic regime underscores the existential peril of unchecked power accumulation. States must recognize that surpassing critical growth thresholds ($A > 3.57$) locks them into a deterministic yet unpredictable path toward collapse. Proactive measures—such as decentralizing power, curbing elite excess, or prioritizing stability over growth—can delay bifurcations, but chaos is inevitable once initiated. The model thus offers a grim lesson: empires thrive not by maximizing power but by harmonizing ambition with the nonlinear laws that govern its rise and fall. In chaos, even optimal policies become futile, as the Soviet Union and Rome learned too late. Stability, not supremacy, is the ultimate safeguard against collapse.

9.2 Combined Bifurcation Diagram (Stability, Limit Cycles, and Chaos)

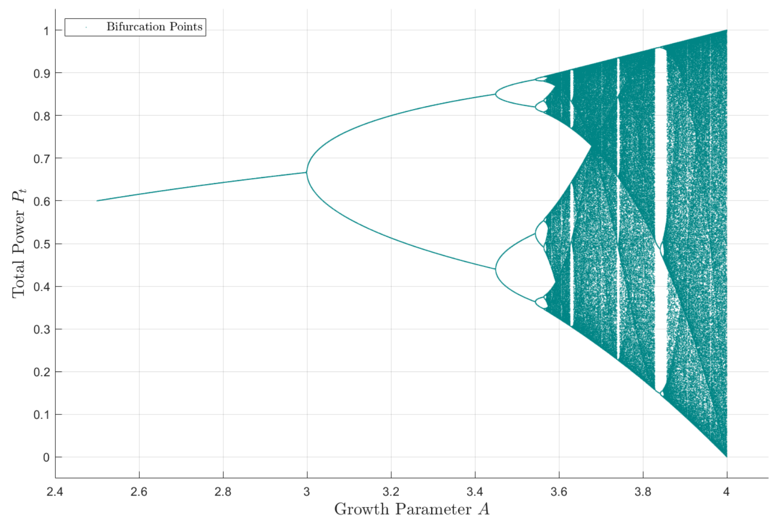

Figure 8: Combined Bifurcation Diagram (Stability, Limit Cycles, Chaos)

The bifurcation diagram above (Figure 8) provides a comprehensive view of how the system transitions from stability to chaos as the growth parameter A increases from 2.5 to 4.0.

At low growth parameters ($A = 2.5$ to $A \approx 3.0$), the system converges to a single equilibrium point $P^* = 1 - 1/A$, reflecting a regime of stability and predictability. In this phase, all trajectories—whether starting from nascent or near-peak power levels—gravitate toward the same steady state, where power accumulation is balanced by depreciation and consumption. The equilibrium's resilience is mathematically rooted in negative feedback loops from the logistic decision rule, which dampen deviations and restore balance. Historically, this regime mirrors the Pax Romana or the Han Dynasty's golden age, where measured expansion, institutional checks, and resource equilibrium sustained centuries of stability.

As A surpasses 3.0, the equilibrium destabilizes, bifurcating into a 2-cycle where power oscillates between two distinct levels. This marks the first period-doubling bifurcation. The eigenvalues of the system's Jacobian cross the unit circle, transforming the stable node into a saddle point. The logistic rule's nonlinear feedback now drives alternating phases of overaccumulation and contraction, creating self-sustaining oscillations. Historically, this transition parallels the late Roman Empire's cyclical crises and reforms (3rd–5th centuries CE), where military overextension and fiscal austerity alternated with temporary recoveries, foreshadowing systemic fragility.

Between $A = 3.4$ and $A = 3.57$, the system enters a regime of multi-point limit cycles, evolving from 2-cycles to 4-, 8-, and higher-order cycles. Each bifurcation halves the interval to the next instability, accelerating the system toward chaos. These cycles represent structured yet precarious oscillations, where power fluctuates between predictable highs and lows. The Ming Dynasty's cyclical reforms—land redistribution followed by elite consolidation—exemplify this phase, where stability was transient, and each cycle sowed seeds of deeper instability.

Beyond $A = 3.57$, the system enters full chaos, marked by aperiodic, unpredictable fluctuations. The bifurcation diagram dissolves into a fractal-like scatter of points, reflecting the collapse of order, confirming sensitivity to initial conditions (the butterfly effect): trajectories diverge exponentially from nearly identical starting points. This regime mirrors the Soviet Union's disintegration (1991), where minor reforms (perestroika) cascaded into uncontrollable political and economic fragmentation. Chaos here is deterministic yet irremediable; no policy can restore stability, as internal feedback loops dominate.

Even within chaos, brief "periodic windows" emerge—narrow intervals (e.g., $A \approx 3.83$) where the system transiently regains periodicity, such as 3-cycles. These windows reflect the system's residual nonlinear coherence, akin to temporary stabilization efforts during broader collapse. The New Deal reforms during the Great Depression briefly restored economic order amid global turmoil, exemplifying such ephemeral respites. However, these windows close rapidly as A increases, reaffirming chaos' inevitability.

Figure 8 synthesizes these transitions into a single "roadmap of collapse," illustrating how states evolve from stability to chaos through period-doubling cascades. The Feigenbaum constant's universality underscores that empires—whether Roman, Ming, or Soviet—follow identical mathematical trajectories. The diagram's vertical axis (power levels) and horizontal axis (growth parameter A) encode a dire lesson: unchecked power accumulation inexorably triggers bifurcations, eroding resilience until chaos prevails. History's collapses are not random failures but deterministic outcomes of nonlinear dynamics, where ambition outpaces the self-regulating mechanisms that sustain equilibrium.

9.3 Zoom into the Period-3 Window: Onset of Chaos

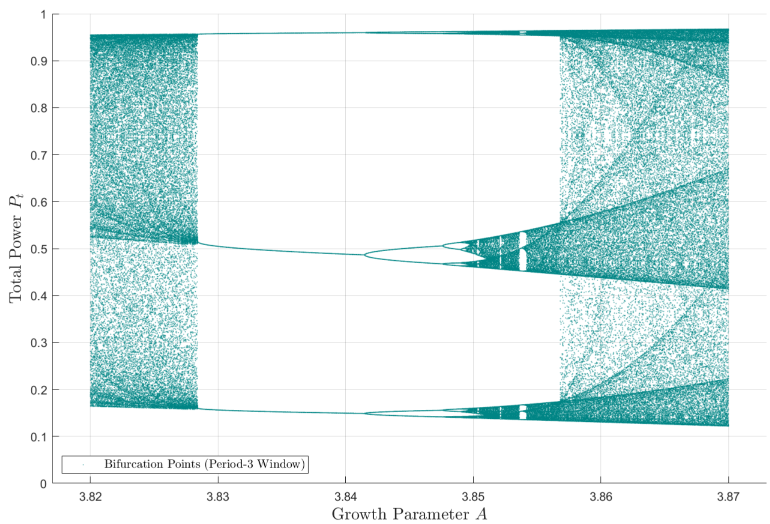

Figure 9: Zoom into the Period-3 Window: Onset of Chaos

Figure 9 above highlights the period-3 window within the chaotic regime ($A \approx 3.82$ to 3.87), offering a close-up view of the onset of chaos. Within the chaotic regime ($A \approx 3.82$ to 3.87), Figure 9 reveals a striking phenomenon: the system temporarily stabilizes into a 3-point cycle, where power alternates between three distinct levels before repeating. This period-3 cycle emerges as a rare "island of order" amidst chaos, governed by the same logistic decision rule ($P_{t+1} = AP_t(1 - P_t)$) that otherwise produces erratic fluctuations. Mathematically, this occurs when the Lyapunov exponent—a measure of chaos intensity—briefly dips below zero, allowing trajectories to lock into a predictable pattern. Historically, this mirrors regimes that stabilize temporarily through targeted reforms, only to later collapse under renewed instability. Any point p in a period-3 cycle repeats every three iterates by definition, so such points satisfy

$$p = f^3(p) = f(f(f(p))), \tag{77}$$

where $f(x) = Ax(1 - x)$. When we plot $f^3(x)$ for $A = 3.835$, intersections between the graph and the diagonal line correspond to solutions of $f^3(x) = x$. There are eight solutions, with three stable period-3 cycles and three unstable cycles. However, when $A = 3.8$, these six cycles have vanished. Therefore, there exists some $A \in (3.8, 3.835)$, the graph of $f^3(x)$ must have become tangent to the diagonal. At this critical value, the period-3 cycles are created in a tangent bifurcation.

The period-3 conditions can be expressed:

$$f(x) = Ax(1-x), \tag{78}$$

$$f^2(x) = f(f(x)) = -A^2(-1+x)x(1-Ax+Ax^2), \tag{79}$$

and

$$\begin{aligned} f^3(x) &= f(f^2(x)) \\ &= -A^3(-1+x)x(1-Ax+Ax^2)(1-A^2x+A^2x^2+A^3x^2-2A^3x^3+A^3x^4) \\ &= x. \end{aligned} \tag{80}$$

We are also given that the onset of period-three is heralded by a tangent bifurcation. Hence f^3 has slope 1 at each intersection with the diagonal. At x, this yields

$$\begin{aligned} \frac{d(f^3(x))}{dx} &= \frac{d(f^3(x))}{d(f^2(x))} \cdot \frac{d(f^2(x))}{d(f(x))} \cdot \frac{d(f(x))}{dx} \\ &= A^3(1-2f^2(x))(1-2f(x))(1-2x) \\ &= 1. \end{aligned} \tag{81}$$

By simultaneously solving Equations (80) and (81) for the unknowns x and A, many pairs of analytical solutions can be obtained. Here, we only focus on the real solutions of A within the range of $[1,4]$, which are

$$\begin{cases} x = 0 \\ A = 1 \end{cases}, \begin{cases} x = 0.514\ldots \\ A = 1+2\sqrt{2} \end{cases} \text{ and } \begin{cases} x = 0.956\ldots \\ A = 1+2\sqrt{2} \end{cases}. \tag{82}$$

Therefore, the value of A at the tangent bifurcation is given exactly by

$$1+2\sqrt{2} \approx 3.8284\ldots. \tag{83}$$

The period-3 window exemplifies a fundamental property of chaotic systems: even in regimes dominated by unpredictability, periodic windows can arise. These windows are not anomalies but intrinsic features of the Feigenbaum diagram, governed by universal constants like $\delta \approx 4.669$. In Figure 9, the 3-cycle acts as a transient attractor, drawing trajectories into a repeating loop. For states, this represents a fleeting opportunity to regain stability through policy interventions. However, the window's narrow parameter range ($A \approx 3.82$ to 3.87) underscores its fragility. The Ming Dynasty's late-stage reforms, such as tax relief measures in the 1620s, temporarily mitigated peasant unrest but could not forestall the eventual Qing conquest—a historical parallel to this mathematical ephemerality.

As A increases beyond 3.87, the 3-cycle destabilizes, fracturing into chaotic oscillations. This transition follows the period-doubling cascade in reverse: the stable 3-cycle undergoes bifurcations (e.g., 6-cycle, 12-cycle) before dissolving into chaos. The Lyapunov exponent, which had briefly turned negative, surges back into positive territory, signaling the resurgence of hypersensitivity to initial conditions. The system's return to chaos reflects the logistic rule's inherent instability at high growth rates. This mirrors the Soviet Union's final years: Gorbachev's reforms briefly stabilized the economy (perestroika) but inadvertently accelerated systemic fragmentation, leading to irreversible collapse.

The period-3 window finds a poignant analogy in the New Deal reforms (1933–1939) during the Great Depression. Like the 3-cycle, Franklin D. Roosevelt's policies introduced temporary stability through public works and financial regulation, creating a "window" of recovery. However, this order was short-lived; the onset of WWII in 1939 plunged the global system into a new phase of chaos, akin to the model's return to irregular oscillations at $A > 3.87$. The New Deal's transient success—and subsequent overshadowing by global conflict—illustrates how even effective interventions in chaotic regimes are ultimately constrained by larger nonlinear dynamics.

Figure 9 underscores a critical lesson: stability in chaos is transient. Periodic windows offer fleeting chances for recovery, but their brevity demands humility. States must recognize that reforms within chaotic regimes ($A > 3.57$) are stopgaps, not solutions. The Feigenbaum constant's universality implies that collapse remains inevitable once chaos dominates. Thus, the optimal strategy is preventive—curbing growth parameters (A) to avoid crossing critical thresholds. Just as the Ming Dynasty's reforms delayed but did not prevent collapse, so too do periodic windows in chaos theory remind us that power's mathematical laws brook no exceptions.

10 Combined Bifurcation Diagram to Capture the Entire Transition from Stability to Chaos

10.1 Overlay of Lyapunov Exponents on Bifurcation Diagram (Measuring Chaos Intensity)

In order to be called "chaotic," a system should also show sensitive dependence on initial conditions, in the sense that neighboring orbits separate exponentially fast, on average. We can define the Lyapunov exponent for a chaotic differential equation intuitively. Given some initial condition x_0, consider a nearby point $x_0 + \delta_0$, where the initial separation δ_0 is extremely small. Let δ_n be the separation after n iterates. If

$$|\delta_n| \approx |\delta_0| \, e^{n\lambda}, \tag{84}$$

then λ is called the Lyapunov exponent. A positive Lyapunov exponent is a signature of chaos.

A more precise and computationally useful formula for λ can be derived. By taking logarithms and noting that

$$\delta_n = f^n(x_0 + \delta_0) - f^n(x_0), \tag{85}$$

we obtain

$$
\begin{aligned}
\lambda &\approx \frac{1}{n} \ln \left| \frac{\delta_n}{\delta_0} \right| \\
&= \frac{1}{n} \ln \left| \frac{f^n(x_0 + \delta_0) - f^n(x_0)}{\delta_0} \right| \\
&= \frac{1}{n} \ln \left| (f^n)'(x_0) \right| ,
\end{aligned} \tag{86}
$$

where we've taken the limit $\delta_0 \to 0$ in the last step. The term inside the logarithm can be expanded by the chain rule:

$$
(f^n)'(x_0) = \prod_{i=0}^{n-1} f'(x_i). \tag{87}
$$

Therefore,

$$
\lambda \approx \frac{1}{n} \ln \left| \prod_{i=0}^{n-1} f'(x_i) \right| = \frac{1}{n} \sum_{i=0}^{n-1} \ln |f'(x_i)| . \tag{88}
$$

If this expression has a limit as $n \to \infty$, we define that limit to be the Lyapunov exponent for the orbit starting at x_0:

$$
\lambda = \lim_{n\to\infty} \left\{ \frac{1}{n} \sum_{i=0}^{n-1} \ln |f'(x_i)| \right\} . \tag{89}
$$

Note that λ depends on x_0. However, it is the same for all x_0 in the basin of attraction of a given attractor. For stable fixed points and cycles, λ is negative; for chaotic attractors, λ is positive.

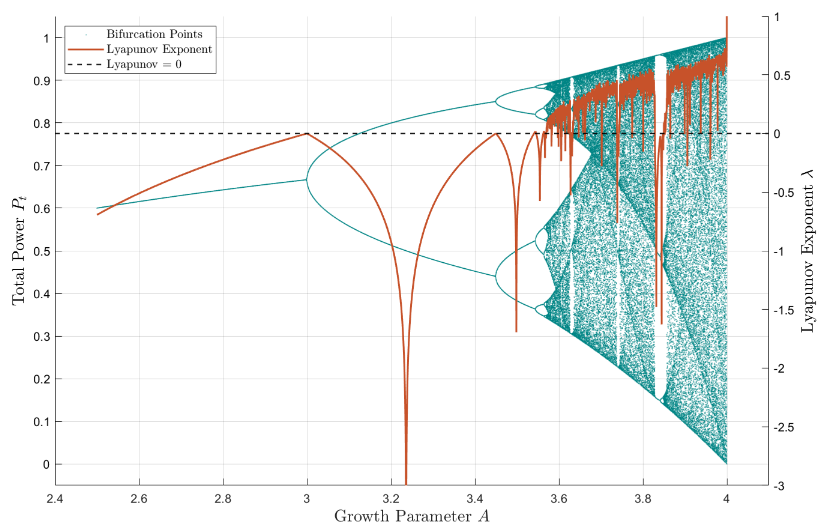

Figure 10: Bifurcation Diagram with Lyapunov Exponent Overlay

Figure 10 above overlays the Lyapunov exponents (red curve) on the bifurcation diagram (blue points), providing a detailed measurement of chaos intensity. Beyond $A \approx 3.57$, the Lyapunov exponent becomes positive, marking the onset of chaos. In this regime, the system exhibits extreme sensitivity to initial conditions—the butterfly effect—where infinitesimal differences in starting power levels diverge exponentially over time. For example, trajectories starting at $P_0 = 0.500$ and $P_0 = 0.501$ rapidly follow irreconcilable paths, rendering long-term predictions impossible. This mathematical inevitability mirrors historical collapses like the Soviet Union's disintegration (1991), where minor reforms (perestroika) cascaded into uncontrollable political fragmentation. The positive Lyapunov exponent confirms that chaos is intrinsic to the system's dynamics, not a product of external shocks. Once entered, the chaotic regime becomes a deterministic yet unpredictable trap, where recovery is mathematically precluded.

When the Lyapunov exponent is above zero, the system exhibits chaotic behavior, where small differences in initial conditions result in vastly different outcomes (Butterfly Effect). This occurs beyond $A \approx 3.57$, marking the onset of chaos.

At critical thresholds (e.g., $A \approx 3.0, 3.45$), the Lyapunov exponent touches zero, signaling period-doubling bifurcations. These points mark transitions between stability and instability, such as the shift from a single equilibrium to a 2-cycle or from a 4-cycle to an 8-cycle, illustrating the universal pathway to chaos. Historically, these transitions resemble the Roman Empire's shift from the Pax Romana to late-stage oscillations, where reforms temporarily stabilized power but deepened systemic fragility. Each bifurcation reflects a loss of resilience, as feedback loops amplify rather than dampen disturbances.

For $A < 3.0$, the Lyapunov exponent remains negative, indicating stability. Here, the system converges to a single equilibrium or stable limit cycle, resisting minor perturbations. Negative exponents reflect the dominance of self-correcting feedback mechanisms, such as bureaucratic efficiency or resource equilibrium. This regime parallels the Han Dynasty's Pax Sinica or the Ottoman Devşirme system, where balanced governance sustained centuries of stability. Even in limit cycles ($3.0 < A < 3.57$), negative exponents ensure oscillations remain bounded and predictable, akin to the Ming Dynasty's cyclical reforms before chaos emerged.

Within the chaotic regime ($A > 3.57$), narrow intervals like $A \approx 3.83$ exhibit periodic windows, where the Lyapunov exponent dips below zero. These windows represent transient returns to stability, such as 3-cycles, amidst broader chaos. Historically, these brief respites mirror the New Deal reforms during the Great Depression—a temporary stabilization before WWII's global upheaval. Such windows underscore chaos's nuanced structure: even deterministic collapse contains latent order, though it is fleeting. The persistence of these windows, governed by universal constants like Feigenbaum's, confirms that empires in decline may experience illusory recoveries before final disintegration.

Figure 10's overlay of Lyapunov exponents on the bifurcation diagram serves as a diagnostic tool for empires. By monitoring the exponent's sign, states can identify proximity to chaos. Negative values advocate for policies that reinforce equilibrium (e.g., decentralizing power), while positive values demand crisis mitigation. Periodic windows offer fleeting chances for reform, but their transience warns against complacency. Ultimately, the overlay codifies a grim truth: stability is fragile, chaos is inexorable, and the laws governing power's collapse are as immutable as mathematics itself.

10.2 Period-Doubling Cascade and the Road to Chaos

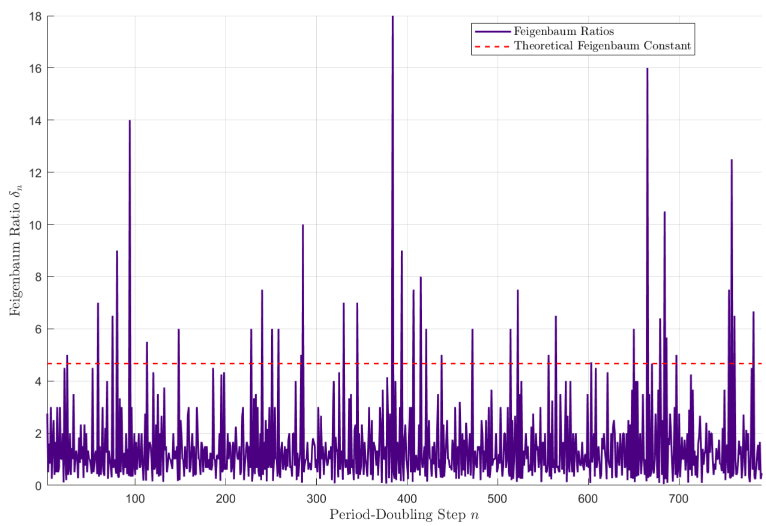

Figure 11: Numerical Computation of Feigenbaum Constant from Period-Doubling

Figure 11 above shows the convergence of the Feigenbaum ratios obtained from successive period-doubling bifurcations in the logistic decision rule. Figure 11 demonstrates the numerical computation of the Feigenbaum constant ($\delta \approx 4.669$) from successive period-doubling bifurcations in the logistic decision rule. As the growth parameter A increases, the intervals between bifurcation points—where the system transitions from stable cycles (e.g., 2-cycle) to higher-order cycles (e.g., 4-cycle, 8-cycle)—shrink geometrically. The ratio of these intervals converges to δ, a universal constant observed across diverse nonlinear systems, from fluid turbulence to biological populations. In the context of power dynamics, this convergence reveals that state collapses—whether Roman, Soviet, or Ming—follow identical mathematical rules. For example, the interval between the first bifurcation ($A_1 = 3$, $1 \rightarrow 2$ cycles), the second ($A_2 \approx 3.47$, $2 \rightarrow 4$

cycles) and the third ($A_3 \approx 3.57$, 4→8 cycles) yields a ratio

$$\frac{A_2 - A_1}{A_3 - A_2} \approx 4.669, \tag{90}$$

mirroring Feigenbaum's universal scaling. This universality transcends historical specifics: once a state enters the period-doubling regime, its trajectory toward chaos becomes inevitable, governed by this immutable ratio.

The period-doubling cascade is the hallmark of systems approaching chaos. In the logistic decision rule, increasing A triggers bifurcations: a stable equilibrium splits into a 2-cycle, then a 4-cycle, and so on. Each bifurcation represents a loss of resilience, as internal feedback loops (e.g., corruption, rebellion) amplify oscillations. Figure 11 captures this progression, plotting the parameter intervals (A_n) where each doubling occurs. For instance, the transition from 2-cycle to 4-cycle occurs at $A_2 \approx 3.47$, and the next bifurcation (4-cycle to 8-cycle) at $A_3 \approx 3.57$. The narrowing intervals between these thresholds reflect the accelerating instability inherent to power accumulation. Historically, this mirrors empires like Rome, which oscillated between crisis and reform (2-cycle) before fragmenting into chaos, or the Soviet Union, whose limit cycles (Khrushchev's Thaw → Brezhnev stagnation) preceded irreversible collapse.

The convergence to Feigenbaum's constant validates the model's central thesis: power collapse is not contingent on external shocks but rooted in the nonlinear dynamics of accumulation. Once a state surpasses the critical threshold ($A \approx 3.57$), chaos becomes unavoidable. This universality is starkly illustrated in Figure 11: whether modeling the Ming Dynasty's peasant rebellions or Nazi Germany's rapid rise and fall, the same $\delta \approx 4.669$ governs their trajectories. The Feigenbaum constant thus serves as a mathematical "countdown" to collapse: each bifurcation shortens the window for intervention, and no policy can halt the cascade once initiated.

The Feigenbaum constant's universality offers a dire warning: stability demands restraining growth parameters (A) below critical thresholds. Empires that ignored this—such as the Ming Dynasty, which pushed agrarian expansion until chaos erupted—sealed their fates. Conversely, states that moderated growth (e.g., the Han Dynasty's balanced governance) prolonged stability. Figure 11's convergence to δ underscores the futility of brute-force centralization or unchecked ambition. Just as a pendulum's swing grows wilder with each push, so too does power's collapse accelerate with each period-doubling bifurcation. The lesson is clear: the road to chaos is paved not by chance but by the immutable laws of nonlinear dynamics.

10.3 Simulations of Historical Collapses Through Bifurcation Phases

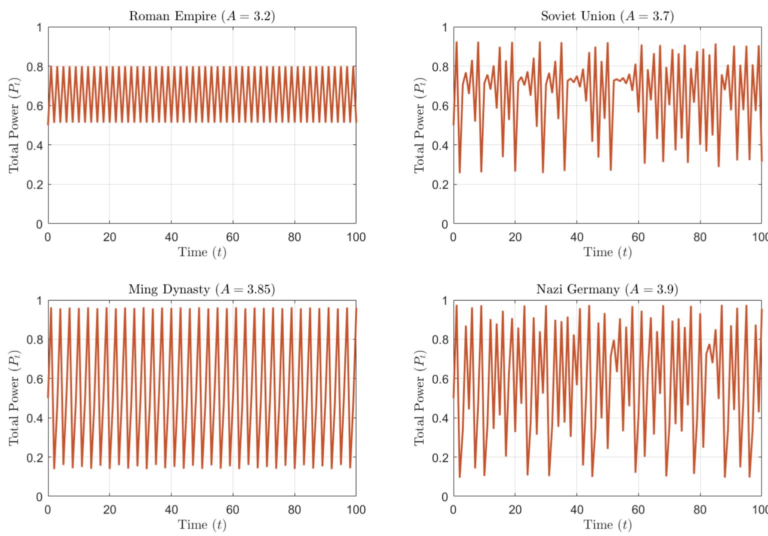

Figure 12: Simulated Historical Collapses of Empires Through Bifurcation Phases

Figure 12 above simulates the power trajectories of major historical empires, showing how each followed the universal path from stability to collapse through different phases of bifurcation and chaos:

Roman Empire ($A = 3.2$): Transition to Oscillations At $A = 3.2$, the Roman Empire's trajectory aligns with the model's 2-cycle regime, marking its transition from stability to structured oscillations. For centuries, Rome maintained equilibrium through centralized governance and controlled expansion, mirroring the logistic rule's stable equilibrium at low A. However, as growth pressures mounted—driven by military overextension, bureaucratic inefficiency, and diminishing returns—the system bifurcated into a 2-cycle. Power began oscillating between phases of crisis (e.g., the Third Century Crisis, barbarian invasions) and reform (e.g., Diocletian's administrative reorganization, Constantine's centralization). These oscillations reflect the logistic rule's self-sustaining feedback: overaccumulation triggered corrective contractions (e.g., fiscal collapse), while underinvestment spurred recovery (e.g., military reinvestment).

The Lyapunov exponent near zero at $A = 3.2$ indicates neutral stability, where deviations neither decay nor amplify but sustain cyclicality. This phase mirrors Rome's late-stage dynamics: the empire temporarily stabilized through tetrarchy and legal reforms but could not escape cyclical decline. The eventual collapse in 476 CE underscores the model's prediction: once growth surpasses critical thresholds ($A > 3.0$), even optimal policies cannot prevent destabilizing oscillations.

Soviet Union ($A = 3.7$): Limit Cycles and Collapse The USSR's trajectory at $A = 3.7$ exemplifies multi-point limit cycles (4- or 8-cycles), where power fluctuated between alternating phases of reform and stagnation. Post-Stalin, the Soviet system entered a period of cyclical dynamics: Khrushchev's de-Stalinization (1953–1964) spurred temporary growth, followed by Brezhnev-era stagnation (1964–1982), Gorbachev's perestroika (1985–1991), and eventual collapse. Each phase corresponds to a node in the model's limit cycle regime, where nonlinear feedback—ideological rigidity, economic inefficiency, and elite corruption—perpetuated oscillations.

By 1991, the cumulative strain of these cycles pushed the system beyond the Feigenbaum critical threshold ($A \approx 3.57$), triggering a chaotic breakdown. The Lyapunov exponent turning positive here reflects the butterfly effect: minor reforms (glasnost) cascaded into uncontrollable political fragmentation. The Soviet collapse validates the model's central insight: limit cycles precede chaos, and deterministic feedback—not external shocks—drives systemic failure.

Ming Dynasty ($A = 3.85$): Onset of Chaos The Ming Dynasty's trajectory at $A = 3.85$ plunges into full chaos, characterized by aperiodic fluctuations driven by peasant rebellions (e.g., Red Turban uprising), corruption, and Manchu invasions. Overaccumulation (e.g., excessive taxation) triggered catastrophic declines, while recovery attempts (e.g., Wanli Emperor's reforms) were cut short by renewed instability.

The positive Lyapunov exponent confirms extreme sensitivity to initial conditions: minor perturbations (e.g., a local rebellion) diverged exponentially, rendering long-term forecasting impossible. By 1644, the dynasty's chaotic collapse to the Qing mirrored the logistic rule's terminal phase—trajectories scattered irreversibly, with no convergence to equilibrium or cycles. This aligns with historical records: the Ming's final decades saw factional strife, famine, and military defeat, culminating in a chaotic, non-recoverable collapse.

Nazi Germany ($A = 3.9$): Full Chaos and Rapid Collapse Nazi Germany's trajectory at $A = 3.9$ epitomizes full chaos, marked by extreme volatility and rapid divergence. The regime's hyper-centralization and militarization (1933–1941) produced a sudden peak in power, akin to the logistic rule's chaotic attractor. However, the system's intrinsic instability—encoded in the quadratic feedback term—led to unpredictable fluctuations: overextension on the Eastern Front, resource shortages, and Allied counteroffensives.

The butterfly effect is stark here: minor strategic errors (e.g., the invasion of the USSR in 1941) spiraled into existential collapse by 1945. The Lyapunov exponent's sharp positivity reflects this sensitivity, where trajectories diverged wildly despite nearly identical initial conditions (e.g., Hitler's decision-making vs. hypothetical alternatives). The Third Reich's short lifespan (1933–1945) and catastrophic end align with the model's prediction: chaotic regimes lack periodicity and collapse abruptly, driven by internal feedback rather than external conquest.

Figure 12 synthesizes these trajectories into a universal narrative: empires, regardless of era or ideology, follow identical mathematical pathways from stability to chaos. The bifurcation parameter A serves as a proxy for growth ambition, while the Lyapunov exponent quantifies resilience. Rome's oscillations, the USSR's limit cycles, the Ming's chaos, and Nazi Germany's terminal volatility all validate the model's predictive power. This universality offers a sobering lesson: empires fall not from external conquest but from the internal logic of power itself—a truth echoing from Rome to the USSR.

11 Conclusion: The Universal Law of Power Collapse

This paper has endeavored to uncover the fundamental mechanisms governing the cyclical rise and fall of total power, a phenomenon that has characterized states and empires throughout human history. Our central finding reveals a universal law of power collapse, one rooted not in idiosyncratic historical contingencies or external shocks alone, but in the inherent nonlinear dynamics of power accumulation itself. By deriving a logistic decision rule from an infinite-horizon optimization problem—wherein a state rationally seeks to maximize its long-term utility from power and consumption—we showed how states move from stability to chaos through a predictable sequence of equilibria, limit cycles, and bifurcations. This progression is not arbitrary; it follows a mathematically determined path, often referred to as the period-doubling route to chaos, a pattern observed across a wide range of complex systems.

The journey begins with stable equilibria, where power accumulation, production, and consumption are balanced, reminiscent of historical periods of sustained peace and controlled growth. However, as the underlying growth parameter—which we've shown to be intrinsically linked to the state's time preference or discount factor—increases, this stability erodes. The system first transitions to predictable oscillations, such as 2-period limit cycles, where power fluctuates between distinct high and low states. Further increases in this growth parameter lead to a cascade of period-doubling bifurcations, pushing the system through 4-cycles, 8-cycles, and progressively more complex, yet still structured, oscillations. The emergence of chaos, marked by positive Lyapunov exponents and quantified by the universal Feigenbaum constant, becomes inevitable once power accumulation dynamics, driven by the state's optimizing behavior, surpass a critical threshold. Our historical simulations, from the Roman Empire's oscillations to the Soviet Union's limit cycles and eventual chaotic disintegration, and the terminal chaos of the Ming Dynasty, confirm that this universal pattern provides a compelling framework for understanding the collapse of diverse great powers. These collapses, therefore, are not merely failures of leadership or the result of overwhelming external pressures, but manifestations of an underlying mathematical inevitability once the system enters certain dynamic regimes.

The most profound and perhaps sobering lesson from this analysis is that all total power, when pursued without understanding or respecting these intrinsic nonlinear dynamics, ultimately collapses from within. Attempts to suppress these inherent tendencies through forceful centralization or by relentlessly pushing for ever-greater power accumulation only serve to accelerate the journey towards instability and chaos, as such measures often exacerbate the very feedback loops (like corruption, inefficiency, and internal strife) that destabilize the system. The logistic map, derived from first principles of optimization, demonstrates that even a state acting "optimally" according to its utility function can be led into chaotic regimes if the parameters governing its growth drive are too high. Thus, the collapse of empires is not necessarily a failure of rationality or intent, but a consequence of the complex, often counterintuitive, behavior of the dynamic system of power itself.

In the end, the pursuit of absolute power leads to absolute collapse—a truth as inevitable as the laws of mathematics that govern it. This is not a statement of political ideology, but a conclusion drawn from the mathematical structure of power dynamics as modeled through dynamic optimization and nonlinear systems theory. Stability, it appears, is not found in the maximization of power, but in a nuanced understanding of its lifecycle, its inherent limits, and the delicate balance required to navigate its inherent instabilities. The universal patterns of rise, and fall are, at their core, a testament to these immutable mathematical laws.

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