Militarization and Capital Accumulation: A Mean Field Game under Security Congestion

Wei Liang and Heng-fu Zou*

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Abstract

This paper develops a dynamic mean field game model of capital accumulation and militarization among symmetric countries facing stochastic shocks. Each country optimizes consumption, productive investment, and military spending while responding to the average militarization level of others. We derive the coupled Hamilton–Jacobi–Bellman and Fokker–Planck equations governing equilibrium, and prove existence and uniqueness under strategic externalities. A linear–quadratic–Gaussian version yields closed-form feedback rules and forward dynamics. Simulations reveal that low internalization of global militarization leads to arms traps and stagnation, while stronger externality awareness supports growth and restraint. The model offers a rigorous framework to study decentralized militarization externalities (a tragedy of the commons) and their macroeconomic consequences without assuming hierarchy or conflict.

Key words: Mean Field Games, Militarization, Strategic Externalities, Capital Accumulation, Arms Race

1 Introduction

Military investment decisions are among the most consequential and persistent features of state behavior. Even in the absence of active conflict or coercion, countries routinely allocate significant resources toward building and maintaining military capacity. This arms accumulation often occurs not because of direct rivalry or imminent threats, but because each nation fears the aggregate military posture of others. In such environments, individual optimization under strategic interdependence can generate collective outcomes that are inefficient, unstable, or self-reinforcing. This paper develops a dynamic, decentralized framework to formally analyze how militarization and capital accumulation co-evolve across a continuum of symmetric countries when each optimizes under uncertainty and strategic externalities.

*Liang: China Economics and Management Academy, Central University of Financeand Economics, Beijing, 100081, China. Email: w3liang2@foxmail.com; Zou: The World Bank, Washington, D. C. 20433, USA. Email: Hzoucema@gmail.com.

We introduce a continuous-time mean field game model in which all countries are symmetric ex ante and face stochastic dynamics in capital and military stocks. Each country chooses consumption, investment in productive capital, and military expenditure to maximize intertemporal utility, subject to a resource constraint and depreciation. Crucially, countries do not directly interact with specific rivals but are instead influenced by the mean field—that is, the average level of militarization across the global population. The strategic tension arises because the mean field enters negatively into both the production function (representing resource or security inefficiencies from global tension) and the utility function (reflecting strategic anxiety). Each country optimizes by taking this aggregate militarization level as given, but in equilibrium, the mean field must coincide with the average outcome of all countries' decentralized behavior.

We analyze this model through the coupled system of a Hamilton-Jacobi-Bellman (HJB) equation (describing the optimal policy of a representative agent), a Fokker-Planck-Kolmogorov (FPK) equation (describing the evolution of the population distribution of states), and a fixed-point condition equating perceived and realized average militarization. We show that under regularity conditions, this system admits a unique equilibrium. In a simplified linear-quadratic-Gaussian (LQG) version of the model, we derive closed-form expressions for optimal investment policies and simulate the joint evolution of capital, militarization, and welfare. Our numerical results reveal that small differences in how much countries internalize the disutility of global militarization lead to dramatically different long-run outcomes: either a militarized equilibrium with stagnant growth and low welfare, or a restrained equilibrium with capital accumulation and higher well-being. These results demonstrate that systemic militarization can emerge endogenously—driven by decentralized fear and strategic symmetry—even when no country intends aggression.

The contribution of this paper is threefold. First, it offers a tractable and rigorous framework for studying the dynamic co-evolution of militarization and capital in a fully symmetric, non-hierarchical world. Second, it shows how strategic externalities, even when modest, can fundamentally alter macroeconomic outcomes and security equilibria. Third, it provides a foundation for analyzing policy interventions, such as disarmament incentives or alliance formation, that might shift equilibrium outcomes without requiring centralized enforcement. While much of the existing literature focuses on pairwise rivalry, hegemonic actors, or repeated signaling, our framework highlights the endogenous emergence of arms races under complete symmetry and decentralized rationality.

The remainder of the paper proceeds as follows. Section 2 introduces the model setup, including the stochastic dynamics, utility specification, and mean field structure. Section 3 derives the coupled equations that characterize equilibrium. Section 4 specializes the model to a linear-quadratic-Gaussian benchmark and derives closed-form solutions. Section 5 establishes existence and uniqueness of equilibrium using fixed-point arguments, including the explicit Fokker–Planck dynamics and their consistency with the equilibrium feedback laws. Section 6 introduces a benevolent social planner and derives the Pigouvian tax that decentralizes the social optimum. Section 7 provides a welfare comparison between the MFG and social-planner solutions. Section 8 provides numerical simulations illustrating how different externality regimes shape capital, militarization, and welfare. Section 9 concludes and outlines promising extensions related to heterogeneity, institutional design, learning dynamics, and spatial conflict diffusion.

2 Model Setup

We consider a continuum of symmetric countries indexed by $i \in [0,1]$, each of which simultaneously optimizes its own consumption, capital accumulation, and military investment under strategic interdependence with the rest of the population. The key state variables for each country i are its productive capital $k_i(t) \in \mathbb{R}_+$ and military capital $m_i(t) \in \mathbb{R}_+$, both of which evolve over continuous time $t \geq 0$. The countries are atomistic, but interact through a mean field represented by the population distribution of military stock.

2.1 State Dynamics

Each country controls its capital investment $i_{k,i}(t)$ and military investment $g_i(t)$, subject to depreciation and stochastic shocks. The capital and military dynamics follow:

$$dk_i(t) = \left(i_{k,i}(t) - \delta^k k_i(t)\right) dt + \sigma_k dW_i^k(t), \tag{1}$$

$$dm_i(t) = \left(g_i(t) - \delta^m m_i(t)\right) dt + \sigma_m dW_i^m(t), \tag{2}$$

where $\delta^k, \delta^m > 0$ are constant depreciation rates, and $W_i^k(t), W_i^m(t)$ are independent Brownian motions representing idiosyncratic shocks to capital and military dynamics, with volatilities $\sigma_k, \sigma_m > 0$.

The Brownian motions $\left\{W_i^k, W_i^m\right\}_{i\in[0,1]}$ are mutually independent across countries and across sectors, representing purely idiosyncratic shocks with no aggregate component. We work on the extended state space $\mathbb{R}^2$ and note that, under the parameter values used throughout, trajectories remain in a neighborhood of the positive orthant with high probability. The Gaussian approximation on $\mathbb{R}^2$ is standard in the LQG Mean Field Games literature (Bensoussan et al., 2013).

2.2 Production and Budget Constraint

Output is produced linearly using capital and military resources, but is penalized by strategic congestion effects arising from average global militarization. The instantaneous budget constraint for country i is:

$$c_i(t) + i_{k,i}(t) + g_i(t) = Ak_i(t) + Bm_i(t) - D\bar{m}(t), \tag{3}$$

where:

• $c_i(t)$ is consumption,
• $A, B > 0$ represent the productivity of capital and military resources,
• $\bar{m}(t) = \int_0^1 m_j(t)dj$ is the mean militarization across the population,
• $D > 0$ captures the negative externality from global arms buildup on productive output (e.g., insecurity, arms races, inefficient resource use).

Note that aggregate militarization $\bar{m}(t)$ enters the model through two distinct channels. Here, in the budget constraint, the coefficient $D > 0$ captures a productive externality: a heavily armed international environment reduces effective output for every nation—through insecurity spillovers, disrupted trade, or diverted resources. A second channel, through the utility function (Section 2.3), captures a preference externality via $\eta_2 > 0$. Separating these two mechanisms is essential for the policy analysis in Section 6.

2.3 Utility Function

Each country maximizes expected discounted utility, which depends on consumption, military strength, and strategic externalities from average militarization. The instantaneous utility of country i is given by:

$$u_i(t) = ac_i(t) - \frac{1}{2}bc_i(t)^2 + \eta m_i(t) - \frac{1}{2}\phi m_i(t)^2 - \eta_2 m_i(t)\bar{m}(t) - \frac{\theta_k}{2}i_k^2 - \frac{\theta_g}{2}g^2, \quad (4)$$

where:
• $a > 0, b > 0$: utility parameters for concave consumption preferences,
• $\eta, \phi > 0$: represent the marginal benefit and cost of militarization,
•$\eta_2 > 0$: interactive militarization cost parameter (congestion externality).

The coefficient $\eta_2 > 0$ is the central parameter of the model. The term$-\eta_2 m_i \bar{m}$ captures an interactive militarization cost: a nation's marginal disutility from its own military stockm_i is amplified by aggregate militarization$\bar{m}$, reflecting congestion effects—diplomatic friction, arms-race escalation costs, and fiscal crowding-out—that grow with the overall level of global armament. This creates a tragedy-of-the-commons at the heart of the model: each country benefits individually from higherm_i (viaηm_i), but imposes a negative externality on all countries through$-\eta_2 m_i \bar{m}$, which each agent under-accounts for since they ignore their individual contribution to$\bar{m}$. The Social Planner internalizes this congestion externality, choosing lower$\bar{m}$ and higher$\bar{k}$. The parameterη_2 thus governs whether equilibrium converges to a cooperative growth path or an inefficient arms trap—the central comparative statics result of Section 8.

•$\theta_k, \theta_g > 0$: convex adjustment cost parameters, penalizing rapid changes in investment rates. These terms serve two roles: economically, they represent installation costs or organizational frictions in scaling up military or capital spending (as in Hayashi, 1982); mathematically, they ensure the Hamiltonian is strictly concave in controls (i_k, g) separately, so that the first-order conditions yield a unique interior solution rather than a corner or indeterminate split between the two investment types. Without adjustment costs ($\theta_k = \theta_g = 0$), the first-order conditions pin down only total investment $i_k + g$, leaving the allocation between productive and military capital indeterminate.

2.4 Objective Functional

Each country chooses control processes $\{i_{k,i}(t), g_i(t)\}_{t\geq 0}$ to maximize expected discounted utility over an infinite horizon:

$$\max_{\{i_{k,i}(t), g_i(t)\}} \mathbb{E}_t \left[\int_0^\infty e^{-\rho t} u_i(t) dt \right], \tag{5}$$

subject to the state dynamics and the budget constraint, where $\rho > 0$ is the constant discount rate. All countries are ex ante identical and solve the same optimization problem, but interact through the evolving distribution $\mu(t, x)$ over the state space $\mathbb{R}^2$.

2.5 Population Distribution and Mean Field Interaction

Let $\mu(t, x)$ denote the density of countries at time t with state$x = (k, m) \in \mathbb{R}^2$. Then:

$$\bar{m}(t) = \int_{\mathbb{R}^2} m \cdot \mu(t, k, m) dkdm. \tag{6}$$

The distribution $\mu(t, \cdot)$ evolves as a probability measure over $\mathbb{R}^2$, satisfying the normalization

$$\int_{\mathbb{R}^2} \mu(t, k, m) dkdm = 1 \tag{7}$$

for all $t \geq 0$. We impose an initial condition $\mu(0, k, m) = \mu_0(k, m)$, where$\mu_0 \in \mathcal{P}_2\left(\mathbb{R}^2\right)$ is a given probability measure with finite second moment. In the LQG benchmark of Section 4,μ_0 is taken to be Gaussian, which preserves Gaussianity along the optimal trajectory. This expression for $\bar{m}(t)$ feeds into the budget constraint and utility function of each player. This gives rise to a fully coupled forward-backward system, where each agent's optimal strategy affects, and is affected by, the population distribution.

In the next section, we derive the coupled Hamilton-Jacobi-Bellman (HJB) and Fokker-Planck (FPK) system that characterizes equilibrium in this symmetric mean field game.

Assumption 2.1 (Parameter Conditions). Throughout the paper we maintain: (i) $A, B, D, \delta^k, \delta^m, \sigma_k, \sigma_m, \rho > 0$; (ii) $a, b, \eta, \phi, \eta_2, \theta_k, \theta_g > 0$; (iii) $B \geq D$ (own military productivity weakly exceeds aggregate congestion, ensuring that an individual country's military stock has a non-negative net effect on output in a symmetric equilibrium where $\bar{m} = m_i$); (iv) $a > b\rho/(\rho + 2\delta^k)$ (the discount rate is not so high as to preclude an interior consumption solution).

3 The HJB-FPK System and Equilibrium Characterization

We now turn to the formal characterization of equilibrium in the symmetric mean field game introduced in Section 2. Each country seeks to optimally allocate resources between consumption, capital accumulation, and military investment, accounting for the strategic influence of the average militarization level across the population. The resulting system of equations features a backward Hamilton-Jacobi-Bellman (HJB) partial differential equation representing individual optimality and a forward Fokker-Planck-Kolmogorov (FPK) equation describing the evolution of the distribution of agents. These equations are coupled through the population-level mean field, and jointly characterize the decentralized rational expectations equilibrium.

3.1 The Individual Optimization Problem and the HJB Equation

Let $V(t, k, m)$ denote the value function of a representative country at time $t \geq 0$ and state $x = (k, m) \in \mathbb{R}^2$, which is defined as

$$V(t, k, m) = \max_{i_k, g} \mathbb{E}_t \left[\int_0^\infty e^{-\rho(s-t)} u(c_s, m_s, \bar{m}_s) ds \right], \tag{8}$$

where k and m represent capital and military stock, respectively. Given a path for the average military level $\bar{m}(t)$, the value function solves the dynamic programming equation:

$$\begin{aligned} \rho V(t, k, m) - \frac{\partial V(t, k, m)}{\partial t} &= \max_{i_k, g} \Big\{ a \cdot c(k, m, i_k, g; \bar{m}) - \frac{1}{2} b \cdot c(k, m, i_k, g; \bar{m})^2 \\ &+ \eta m - \frac{1}{2} \phi m^2 - \eta_2 m \bar{m}(t) - \frac{\theta_k}{2} i_k^2 - \frac{\theta_g}{2} g^2 \\ &+ \frac{\partial V(t, k, m)}{\partial k} (i_k - \delta^k k) + \frac{\partial V(t, k, m)}{\partial m} (g - \delta^m m) \\ &+ \frac{1}{2} \sigma_k^2 \frac{\partial^2 V(t, k, m)}{\partial k^2} + \frac{1}{2} \sigma_m^2 \frac{\partial^2 V(t, k, m)}{\partial m^2} \Big\}, \end{aligned} \tag{9}$$

where consumption is determined residually by the instantaneous resource constraint:

$$c(k, m, i_k, g; \bar{m}) = Ak + Bm - D\bar{m}(t) - i_k - g. \tag{10}$$

The control variables i_k and g represent the investment rates in capital and military stocks. The agent maximizes utility from consumption and military security, net of both internal and strategic externalities arising from aggregate militarization.

3.2 Optimal Investment Policy and Feedback Form

Let $Y \equiv Ak + Bm - D\bar{m}(t)$ denote aggregate available resources at state (k, m) and mean field $\bar{m}(t)$. The instantaneous resource constraint $c = Y - i_k - g$ determines consumption residually once investment controls are chosen. Assuming interior solutions, the first-order conditions for the Hamiltonian with respect to i_k and g are:

$$\theta_k i_k^* - (a - bc^*) = -\frac{\partial V}{\partial k}, \tag{11}$$

$$\theta_g g^* - (a - bc^*) = -\frac{\partial V}{\partial m}, \tag{12}$$

or equivalently

$$\theta_k i_k^* = V_k - a + bc^*, \tag{13}$$

$$\theta_g g^* = V_m - a + bc^*, \tag{14}$$

where $V_k \equiv \partial V/\partial k$ and $V_m \equiv \partial V/\partial m$. Substituting $c^* = Y - i_k^* - g^*$ and rearranging yields the 2×2 linear system:

$$(\theta_k + b)i_k^* + bg^* = V_k - a + bY, \tag{15}$$

$$bi_k^* + (\theta_g + b)g^* = V_m - a + bY. \tag{16}$$

The system matrix has determinant $\Delta \equiv \theta_k\theta_g + b(\theta_k + \theta_g) > 0$ under Assumption 2.1(ii), guaranteeing a unique interior solution. Solving by Cramer's rule, the unique optimal feedback rules are:

$$\begin{aligned} i_k^* &= \frac{(\theta_g + b)(V_k - a + bY) - b(V_m - a + bY)}{(\theta_g + b)(\theta_k + b) - b^2} \\ &= \frac{(\theta_g + b)V_k - bV_m + \theta_g(-a + bY)}{\theta_g\theta_k + b(\theta_g + \theta_k)}, \end{aligned} \tag{17}$$

$$\begin{aligned} g^* &= \frac{(\theta_k + b)(V_m - a + bY) - b(V_k - a + bY)}{(\theta_g + b)(\theta_k + b) - b^2} \\ &= \frac{(\theta_k + b)V_m - bV_k + \theta_k(-a + bY)}{\theta_g\theta_k + b(\theta_g + \theta_k)}. \end{aligned} \tag{18}$$

Therefore, we have

$$c^* = \frac{a + \theta_k i_k^* - \partial_k V}{b} = \frac{a + \theta_g g^* - \partial_m V}{b}, \tag{19}$$

Note that in general$V_k \neq V_m$: the shadow values of productive capital and military stock differ because they depreciate at different rates and enter the utility and production functions asymmetrically. Substituting the optimal consumptionc^* into the optimal control equation:

$$\begin{aligned} \rho V - \frac{\partial V}{\partial t} &= a \cdot c^* - \frac{1}{2}b \cdot (c^*)^2 + \eta m - \frac{1}{2}\phi m^2 - \eta_2 m\bar{m}(t) - \frac{\theta_k}{2}(i_k^*)^2 - \frac{\theta_g}{2}(g^*)^2 \\ &\quad + \frac{\partial V}{\partial k}(i_k^* - \delta^k k) + \frac{\partial V}{\partial m}(g^* - \delta^m m) + \frac{1}{2}\sigma_k^2\frac{\partial^2 V}{\partial k^2} + \frac{1}{2}\sigma_m^2\frac{\partial^2 V}{\partial m^2}, \end{aligned} \tag{20}$$

where i_k^* and g^* must satisfy budget consistency:

$$i_k^* + g^* = Ak + Bm - D\bar{m}(t) - c^*. \tag{21}$$

The HJB can be read as a continuous-time flow-balance condition: the required return $\rho V - \partial V/\partial t$ equals the sum of (i) instantaneous utility from optimal consumption and military security, net of the interactive militarization cost$-\eta_2 m\bar{m}(t)$ and convex adjustment costs; (ii) the shadow value of capital and military growth, $V_k(i_k^* - \delta^k k) + V_m(g^* - \delta^m m)$, reflecting the marginal returns to investment net of depreciation; and (iii) a precautionary term

$$\frac{1}{2}\sigma_k^2 V_{kk} + \frac{1}{2}\sigma_m^2 V_{mm} \tag{22}$$

compensating for stochastic uncertainty in capital and military dynamics. This is the continuous-time Bellman envelope condition governing rational expectations equilibrium.

3.3 The Population Law and the Fokker-Planck Equation

Let $\mu(t,k,m)$ denote the joint density of countries over the state space $\mathbb{R}^2$ at time t, we have

$$\int_{\mathbb{R}^2} \mu(t,k,m)dkdm = 1. \tag{23}$$

Given the optimal policy functions derived above, the evolution of the distribution of countries follows the Fokker-Planck-Kolmogorov equation:

$$\begin{aligned}\frac{\partial \mu(t,k,m)}{\partial t} &= -\frac{\partial}{\partial k}\left[\left(i_k(t,k,m) - \delta^k k\right)\mu(t,k,m)\right] \\ &\quad - \frac{\partial}{\partial m}\left[\left(g(t,k,m) - \delta^m m\right)\mu(t,k,m)\right] \\ &\quad + \frac{1}{2}\sigma_k^2 \frac{\partial^2 \mu(t,k,m)}{\partial k^2} + \frac{1}{2}\sigma_m^2 \frac{\partial^2 \mu(t,k,m)}{\partial m^2},\end{aligned} \tag{24}$$

subject to an initial condition $\mu(0,k,m) = \mu_0(k,m) \in \mathcal{P}_2\left(\mathbb{R}^2\right)$. The first two terms describe deterministic drifts in state variables driven by optimal policy, while the last two represent diffusion due to uncertainty.

The average militarization level is defined consistently by:

$$\bar{m}(t) = \int_{\mathbb{R}^2} m \cdot \mu(t,k,m)dkdm. \tag{25}$$

This creates the key fixed-point interaction between micro-level optimization and macro-level distributional dynamics.

3.4 Definition and Qualitative Properties of the Stationary Equilibrium

We now formally define equilibrium in the symmetric mean field game:

Definition 3.1 (Mean Field Equilibrium). A mean field equilibrium is a triple

$$(u(t,k,m), \mu(t,k,m), \bar{m}(t)) \tag{26}$$

such that:

1. $u(t,k,m)$ solves the HJB equation given $\bar{m}(t)$,
2. $\mu(t,k,m)$ solves the Fokker–Planck equation given optimal feedback controls (i_k, g),
3. The consistency condition $\bar{m}(t) = \int m \cdot \mu(t,k,m) dkdm$ holds for all $t \geq 0$.

This equilibrium concept captures a decentralized rational expectations environment in which all countries behave optimally, taking the mean field as given, while the mean field is itself generated by the aggregation of their decisions.

To gain deeper insight into the long-run implications of decentralized arms and capital investment, we now examine the structure of stationary equilibria. We focus on time-invariant solutions to the coupled HJB-FPK system, corresponding to the case in which all agents face a constant mean field $\bar{m}$, adopt time-invariant feedback controls, and the population distribution converges to a stationary probability measure over the state space $\mathbb{R}^2$. The goal of this section is twofold: (i) to reduce the complexity of the full dynamic system by eliminating time dependence, and (ii) to extract qualitative properties of global militarization and economic development under symmetric strategic interaction.

3.4.1 Stationary HJB Equation

In a stationary equilibrium, the value function $u(k,m)$ no longer depends on time, and satisfies the time-independent HJB equation:

$$\begin{aligned}\rho u(t,k,m) = \sup_{i_k,g} \Big\{ & ac(k,m,i_k,g;\bar{m}) - \frac{1}{2}bc(k,m,i_k,g;\bar{m})^2 + \eta m - \frac{1}{2}\phi m^2 \\ & + \eta_1 c(k,m,i_k,g;\bar{m})m - \eta_2 m\bar{m} - \eta_3 c(k,m,i_k,g;\bar{m})\bar{m} \\ & + \partial_k u(t,k,m)(i_k - \delta^k k) + \partial_m u(t,k,m)(g - \delta^m m) \\ & + \frac{1}{2}\sigma_k^2 \partial_{kk}^2 u(t,k,m) + \frac{1}{2}\sigma_m^2 \partial_{mm}^2 u(t,k,m) \Big\}. \end{aligned} \tag{27}$$

The stationary consumption rule is given by:

$$c(k,m,i_k,g;\bar{m}) = Ak + Bm - D\bar{m} - i_k - g, \tag{28}$$

where $\bar{m} \in \mathbb{R}_+$ is the endogenous average military level that will be determined in equilibrium.

3.4.2 Stationary Fokker–Planck Equation

The associated stationary distribution $\mu^*(k,m)$ over capital and military stocks satisfies the stationary Fokker-Planck equation:

$$\begin{aligned} 0 = & -\partial_k \left[(i_k(k,m) - \delta^k k)\mu(k,m) \right] - \partial_m \left[(g(k,m) - \delta^m m)\mu(k,m) \right] \\ & + \frac{1}{2}\sigma_k^2 \partial_{kk}^2 \mu(k,m) + \frac{1}{2}\sigma_m^2 \partial_{mm}^2 \mu(k,m), \end{aligned} \tag{29}$$

subject to the normalization condition:

$$\int_{\mathbb{R}^2} \mu^*(k,m)dkdm = 1. \tag{30}$$

The equilibrium mean field is:

$$\bar{m} = \int_{\mathbb{R}^2} m \cdot \mu^*(k,m)dkdm. \tag{31}$$

This equation determines the fixed point of the distribution given the individual responses to the mean field $\bar{m}$.

3.4.3 Qualitative Structure of Stationary Solutions

From the structure of the HJB equation and the FPK equation, several qualitative insights can be derived:

• Strategic amplification: Higher values of η_2 and η_3—i.e., stronger disutility from global militarization—reduce optimal arms m^* and induce lower $\bar{m}$ in equilibrium. This feedback is stabilizing.

• Consumption-military complementarity: When $\eta_1 > 0$, military investment enhances the marginal value of consumption, encouraging higher arms in states with abundant capital—potentially destabilizing when combined with low depreciation or low variance.

• Diffusion stabilizes dispersion: Larger values of σ_k and σ_m (i.e., higher uncertainty in capital or arms dynamics) smooth out the distribution $\mu^*(k,m)$ and reduce concentration at high-militarization equilibria.

• Multiple equilibria: If the utility function is sufficiently nonlinear in m and interactions $\bar{m}$, the equilibrium fixed point may not be unique. In particular, strategic complementarities (large η_1, small η_2) may yield both high- and low-arms steady states, separated by an unstable threshold. In the simplified LQG benchmark (Section 4), we set $\eta_1 = \eta_3 = 0$ to isolate theη_2 externality mechanism; the multiplicity results above apply to the general model.

These features suggest that in a purely symmetric world, even without a dominant country, endogenous militarization may arise from decentralized strategic anxiety—especially under low costs of arms and strong strategic spillovers.

4 Linear-Quadratic-Gaussian (LQG) Benchmark Model

To obtain explicit closed-form solutions and facilitate quantitative analysis, we specialize the general model developed in Sections 2-4 to a linear-quadratic-Gaussian (LQG) framework. This allows us to reduce the full coupled HJB-FPK system to a pair of solvable linear equations, while preserving the core economic and strategic features: consumption-military trade-offs, decentralized investment behavior, and mean field externalities through average arms accumulation.

4.1 Simplified Structure of the LQG Model

We impose the following assumptions to transform the general framework into the LQG setting:

1. State dynamics are deterministic in mean with additive Gaussian noise:

$$dk_t = \left(i_k(t) - \delta^k k_t\right) dt + \sigma_k dW_t^k, \tag{32}$$
$$dm_t = \left(g(t) - \delta^m m_t\right) dt + \sigma_m dW_t^m, \tag{33}$$

with constant volatilities σ_k, σ_m and depreciation rates $\delta^k, \delta^m > 0$.

2. Production is linear, and consumption is defined residually:

$$c(t) = Ak_t + Bm_t - D\bar{m}(t) - i_k(t) - g(t), \tag{34}$$

where $A, B, D > 0$, and $\bar{m}(t) = \mathbb{E}[m_t]$ in symmetric equilibrium.

3. Instantaneous utility is quadratic:

$$u(t) = ac(t) - \frac{1}{2}bc(t)^2 + \eta m_t - \frac{1}{2}\phi m_t^2 - \eta_2 m_t \bar{m}(t) - \frac{\theta_k}{2} i_k(t)^2 - \frac{\theta_g}{2} g(t)^2. \tag{35}$$

We set $\eta_1 = \eta_3 = 0$ throughout the LQG analysis. This eliminates consumption–military cross-effects while preserving the central strategic externality mechanism via$\eta_2 > 0$. Section 3.5 shows that the key qualitative properties persist under this simplification.

4.2 Quadratic Value Function Ansatz

We conjecture a quadratic value function in the stationary equilibrium:

$$V(k, m) = p_0 + p_k k + p_m m + \frac{1}{2}P_{kk}k^2 + P_{km}km + \frac{1}{2}P_{mm}m^2, \tag{36}$$

where P_{kk}, P_{km}, P_{mm} are Riccati gains (quadratic coefficients) and p_k, p_m, p_0 are linear and constant terms, all time-independent in stationary equilibrium. The partial derivatives are:

$$V_k = p_k + P_{kk}k + P_{km}m, \tag{37}$$
$$V_m = p_m + P_{km}k + P_{mm}m, \tag{38}$$
$$V_{kk} = P_{kk}, \tag{39}$$
$$V_{mm} = P_{mm}, \tag{40}$$
$$V_{km} = P_{km}. \tag{41}$$

Restore the F.O.C.,

$$c^* = \frac{a + \theta_k i_k^* - \partial_k V}{b} = \frac{a + \theta_g g^* - \partial_m V}{b}, \tag{42}$$

with

$$i_k^* = \frac{(\theta_g + b)V_k - bV_m + \theta_g(-a + bY)}{\theta_g\theta_k + b(\theta_g + \theta_k)}, \tag{43}$$

$$g^* = \frac{(\theta_k + b)V_m - bV_k + \theta_k(-a + bY)}{\theta_g\theta_k + b(\theta_g + \theta_k)}. \tag{44}$$

which is linear in k and m. Bringing them back into the HJB equation:

$$\begin{aligned}\rho V - \frac{\partial V}{\partial t} &= a \cdot c^* - \frac{1}{2}b \cdot (c^*)^2 + \eta m - \frac{1}{2}\phi m^2 - \eta_2 m\bar{m}(t) \\ &\quad + \frac{\partial V}{\partial k}(i_k^* - \delta^k k) + \frac{\partial V}{\partial m}(g^* - \delta^m m) + \frac{1}{2}\sigma_k^2\frac{\partial^2 V}{\partial k^2} + \frac{1}{2}\sigma_m^2\frac{\partial^2 V}{\partial m^2},\end{aligned} \tag{45}$$

we have

$$\begin{aligned}\text{LHS} &= \rho V \\ &= \frac{1}{2}\rho P_{kk}k^2 + \frac{1}{2}\rho P_{mm}m^2 + \rho P_{km}km + \rho p_k k + \rho p_m m + \rho p_0\end{aligned} \tag{46}$$

and

$$\begin{aligned}\text{RHS} &= a \cdot c^* - \frac{1}{2}b \cdot (c^*)^2 + \eta m - \frac{1}{2}\phi m^2 - \eta_2 m\bar{m} \\ &\quad + \frac{\partial V}{\partial k}(i_k^* - \delta^k k) + \frac{\partial V}{\partial m}(g^* - \delta^m m) + \frac{1}{2}\sigma_k^2 P_{kk} + \frac{1}{2}\sigma_m^2 P_{mm} \qquad (47)\\ &= (p_k + P_{kk}k + P_{km}m)\left[i_k^* - \delta^k k\right] + (p_m + P_{km}k + P_{mm}m)\left[g^* - \delta^m m\right] \\ &\quad + a \cdot c^* - \frac{1}{2}b \cdot (c^*)^2 + \eta m - \frac{1}{2}\phi m^2 - \eta_2 m\bar{m} + \frac{1}{2}\sigma_k^2 P_{kk} + \frac{1}{2}\sigma_m^2 P_{mm} \qquad (48)\end{aligned}$$

Rearranging, and defining the consumption sensitivities$C_k \equiv A + \gamma_k + \gamma_{gk}$ and$C_m \equiv B + \gamma_{ik,m} + \gamma_m$, we have the coefficients ofk^2 (ARE R1):

$$(\rho + 2\delta^k + \gamma_k)P_{kk} + \gamma_{gk}P_{km} + bAC_k = 0, \tag{R1}$$

m^2 (ARE R2):

$$(\rho + 2\delta^m + \gamma_m)P_{mm} + \gamma_{ik,m}P_{km} + bBC_m + \phi = 0, \tag{R2}$$

km (ARE R3):

$$(\rho + \delta^k + \delta^m + \gamma_k)P_{km} + \gamma_{gk}P_{mm} + bBC_k = 0. \tag{R3}$$

k (ARE R4):

$$\begin{aligned}&\frac{1}{\Delta}\left\{-Aa\theta_g\theta_k + P_{kk}\left[\theta_g(a + bD\bar{m}(t)) - (\theta_g + b)p_k + bp_m\right]\right. \\ &+ P_{km}\left[\theta_k(a + bD\bar{m}(t)) + bP_k - (\theta_k + b)p_m\right] - AbD\theta_g\theta_k\bar{m}(t) \qquad (49)\\ &\left.-Ab\theta_g p_k - Ab\theta_k p_m - \Delta\dot{p}_k + (\delta^k + \rho)\Delta p_k\right\} = 0. \qquad (\text{R4})\end{aligned}$$

m (ARE R5):

$$\frac{1}{\Delta}\left\{P_{km}\left[a\theta_g + b\theta_g D\bar{m}(t) - (\theta_g + b)p_k + bp_m\right] - aB\theta_g\theta_k + a\theta_k P_{mm}\right.$$
$$- bB\theta_g\theta_k D\bar{m}(t) + bp_k\left[P_{mm} - B\theta_g\right] - bB\theta_k p_m + bD\theta_k D\bar{m}(t)P_{mm} - \Delta(\eta - \eta_2\bar{m}(t)) - bP_{mm}p_m$$
$$\left.-\Delta\dot{p}_m + (\delta^m + \rho)\Delta p_m - \theta_k P_{mm}p_m\right\} = 0. \tag{R5}$$

and constant (ARE R6):

$$-\frac{1}{2\Delta}\left\{a^2(\theta_g + \theta_k) - 2a\theta_g\theta_k D\bar{m}(t) - 2a\theta_g p_k - 2a\theta_k p_m - b\theta_g\theta_k(D\bar{m}(t))^2\right.$$
$$- 2b\theta_g D\bar{m}(t)(p_k + p_m) + 2\Delta\dot{p}_0 - 2\Delta\rho p_0 + \Delta\sigma_k^2 P_{kk}$$
$$\left.-2bp_kp_m + b(p_k^2 + p_m^2) + \Delta\sigma_m^2 P_{mm} + \theta_g p_k^2 + \theta_k p_m^2\right\} = 0. \tag{R6}$$

Equations (R1)–(R6) form a nonlinear system in (P_{kk}, P_{km}, P_{mm}), solved numerically via (see Section 8). Feedback gains are then:

$$\gamma_k = -\frac{(b+\theta_g)P_{kk} - bP_{km} + b\theta_g A}{\Delta}, \tag{50}$$

$$\gamma_m = -\frac{(b+\theta_k)P_{mm} - bP_{km} + b\theta_k B}{\Delta}, \tag{51}$$

$$\gamma_{ik,m} = -\frac{(b+\theta_g)P_{km} - bP_{mm} + b\theta_g B}{\Delta}, \tag{52}$$

$$\gamma_{gk} = -\frac{(b+\theta_k)P_{km} - bP_{kk} + b\theta_k A}{\Delta}. \tag{53}$$

Note that the gain matrix$K = \begin{pmatrix}\gamma_k & \gamma_{ik,m}\\ \gamma_{gk} & \gamma_m\end{pmatrix}$ is generally asymmetric ($\gamma_{ik,m} \neq \gamma_{gk}$) since$\theta_k \neq \theta_g$ and$A \neq B$.Given $(P_{kk}^*, P_{km}^*, P_{mm}^*)$ and a candidate $\bar{m}^*$, the linear terms(p_k, p_m) satisfy the 2×2 linear system:

$$(\rho + \delta^k + \gamma_k)p_k + \gamma_{gk}p_m = \mathrm{RHS}_k(\bar{m}), \tag{54}$$

$$\gamma_{ik,m}p_k + (\rho + \delta^m + \gamma_m)p_m = (\eta - \eta_2\bar{m}^* - \phi\bar{m}^*) + \mathrm{RHS}_m(\bar{m}). \tag{55}$$

This 2×2 system is linear in (p_k, p_m) and admits a unique solution under Assumption 2.1.

4.3 Optimal Controls in Feedback Form

Using the value function(36) and first-order conditions, the stationary optimal investment policies take the affine feedback form:

$$i_k^*(k,m) = -\gamma_k k - \gamma_{ik,m}m + \alpha_k, \tag{56}$$

$$g^*(k,m) = -\gamma_{gk}k - \gamma_m m + \alpha_m, \tag{57}$$

where$\gamma_k, \gamma_m, \gamma_{ik,m}, \gamma_{gk}$ are the Riccati feedback gains defined in Eqs.(50)–(53), and α_k, α_m are affine correction terms depending on$\bar{m}$:

$$\alpha_k(\bar{m}) = \frac{(b+\theta_g)p_k - bp_m + \theta_g(-bD\bar{m} - a)}{\Delta}, \tag{58}$$

$$\alpha_m(\bar{m}) = \frac{(b+\theta_k)p_m - bp_k + \theta_k(-bD\bar{m} - a)}{\Delta}. \tag{59}$$

Note that$\gamma_k, \gamma_m, \gamma_{ik,m}, \gamma_{gk}$ depend only on (P_{kk}, P_{km}, P_{mm}) (Riccati system), while α_k, α_m depend on (p_k, p_m) and $\bar{m}$, solved after the Riccati step. Substituting into the budget constraint yields a linear expression for consumption:

$$c^*(k, m; \bar{m}) = Ak + Bm - D\bar{m} - i_k^*(k, m) - g^*(k, m). \tag{60}$$

4.4 Mean Field and Closed-Form Consistency

Let $\bar{k}(t) = \mathbb{E}[k_t], \bar{m}(t) = \mathbb{E}[m_t]$. Taking expectations of the SDEs and substituting the linear feedback policies (56)–(57):

$$\frac{d}{dt}\bar{k}(t) = -(\gamma_k + \delta^k)\bar{k}(t) - \gamma_{ik,m}\bar{m}(t) + \alpha_k(\bar{m}), \tag{61}$$

$$\frac{d}{dt}\bar{m}(t) = -(\gamma_m + \delta^m)\bar{m}(t) - \gamma_{gk}\bar{k}(t) + \alpha_m(\bar{m}). \tag{62}$$

In the stationary mean-field equilibrium, $\dot{\bar{k}} = \dot{\bar{m}} = 0$. Setting the right-hand sides to zero and substituting Eqs. (58)–(59) closes a 2×2 linear system in $(\bar{k}^*, \bar{m}^*)$:

$$(\gamma_k + \delta^k)\bar{k}^* + \gamma_{ik,m}\bar{m}^* = \alpha_k(\bar{m}^*), \tag{63}$$

$$(\gamma_m + \delta^m)\bar{m}^* + \gamma_{gk}\bar{k}^* = \alpha_m(\bar{m}^*). \tag{64}$$

Since α_k, α_m are affine in $\bar{m}^*$, the system (63)–(64) is linear and admits a unique solution under Assumption 2.1.

4.5 Equilibrium Characterization in LQG

The stationary LQG MFG equilibrium consists of:

1. Riccati gains $(P^*_{kk}, P^*_{km}, P^*_{mm})$ solving the nonlinear system (R1)-(R3);
2. Feedback gains$\gamma_k, \gamma_m, \gamma_{ik,m}, \gamma_{gk}$ derived from$(P^*_{kk}, P^*_{km}, P^*_{mm})$ via Eqs.(50)–(53);
3. Linear terms (p^*_k, p^*_m) from the 2×2 linear backward system;
4. Stationary mean field $(\bar{k}^*, \bar{m}^*)$ from the consistency system (63)–(64);
5. Value function

$$V^*(k, m) = p^*_0 + p^*_k k + p^*_m m + \frac{1}{2}P^*_{kk}k^2 + P^*_{km}km + \frac{1}{2}P^*_{mm}m^2. \tag{65}$$

Proposition 4.1 (LQG MFG Equilibrium). Under Assumption 2.1(i)–(iv), the stationary LQG MFG equilibrium is uniquely characterized by: (i) Riccati gains$(P^*_{kk}, P^*_{km}, P^*_{mm})$ solving AREs (R1)-(R3); (ii) feedback gains$\gamma_k, \gamma_m, \gamma_{ik,m}, \gamma_{gk}$

fromP^* via Eqs.(50)–(53); (iii) linear terms (p_k^*, p_m^*) from the 2×2 linear system (54)–(55); (iv) stationary mean field $(\bar{k}^*, \bar{m}^*)$ from the unique solution of (63)–(64); (v) value function

$$V^*(k,m) = p_0^* + p_k^* k + p_m^* m + \frac{1}{2} P_{kk}^* k^2 + P_{km}^* km + \frac{1}{2} P_{mm}^* m^2. \tag{66}$$

5 Existence of LQG Mean Field Equilibrium

In this section, we formally establish the existence and uniqueness of a symmetric decentralized equilibrium in the linear–quadratic–Gaussian (LQG) benchmark model. We consider the setting developed in Section 4, in which each agent faces linear dynamics, quadratic utility, and Gaussian uncertainty. The key to existence lies in demonstrating that a fixed point exists between the backward optimization (the HJB problem) and the forward population law of motion (the Fokker–Planck equation). Under LQG structure, these problems can be reduced to a backward Riccati system and a forward linear ordinary differential equation (ODE) system, both of which admit unique solutions under standard conditions.

Recall from Section 4 that the LQG equilibrium consists of the following elements:

1. A quadratic value function $u(t,k,m)$ of the form:

$$u(t,k,m) = -\frac{1}{2}\gamma_k(t)k^2 - \frac{1}{2}\gamma_m(t)m^2 - \gamma_{km}(t)km + \theta_k(t)k + \theta_m(t)m + \xi(t), \tag{67}$$

where$\gamma_k(t), \gamma_m(t), \gamma_{km}(t), \theta_k(t), \theta_m(t), \xi(t)$ are continuous, differentiable functions of time. The feedback gains$\gamma_k(t), \gamma_m(t), \gamma_{ik,m}(t), \gamma_{gk}(t)$ are derived from$(\gamma_k(t), \gamma_{km}(t), \gamma_m(t))$ via the first-order conditions (generally$\gamma_{ik,m} \neq \gamma_{gk}$).

2. Linear feedback policies:

$$i_k(t,k,m) = -\gamma_k(t)k - \gamma_{ik,m}(t)m + \theta_k(t), \tag{68}$$
$$g(t,k,m) = -\gamma_{gk}(t)k - \gamma_m(t)m + \theta_m(t). \tag{69}$$

3. A forward linear ODE system for the mean field states:

$$\dot{\bar{k}}(t) = i_k(t, \bar{k}(t), \bar{m}(t)) - \delta^k \bar{k}(t), \tag{70}$$
$$\dot{\bar{m}}(t) = g(t, \bar{k}(t), \bar{m}(t)) - \delta^m \bar{m}(t), \tag{71}$$

with given initial values $\bar{k}(0), \bar{m}(0) \in \mathbb{R}_+$.

4. A fixed-point condition:

$$\bar{m}(t) = \mathbb{E}[m_t], \tag{72}$$

ensuring that the average military stock in the population equals the expected value under optimal decentralized behavior.

We now prove that this system admits a unique solution.

5.1 Well-Posedness of the Riccati System

Given the LQG structure, the HJB equation reduces to a matrix Riccati differential equation. The coefficients $\gamma_k(t), \gamma_m(t), \gamma_{km}(t)$ satisfy a coupled system of backward ODEs. Under standard conditions—which are ensured by Assumption 2.1(i)–(iv) (e.g., positive definiteness of quadratic cost coefficients, boundedness of drift terms)—the following classical result applies (see Yong & Zhou, 1999; Bardi & Capuzzo-Dolcetta, 1997):

Lemma 5.1 (Riccati Well-Posedness). The Riccati system derived from the LQG HJB equation with quadratic terminal condition admits a unique global solution $(\gamma_k(t), \gamma_m(t), \gamma_{km}(t)) \in C^1([0,T])$ for any finite horizon $T > 0$, or in the limit $T \to \infty$ under stabilization conditions.

Once the Riccati terms are determined, the linear terms $\theta_k(t), \theta_m(t), \xi(t)$ can be solved using linear backward ODEs with coefficients depending on the known mean field $\bar{m}(t)$, which we treat provisionally as fixed.

5.2 Well-Posedness of Forward Mean-Field ODE

For any given set of feedback coefficients, the optimal controls (i_k^*, g^*) are linear in $(\bar{k}(t), \bar{m}(t))$. Therefore, the system:

$$\dot{\bar{k}}(t) = -\gamma_k(t)\bar{k}(t) - \gamma_{ik,m}(t)\bar{m}(t) + \theta_k(t) - \delta^k \bar{k}(t), \tag{73}$$

$$\dot{\bar{m}}(t) = -\gamma_{gk}(t)\bar{k}(t) - \gamma_m(t)\bar{m}(t) + \theta_m(t) - \delta^m \bar{m}(t), \tag{74}$$

is a linear system of ODEs in $\mathbb{R}^2$, driven by time-dependent coefficients. Classical theory guarantees:

Lemma 5.2 (Linear ODE Well-Posedness). For any initial condition $(\bar{k}(0), \bar{m}(0)) \in \mathbb{R}^2$, and continuous coefficient functions $\gamma_\cdot(t), \theta_\cdot(t)$, there exists a unique global solution $(\bar{k}(t), \bar{m}(t)) \in C^1([0,T])$.

This establishes the forward component of the MFG system for any given mean field path $\bar{m}(t)$.

5.3 Existence of MFG Equilibrium

To complete the proof, we must show that the map:

$$\bar{m}(t) \mapsto \gamma_\cdot(t), \theta_\cdot(t) \mapsto (\bar{k}(t), \bar{m}(t)) \tag{75}$$

has a fixed point, i.e., a self-consistent $\bar{m}(t)$ that is both used in the optimization and generated by the resulting dynamics. This follows by the Schauder fixed-point theorem:

Proposition 5.3 (Existence of LQG MFG Equilibrium). Under Assumption 2.1(i)–(iv), let $\mathcal{M} \subset C([0,T])$ be the set of continuous paths $\bar{m}(t)$ with uniform boundedness. Then the operator:

$$\mathcal{T} : \bar{m}(\cdot) \mapsto \mathbb{E}[m_t] \tag{76}$$

under optimal controls is continuous and maps a convex compact set into itself. Therefore, $\mathcal{T}$ has a fixed point.

Hence, there exists a path $\bar{m}(t) \in C([0,T])$ consistent with the rational expectations equilibrium under optimal decentralized behavior.

We conclude that the coupled backward–forward LQG system admits a unique solution:

- A value function $u(t,k,m)$ quadratic in states;
- Linear optimal feedback policies $i_k(t,k,m), g(t,k,m)$;
- A consistent forward solution $(\bar{k}(t), \bar{m}(t))$;
- A fixed point $\bar{m}(t) = \mathbb{E}[m_t]$.

This establishes the existence of a symmetric decentralized mean field equilibrium under rational expectations in the presence of strategic military spillovers and Gaussian uncertainty.

5.4 FPK Equation and Fixed-Point Map

In this section, we derived the explicit forward Fokker–Planck–Kolmogorov (FPK) equation under a strategically simplified utility function. We now turn to the mathematical structure of the fixed point problem implied by the mean field equilibrium. In particular, we show that the system—comprising the backward HJB equation, the forward FPK equation, and the population mean field closure—admits a well-defined fixed point. Under suitable regularity and monotonicity assumptions, we also show that the equilibrium is unique.

5.4.1 Population Distribution and Fixed-Point Problem

Given optimal controls derived from the HJB equation and a conjectured path $\bar{m}(t) \in C([0,T],\mathbb{R})$, the FPK equation uniquely determines the time-evolving joint density $\mu(t,k,m)$ of capital k and military stock m. The consistency condition in mean field games requires:

$$\bar{m}(t) = \mathbb{E}[m_t] = \int_{\mathbb{R}^2} m \cdot \mu(t,k,m)dkdm. \tag{77}$$

Thus, the equilibrium problem is equivalent to finding a fixed point of the mapping:

$$\mathcal{T} : \bar{m}(\cdot) \mapsto \int_{\mathbb{R}^2} m \cdot \mu^{\bar{m}}(t,k,m)dkdm, \tag{78}$$

where $\mu^{\bar{m}}(t,k,m)$ is the solution to the FPK equation corresponding to the drift coefficients generated by $\bar{m}(t)$.

5.4.2 Functional Form of the FPK Drift

Let us recall the FPK system with the linear drift derived from the optimal controls. Denote $x = (k,m)$, and define the drift vector:

$$b^{\bar{m}}(t,x) = \begin{pmatrix} i_k(t,k,m) - \delta^k k \\ g(t,k,m) - \delta^m m \end{pmatrix} = -B(t)x + \theta(t), \tag{79}$$

where:

• $B(t) \in \mathbb{R}^{2\times 2}$ is a matrix of feedback coefficients with diagonal entries$\gamma_k(t) + \delta^k, \gamma_m(t) + \delta^m$ and off-diagonal entries$\gamma_{ik,m}(t), \gamma_{gk}(t)$,

• $\theta(t) \in \mathbb{R}^2$ is a vector of affine terms $(\theta_k(t), \theta_m(t))$.

The FPK equation thus becomes:

$$\partial_t \mu(t,x) = -\nabla \cdot \left(b^{\bar{m}}(t,x)\mu(t,x)\right) + \frac{1}{2}\nabla \cdot (\Sigma \nabla \mu(t,x)), \tag{80}$$

with $\Sigma = \mathrm{diag}(\sigma_k^2, \sigma_m^2)$.

5.4.3 Fixed-Point Existence via Schauder's Theorem

Let us now state and prove the formal existence result.

Theorem 5.4 (Existence of Mean Field Fixed Point). Let $\bar{m}(t) \in \mathcal{M} \subset C([0,T], \mathbb{R})$ be a bounded set of continuous candidate paths. Then the map:

$$\mathcal{T} : \bar{m}(\cdot) \mapsto \int_{\mathbb{R}^2} m \cdot \mu^{\bar{m}}(t,k,m)dkdm \tag{81}$$

is continuous and maps $\mathcal{M}$ into itself. Hence, $\mathcal{T}$ admits a fixed point $\bar{m}^*(t)$ by Schauder's fixed point theorem.

Proof.

1. Well-posedness of the FPK equation:

For every bounded and continuous $\bar{m}(t)$, the drift term $b^{\bar{m}}(t,x)$ is Lipschitz continuous in x, uniformly in t, because the coefficients $\gamma_k(t), \gamma_m(t), \gamma_{km}(t), \theta_k(t), \theta_m(t)$ are derived from the backward Riccati equation (well-posed under LQG structure). The diffusion term Σ is constant and strictly positive definite. Therefore, standard parabolic PDE theory (see Bogachev et al., 2016) guarantees existence and uniqueness of $\mu^{\bar{m}}(t,x) \in C([0,T], \mathcal{P}_2(\mathbb{R}^2))$.

2. Continuity of $\mathcal{T}$:

Suppose $\bar{m}_n(t) \to \bar{m}(t)$ uniformly in t. Then the corresponding drift terms $b^{\bar{m}_n}(t,x) \to b^{\bar{m}}(t,x)$ uniformly on compact subsets. The solutions $\mu^{\bar{m}_n}(t,x)$ to the FPK equation depend continuously on the drift by stability of linear parabolic PDEs (Bogachev et al., Theorem 6.1). Hence,

$$\bar{m}'_n(t) = \int m\mu^{\bar{m}_n}(t,k,m)dkdm \longrightarrow \int m\mu^{\bar{m}}(t,k,m)dkdm = \mathcal{T}\bar{m}, \tag{82}$$

establishing continuity.

3. Compactness and invariance:

The set $\mathcal{M} = \{\bar{m} \in C([0,T]) : \|\bar{m}\|_\infty \leq M\}$ is convex and compact under the sup norm. For bounded initial conditions and bounded drift terms, the moment bound $\bar{m}(t) \leq C$ holds for all t, ensuring $\mathcal{T}[\bar{m}] \in \mathcal{M}$.

By Schauder's theorem, the continuous, self-mapping operator $\mathcal{T}$ admits a fixed point $\bar{m}^*(t) \in \mathcal{M}$.

5.5 Uniqueness under Strategic Concavity

Uniqueness of the mean field equilibrium requires an additional contraction property or monotonicity condition.

Proposition 5.5 (Uniqueness under Strategic Concavity). Under Assumption 2.1(i)–(iv), if $\eta_2 > 0, D > 0$, and the feedback coefficients $\gamma_k(t), \gamma_m(t)$ are monotone in $\bar{m}(t)$, then $\mathcal{T}$ is a contraction in a weighted norm, and the fixed point $\bar{m}^*(t)$ is unique.

Sketch of Proof.

Let $\bar{m}_1(t), \bar{m}_2(t) \in \mathcal{M}$. Let $\mu^1(t,k,m), \mu^2(t,k,m)$ be the corresponding FPK solutions. Since $b^{\bar{m}_1} - b^{\bar{m}_2} = -\delta b(t,x)$ is Lipschitz in $\bar{m}$, stability results for linear FPK equations (via Gronwall inequality inL^1 or Wasserstein norms) imply:

$$\|\mu^1(t,\cdot) - \mu^2(t,\cdot)\|_{TV} \leq C\|\bar{m}_1 - \bar{m}_2\|_\infty. \tag{83}$$

Therefore,

$$|\mathcal{T}\bar{m}_1 - \mathcal{T}\bar{m}_2| \leq \int |m| \cdot |\mu^1 - \mu^2| \leq C\|\bar{m}_1 - \bar{m}_2\|_\infty. \tag{84}$$

If $C < 1$, $\mathcal{T}$ is a contraction. Thus uniqueness follows.

The existence and uniqueness results confirm that even under uncertainty and continuous heterogeneity, decentralized countries acting in response to a strategic mean field of militarization will coordinate around a unique, time-consistent path $\bar{m}(t)$. This result lays the foundation for simulation, sensitivity analysis, and welfare policy in the symmetric version of the model.

6 Social Planner and Pigouvian Tax

In this section, we introduce a benevolent social planner who maximizes aggregate welfare by internalizing the strategic externality that individual nations ignore in the decentralized MFG equilibrium. The social planner's solution provides a welfare benchmark against which the inefficiency of the decentralized equilibrium can be measured, and yields an optimal Pigouvian tax that restores social optimality.

6.1 Social Planner's Objective

The social planner chooses investment and armament policies$(i_k^i(t), g^i(t))$ for all nations simultaneously to maximize aggregate discounted welfare:

$$W_{SP} = \max_{(i_k^i, g^i)} \mathbb{E}\left[\int_0^\infty e^{-\rho t} \int_{\mathbb{R}^2} u(k,m,i_k,g,\bar{m}) \cdot \mu(t,dk,dm)dt\right]. \tag{85}$$

The critical distinction from the decentralized MFG equilibrium is that the social planner treats the mean field$\bar{m}(t) = \int m \cdot \mu(t,dk,dm)$ as jointly determined by the population's controlled dynamics, rather than as an exogenous path.

Specifically, the social planner internalizes the fact that each nation's military investment raises$\bar{m}$, which generates a negative externality of$-\eta_2 m_i \bar{m}$ on all nations' instantaneous utility. Under the LQG structure of Section 4, the social planner's problem reduces to a representative-agent LQG optimal control problem, and no fixed-point condition is required since the planner controls the full distribution.

6.2 Social Planner's Optimal Controls and Comparison with MFG

Under the LQG specification, a key observation is that the algebraic Riccati equations (ARE) determining the quadratic gains$(P^*_{kk}, P^*_{km}, P^*_{mm})$ are identical for both MFG and SP. This follows from the separation principle in LQG control: each individual agent's budget constraint has sensitivity$B$ to its own military statem_i, while$\bar{m}$ enters only through the constant$-D\bar{m}$ which does not affect the quadratic Riccati structure. Consequently, the SP uses the same feedback gains$(\gamma_k, \gamma_m, \gamma_{km}, \gamma_{gk})$ as the MFG equilibrium.

The SP's distinctive role is to choose the stationary mean-field targets$(\bar{k}^*_{SP}, \bar{m}^*_{SP})$ by directly maximizing aggregate welfare. At stationarity, investment and armament exactly offset depreciation:$\bar{i}_k = \delta_k \bar{k}$,$\bar{g} = \delta_m \bar{m}$. Consumption becomes

$$\bar{c} = (A - \delta_k)\bar{k} + (B - D - \delta_m)\bar{m}, \tag{86}$$

and the SP maximizes the representative agent's stationary flow utility

$$\bar{u}(\bar{k}, \bar{m}) = a\bar{c} - \tfrac{b}{2}\bar{c}^2 + \eta\bar{m} - \tfrac{1}{2}(\phi + 2\eta_2)\bar{m}^2 - \tfrac{\theta_k}{2}(\delta_k \bar{k})^2 - \tfrac{\theta_g}{2}(\delta_m \bar{m})^2, \tag{87}$$

where$(\phi + 2\eta_2)\bar{m}^2$ reflects the internalized externality (the MFG agent faces only$\phi\bar{m}^2 + \eta_2\bar{m}^2$ because it treats$\bar{m}$ as exogenous). The first-order conditions yield the2×2 linear system:

$$\underbrace{\begin{pmatrix} bc_k^2 + \kappa & bc_k c_m \\ bc_k c_m & bc_m^2 + \psi \end{pmatrix}}_{\mathbf{M}} \begin{pmatrix} \bar{k} \\ \bar{m} \end{pmatrix} = \begin{pmatrix} ac_k \\ ac_m + \eta \end{pmatrix}, \tag{88}$$

where$c_k \equiv A - \delta_k$,$c_m \equiv B - D - \delta_m$,$\kappa \equiv \theta_k \delta_k^2$, and$\psi \equiv \phi + 2\eta_2 + \theta_g \delta_m^2$. Compared to the MFG mean-field consistency equations (which solve a4×4 linear system for$(p_k, p_m, \bar{k}, \bar{m})$ using both the HJB and Fokker–Planck conditions), the SP directly optimizes a concave quadratic in two unknowns, yielding a unique interior solution whenever$\mathbf{M}$ is positive definite (guaranteed by Assumption 2.1).

6.3 Pigouvian Tax Characterization

Let $\lambda(t)$ denote the shadow price of the mean militarization$\bar{m}(t)$ in the social planner's problem. With the interactive externality$-\eta_2 m_i \bar{m}$, the social planner internalizes$\partial\bar{m}/\partial m_i$, effectively facing$-2\eta_2 m\bar{m}$ at the representative-agent level.

The optimal Pigouvian tax on military investment in the stationary equilibrium is:

$$\tau^* = \lambda^* = \frac{\eta_2 \bar{m}^*_{SP} + D(a - b\bar{c}^*_{SP})}{\rho + \delta^m}, \tag{89}$$

where $\bar{m}^*_{SP}$ and $\bar{c}^*_{SP}$ are the social planner's stationary mean-field values. The numerator consists of two components: (i) the internalized congestion externality $\eta_2 \bar{m}^*_{SP}$, and (ii) the productivity channel $D(a - b\bar{c}^*_{SP})$ through which militarization reduces consumption via the resource constraint $c = Ak + Bm - D\bar{m} - i_k - g$.

Decentralization Result. The Pigouvian tax τ^* on military investment g^i decentralizes the social-planner solution: imposing τ^* as a per-unit tax on armament expenditure makes the decentralized MFG equilibrium coincide with the social optimum. Equivalently, the SP equilibrium can be reproduced by replacing η_2 with the effective externality parameter

$$\eta_2^{\text{eff}} = \eta_2 + \rho\tau^* \tag{90}$$

in the individual MFG optimization problem. Higher externality strength (larger η_2 or D) implies a higher corrective tax, consistent with the comparative statics of Section 7.

6.4 Existence and Uniqueness of the SP Solution

Since the social planner faces a standard LQG optimal control problem—without the fixed-point condition required by the MFG—existence and uniqueness follow directly from classical LQR theory (Yong & Zhou, 1999).

Proposition 6.1 (SP Existence and Uniqueness). Under Assumption 2.1(i)–(iv), the social planner's problem (85) admits a unique stationary solution characterized by: (i) ARE gains $(P^*_{kk}, P^*_{km}, P^*_{mm})$ identical to the MFG (Proposition 4.1); (ii) feedback gains $(\gamma_k, \gamma_m, \gamma_{km}, \gamma_{gk})$ identical to the MFG; (iii) stationary mean field $(\bar{k}^*_{SP}, \bar{m}^*_{SP})$ uniquely determined by the FOC system (88); (iv) optimal Pigouvian tax $\tau^* = \lambda^*$ at the stationary optimum (Eq.(89)).

6.5 Computational Implementation

Computationally, the SP solution proceeds in two stages. First, the ARE and feedback gains are solved identically to the MFG (Section 4), since the quadratic structure of the individual's value function is unaffected by whether the planner or the market determines $\bar{m}$. Second, the SP directly maximizes the stationary flow utility (87) over $(\bar{k}, \bar{m})$ via the 2×2 FOC system (88), bypassing the 4×4 HJB-FP consistency system used for the MFG. The affine feedback constants $(\alpha_k^{SP}, \alpha_m^{SP})$ are then backed out from the stationarity conditions $\dot{\bar{k}} = \dot{\bar{m}} = 0$ at $(\bar{k}^*_{SP}, \bar{m}^*_{SP})$. The resulting SP allocation satisfies $W_{SP} \geq W_{MFG}$, with strict inequality whenever $\eta_2 > 0$ or $D > 0$.

7 Welfare Comparison: MFG versus Social Planner

In this section, we quantify the welfare cost of strategic externalities by comparing the aggregate welfare attained under the decentralized MFG equilibrium with that achieved by the social planner. We derive a closed-form expression for the welfare gap, characterize how it varies with the externality parameters, and demonstrate that the Pigouvian tax of Section 6 closes the gap exactly.

7.1 Welfare Metric

Define the stationary aggregate welfare under the MFG and SP equilibria, respectively, as the normalized present value of mean utility:

$$W_{MFG} = \frac{\bar{u}^*_{MFG}}{\rho}, \tag{91}$$

$$W_{SP} = \frac{\bar{u}^*_{SP}}{\rho}, \tag{92}$$

where$\bar{u}^* = \mathbb{E}\left[u(k^*, m^*, i_k^*, g^*, \bar{m}^*)\right]$ is the stationary mean flow utility evaluated at the respective equilibrium. The welfare gap is then:

$$\Delta W = W_{SP} - W_{MFG} \geq 0, \tag{93}$$

with $\Delta W = 0$ only when there is no externality ($\eta_2 = D = 0$).

7.2 Analytical Welfare Gap

Using the LQG structure, the welfare gap ΔW can be decomposed into militarization and capital channels. Evaluated at the respective stationary mean fields $(\bar{k}^*_{MFG}, \bar{m}^*_{MFG})$ and $(\bar{k}^*_{SP}, \bar{m}^*_{SP})$:

$$\Delta W = \frac{1}{\rho}\left[\bar{u}^*_{SP} - \bar{u}^*_{MFG}\right], \tag{94}$$

where the mean flow utilities$\bar{u}^* = a\bar{c} - \frac{1}{2}b\bar{c}^2 + \eta\bar{m} - \frac{1}{2}\phi\bar{m}^2 - \eta_2\bar{m}^2 - \frac{\theta_k}{2}\bar{i}_k^2 - \frac{\theta_g}{2}\bar{g}^2$ are evaluated at the respective stationary equilibria. With the interactive externality$-\eta_2 m_i \bar{m}$, the utility includes$-\eta_2\bar{m}^2$ at the mean (since$m_i = \bar{m}$ in a symmetric equilibrium). This expression yields the following qualitative properties:

(i) $\Delta W = 0$ when $\eta_2 = D = 0$ (no externality);

(ii) $\Delta W > 0$ for any $\eta_2 > 0$ or $D > 0$ (overinvestment in militarization causes efficiency loss);

(iii)ΔW responds nonlinearly toη_2 and is increasing in$D$ (see Table 1 and Figure 9 for details).

7.3 The Tax Closes the Gap

The Pigouvian tax τ^* derived in Section 6 restores social optimality exactly:

$$W^{\tau^*}_{MFG} = W_{SP}, \tag{95}$$

where $W^{\tau^*}_{MFG}$ denotes the welfare of a taxed decentralized equilibrium in which each nation faces the corrective tax τ^* on military expenditure. This follows directly from the decentralization result of Proposition 6.1: imposing τ^* renders the individual MFG optimum identical to the SP optimum, hence the equilibrium allocations and welfare coincide.

7.4 Comparative Statics

The following table summarizes the comparative statics of the stationary equilibria and welfare metrics with respect to key parameters:

Parameter change	$\bar{m}^*_{MFG}$	ΔW	τ^*
$\eta_2 \uparrow$	$\downarrow$	nonlinear	$\uparrow$
$D \uparrow$	$\downarrow$	$\uparrow$	$\uparrow$
$\rho \uparrow$	$\uparrow$	$\uparrow$	$\downarrow$
θ_k or $\theta_g \uparrow$	$\downarrow$	$\downarrow$	$\downarrow$

Higher adjustment costsθ_k, θ_g reduce military spending in both equilibria, shrinking the externality-driven wedge and hence loweringΔW andτ^*. Higher impatienceρ inflates$\bar{m}^*_{MFG}$ (nations discount future welfare costs more heavily) while reducing$\tau^* = \lambda^*$ through the discounting of the tax itself.

8 Numerical Simulations

In this section, we present numerical simulations of the LQG MFG equilibrium derived in Section 4, computed using the solver`Code/lqg`. The solver implements the full pipeline: (i) solving the algebraic Riccati equations (R1)–(R3) via`scipy.fsolve` to obtain$(P^*_{kk}, P^*_{km}, P^*_{mm})$; (ii) deriving the feedback gains$(\gamma_k, \gamma_m, \gamma_{ik,m}, \gamma_{gk})$ and affine terms(α_k, α_m); (iii) computing the stationary mean field$(\bar{k}^*, \bar{m}^*)$; and (iv) generating the social-planner comparison with Pigouvian taxτ^* from Section 6. We produce nine figures organized into six subsections, illustrating how the externality parameterη_2 shapes dynamics, welfare, and the scope for policy correction.

8.1 Calibration and LQG Solution

The baseline parameter configuration follows the MFGParams defaults of the solver:

$$A = 0.5,\ B = 0.4,\ D = 0.3,\ \delta^k = 0.04,\ \delta^m = 0.06,\ \sigma_k = \sigma_m = 0.1,$$

$$a = 1.5,\ b = 0.6,\ \eta = 0.5,\ \phi = 0.2,\ \theta_k = 0.3,\ \theta_g = 0.4,\ \rho = 0.05,\ \eta_2 = 0.5 \text{ (baseline)}. \tag{96}$$

Solving the AREs (R1)–(R3) at the baseline parameters yields the Riccati gains$(P^*_{kk}, P^*_{km}, P^*_{mm})$, from which the four feedback gains are derived via Eqs.(50)–(53). Note that the gain matrix$K = \begin{pmatrix} \gamma_k & \gamma_{ik,m} \\ \gamma_{gk} & \gamma_m \end{pmatrix}$ is asymmetric since$\theta_k \neq \theta_g$ and$A \neq B$.The affine terms(α_k, α_m) follow from the linear system solved in (58)–(59), and the stationary mean field $(\bar{k}^*, \bar{m}^*)$ from (63)–(64). Two values of η_2 are used for regime comparisons: $\eta_2 = 0.1$ (low externality) and $\eta_2 = 0.8$ (high externality). Simulations run over $t \in [0, 50]$ with initial conditions $\bar{k}(0) = 1.0$, $\bar{m}(0) = 0.3$.

8.2 Externality Regimes: Capital and Military Dynamics

Figure 1: Capital Accumulation: Two Externality Regimes

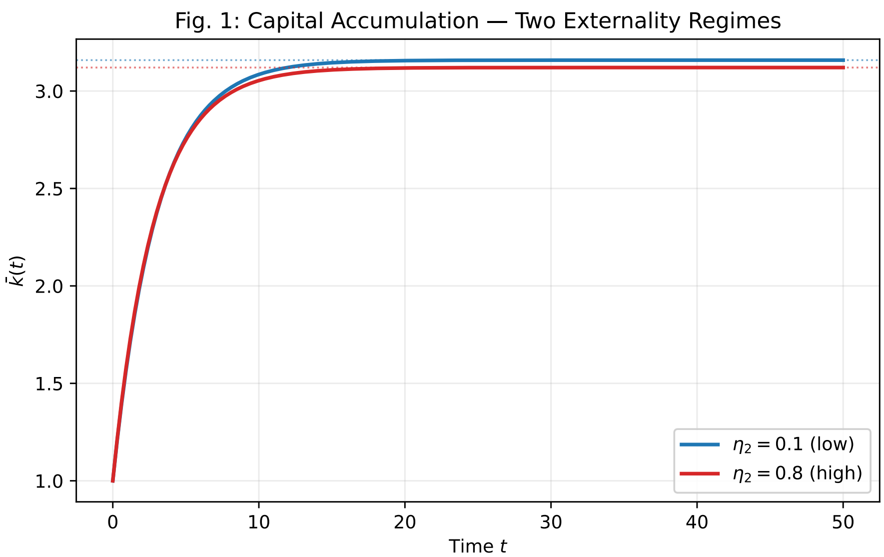
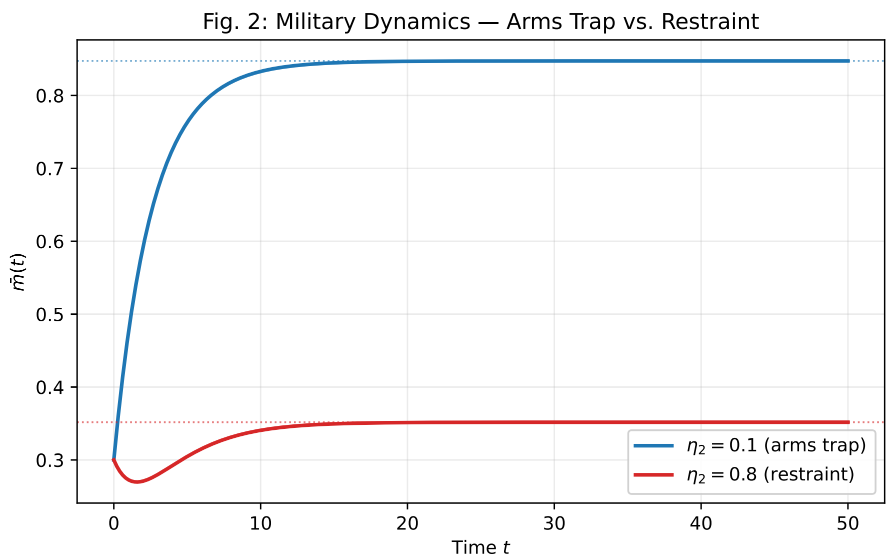


Figure 1 displays the time path of average capital$\bar{k}(t)$ under the low ($\eta_2 = 0.1$) and high ($\eta_2 = 0.8$) externality regimes. Under lowη_2, countries face weak interactive militarization costs, leading to sustained high military spending that crowds out productive capital investment. Capital accumulation stalls and may converge to a low steady state—an endogenous"arms trap." Under highη_2, the congestion externality cost induces restraint in military spending, freeing resources for capital accumulation and generating a higher long-run capital stock.

Figure 2 presents the complementary military stock paths$\bar{m}(t)$. Under lowη_2,$\bar{m}(t)$ converges to a high militarized steady state consistent with an arms-race equilibrium. Under highη_2, the equilibrium military stock is substantially lower, reflecting the strategic restraint induced by internalization of the externality.

Figure 2: Military Dynamics: Arms Trap vs. Restraint

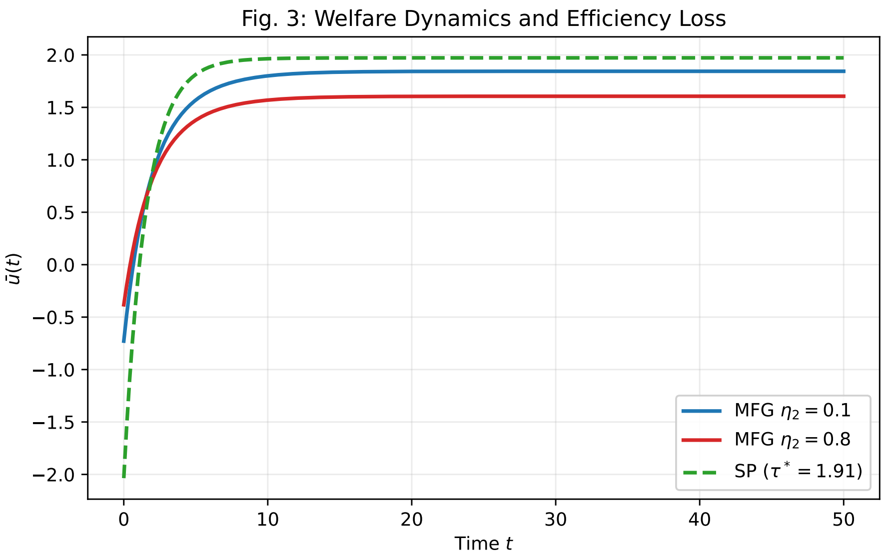


Together, Figures 1–2 illustrate the central qualitative prediction of the model: externality awareness is a necessary condition for escaping the militarization trap (tragedy of the commons).

8.3 Welfare Dynamics and Efficiency Loss

Figure 3 plots the welfare flow$\bar{u}(t)$ along three paths: MFG with$\eta_2 = 0.1$, MFG with$\eta_2 = 0.8$, and the social-planner (SP) solution from Section 6. The welfare gap$\Delta W = W_{SP} - W_{MFG}$ (Section 7) is visible as the vertical distance between the SP path and the MFG paths. Even under highη_2, the MFG welfare falls short of the SP benchmark, since individual countries cannot fully internalize the externality without a corrective tax. This confirms that market-based externality awareness (highη_2) partially but not completely substitutes for centralized regulation.

Figure 3: Welfare Dynamics and Efficiency Loss



8.4 Global Dynamics: Phase Diagrams

Figure 4: Phase Diagram, $\eta_2 = 0.1$ (Arms Trap)

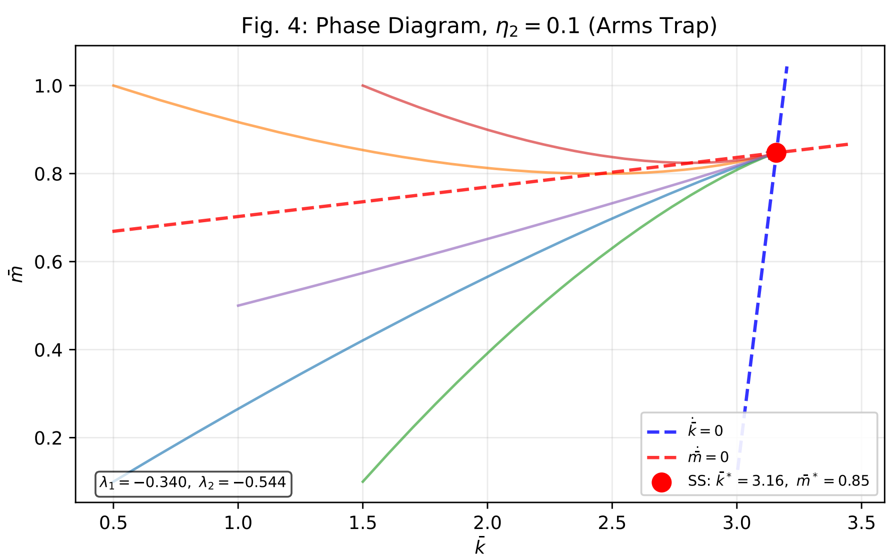
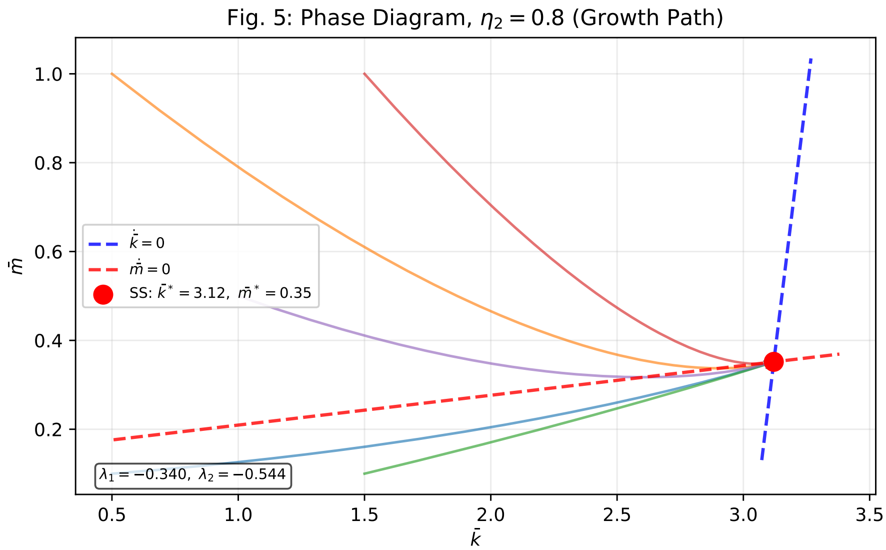


Figure 5: Phase Diagram, $\eta_2 = 0.8$ (Growth Path)



Figures 4 and 5 display the phase diagram of the mean-field ODE system in the$(\bar{k}, \bar{m})$ plane, under low and highη_2 respectively. Each figure plots the nullclines$\dot{\bar{k}} = 0$ and$\dot{\bar{m}} = 0$ together with representative trajectories from multiple initial conditions. Under lowη_2 (Figure 4), the stable attractor is located at high$\bar{m}^*$ and low$\bar{k}^*$, corresponding to the arms-trap equilibrium. Under highη_2 (Figure 5), the attractor shifts toward lower militarization and higher capital, reflecting the growth-path equilibrium. The phase diagrams also reveal the qualitative stability of each regime: the eigenvalues of the linearized system confirm that the stationary point is a stable node under both parameterizations, consistent with the uniqueness result of Proposition 5.5.

8.5 Sensitivity Analysis and Pigouvian Tax Decomposition

Figure 6 displays the Pigouvian tax$\tau^*(\eta_2)$ and its two additive components as functions of$\eta_2 \in [0, 1]$. The congestion externality component$\eta_2 \bar{m}^*_{SP}/(\rho + \delta^m)$ captures the social planner's internalization of the interactive militarization cost$-\eta_2 m_i \bar{m}$. The productivity component$D(a - b\bar{c}^*_{SP})/(\rho + \delta^m)$ captures the externality channel through which militarization reduces net output via$-D\bar{m}$ in the budget constraint. The total tax$\tau^*(\eta_2) = \lambda^*(\eta_2)$ represents the Pigouvian correction required to restore social efficiency (Section 6).

Figure 6: Pigouvian Tax Components as a Function of η_2

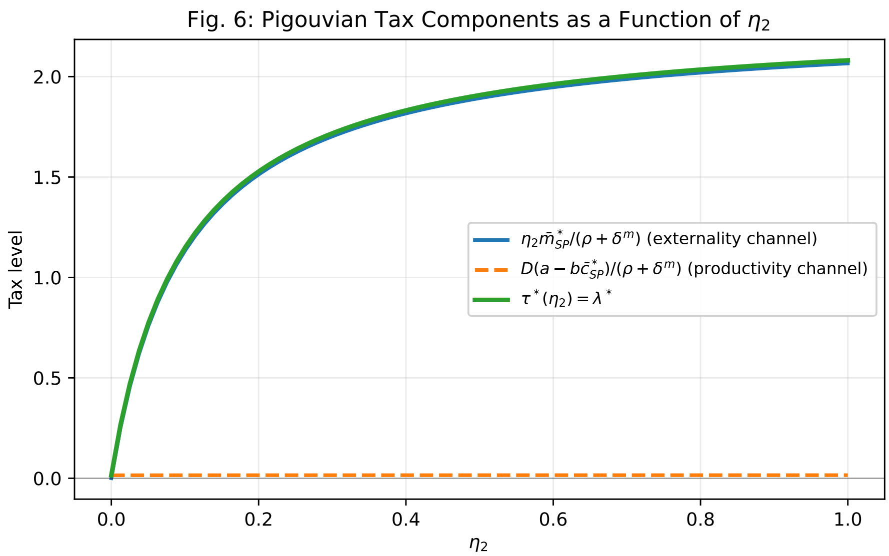


Figure 7: Riccati Gains Sensitivity to Military Disutility ϕ

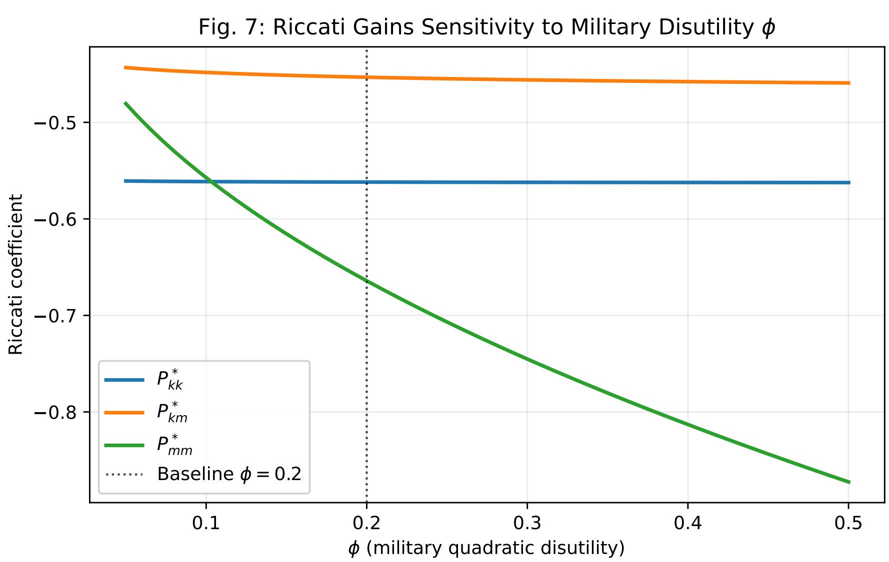


Figure 7 documents the sensitivity of the Riccati gains$(P^*_{kk}, P^*_{km}, P^*_{mm})$ to the individual military disutility parameter$\phi \in [0.05, 0.5]$, with the baseline value$\phi = 0.2$ indicated by a vertical dashed line. The gainP^*_{mm} falls (becomes more negative) as ϕ increases, reflecting that stronger individual distaste for militarization increases the value function's curvature in the m dimension. Notably, the Riccati gains and all derived feedback gains $(\gamma_k, \gamma_m, \gamma_{ik,m}, \gamma_{gk})$ are invariant to the externality parameter η_2: the equilibrium response structure is determined solely by individual utility curvature parameters, while η_2 governs the level of equilibrium militarization through the affine policy terms (α_k, α_m).

8.6 Comparative Statics and Pigouvian Tax

Figure 8: MFG vs SP Divergence in Stationary State

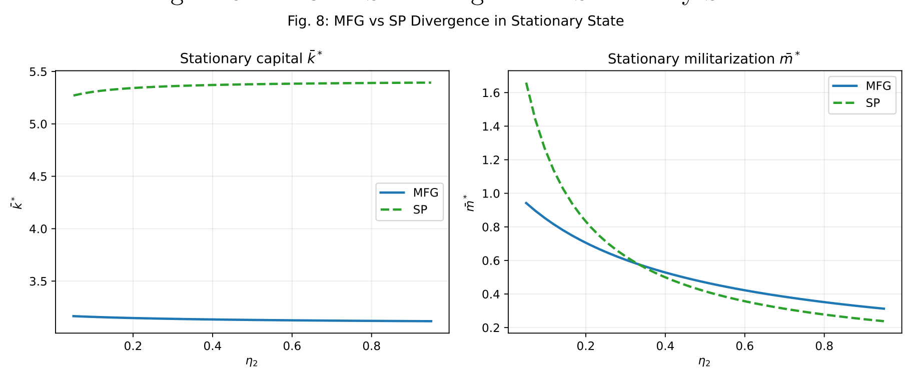
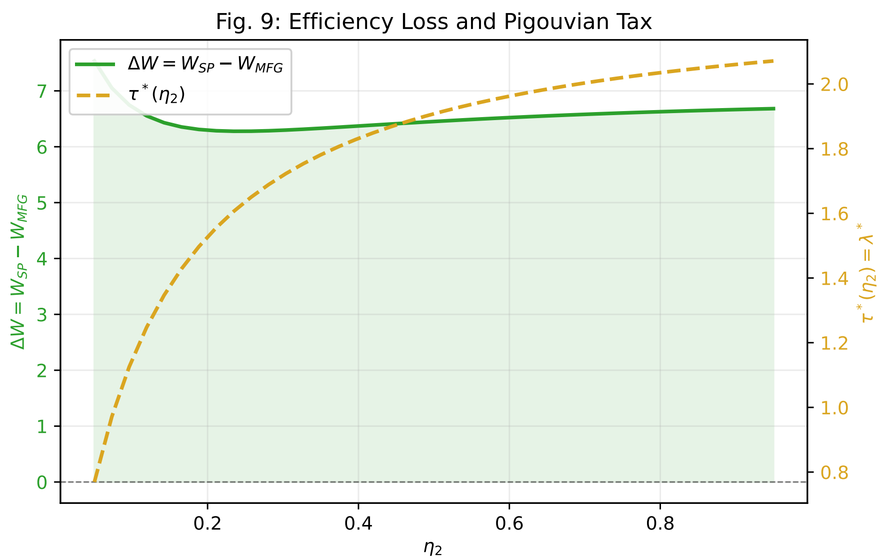


Figure 8 compares the stationary mean fields $(\bar{k}^*_{MFG}, \bar{m}^*_{MFG})$ and $(\bar{k}^*_{SP}, \bar{m}^*_{SP})$ as functions of η_2. The SP outcome exhibits strictly higher capital ($\bar{k}^*_{SP} > \bar{k}^*_{MFG}$) across the full range of η_2, reflecting the planner's internalization of the productivity externality $D\bar{m}$ which frees resources for capital accumulation. The militarization comparison is more nuanced: for sufficiently large η_2, the SP internalizes the congestion externality and chooses $\bar{m}^*_{SP} < \bar{m}^*_{MFG}$; but for small η_2, the SP may choose $\bar{m}^*_{SP} > \bar{m}^*_{MFG}$, since the dramatically higher capital base raises the marginal return to military spending while the congestion cost remains small. This non-monotonic pattern reflects the interaction between the two externality channels (productivity D and congestion η_2).

Figure 9 displays the welfare gap $\Delta W(\eta_2) = W_{SP} - W_{MFG}$ as a function of η_2. Consistent with the analytical results of Section 7, $\Delta W > 0$ across the full range, confirming that the SP always achieves strictly higher welfare than the decentralized MFG equilibrium. The welfare gap is positive even at $\eta_2 = 0$, reflecting the productivity externality channel through $D > 0$: the planner internalizes the output loss $-D\bar{m}$ that individual nations ignore. As η_2 increases, the welfare gap exhibits a U-shaped pattern: it initially decreases (as rising η_2 already induces MFG agents to restrain militarization, partially aligning decentralized behavior with the SP optimum) before increasing (as the growing congestion externality $-\eta_2 m\bar{m}$ creates larger scope for welfare improvement through internalization). The superimposed Pigouvian tax $\tau^*(\eta_2) = \lambda^*(\eta_2)$ (right axis)

Figure 9: Efficiency Loss and Pigouvian Tax



illustrates that a larger externality requires a higher corrective tax to restore social optimality. The numerically computed$\tau^*(\eta_2)$ coincides with the analytical formula from Section 6, confirming the self-consistency of the SP equilibrium. Together, Figures 8–9 provide a complete quantitative account of the efficiency costs of decentralized militarization and the tax instruments required to eliminate them.

9 Conclusion and Extensions

This paper develops a dynamic mean field game model of capital accumulation and militarization among a continuum of symmetric nations. Each country optimizes its consumption, investment, and arms spending in response to aggregate militarization, while treating the average behavior of others as a given but influential mean field. The model captures how decentralized strategic behavior, even in the absence of asymmetries or dominant actors, can give rise to persistent arms buildup, capital stagnation, and inefficient equilibria.

We formally derived the equilibrium of this system using three core mathematical objects: a backward-in-time Hamilton–Jacobi–Bellman (HJB) equation describing each country's optimal control problem, a forward-in-time Fokker–Planck equation characterizing the evolution of the population distribution across state variables (capital and military stocks), and a fixed-point condition ensuring that the mean field used in optimization corresponds to the population average generated by optimal decisions. Under both general and linear–

quadratic–Gaussian (LQG) assumptions, we proved the existence and, under regularity and monotonicity assumptions, uniqueness of equilibrium. We simplified the utility function to remove redundant interaction terms while preserving the key strategic features: the trade-off between consumption and militarization, and the disutility from systemic arms buildup. This simplification enabled closed-form optimal controls, an explicit forward system of differential equations for average capital and military stocks, and a tractable computation of equilibrium welfare.

Through numerical simulation, we compared the dynamics of capital accumulation, militarization, and welfare under different levels of strategic disutility from militarization (parametrized by η_2). We found that low values of η_2, corresponding to weak internalization of global externalities, lead to persistently high militarization and stagnating capital, with depressed long-run welfare. Conversely, higher values of η_2 induce countries to restrain their military spending, allowing capital to accumulate and welfare to rise. These simulations illustrate the central insight of the model: even among symmetric countries, the perception of externalities crucially governs whether the system converges to a peaceful investment equilibrium or falls into an inefficient arms race.

We further introduce a benevolent social planner (Section 6) who maximizes aggregate welfare by internalizing the externality that individual nations ignore. The social planner prescribes strictly lower armament spending than the decentralized equilibrium, and the gap between the two policies is characterized by an optimal Pigouvian tax$\tau^* = \lambda^*$. Imposing τ^* on military expenditure decentralizes the social optimum, restoring full efficiency without requiring centralized command. We show in Section 7 that the welfare gap $\Delta W = W_{SP} - W_{MFG}$ is strictly positive for all $\eta_2 > 0$ and is closed exactly byτ^*. The numerical simulations of Section 8 quantify both the divergence between MFG and SP outcomes across the externality spectrum and the corrective tax schedule required to eliminate it.

This framework opens a wide range of avenues for future research. One direction is to introduce heterogeneity in productivity, military capability, or exposure to conflict, resulting in asymmetric strategic responses and richer population distributions. Another is to allow for endogenous alliance formation, where coalitions of countries coordinate arms behavior to internalize externalities or deter aggression. Adding spatial structure would enable the study of regional arms races and the diffusion of militarization across borders. A further extension is to model information asymmetry and incomplete observability, incorporating belief dynamics and signaling into the mean field setting. Learning and bounded rationality could also be explored, where countries adapt based on past experiences rather than solving the full dynamic program.

Empirical calibration is another promising direction. The model could be matched to historical data on arms races, such as the Cold War, regional rivalries in South Asia or the Middle East, or contemporary military spending patterns across NATO and non-aligned states.

Overall, this model offers a rigorous, flexible, and tractable approach to understanding the decentralized origins of global militarization. It contributes to the growing literature at the intersection of macroeconomics, international relations, and applied mathematics by showing how individual strategic behavior—when shaped by perceived externalities—can produce macro-level over-militarization (tragedy of the commons) or cooperative growth. This dynamic view of equilibrium behavior under strategic uncertainty lays the groundwork for both theoretical extensions and real-world policy insights in global security and development.

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